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A Stochastic Delay Model for Pricing Corporate Liabilities

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We suppose that the price of a firm follows a nonlinear stochastic delay differential equation. We also assume that any claim whose value depends on firm value and time follows a nonlinear stochastic delay differential equation. Using self-financed strategy and replication we are able to derive a random partial differential equation (RPDE) satisfied by any corporate claim whose value is a function of firm value and time. Under specific final and boundary conditions, we solve the RPDE for the debt value and loan guarantees within a single period and homogeneous class of debt. We then analyze the risk structure of a levered firm. We also evaluate loan guarantees in the presence of more than one debt. Furthermore, we perform numerical simulations for specific companies and compare our results with existing models.

DEDICATION

To my kids, my dad, my late mom, to my brothers and sisters, to Adelaide Ngocheu for their conviction to see me embark on this fun journey. They all have been a constant source of encouragement.

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INTRODUCTION

The valuation of corporate claims has always been an important topic for finance researchers. On the one hand, bond issuers would like to know what factors affect prices and yields as yields represent their cost of capital. On the other hand, prospective bond buyers are interested in knowing how sensitive yields and yield spreads are to various relevant factors (e.g. leverage) as they develop investment strategies.

Due to the remarkable growth of the credit derivatives market, the interest in corporate claim value models and risk structure has recently increased. This growth is explained by the need of better prediction models to fit real market data.

Financial distress tends to be an important factor in many corporate decisions. The two main sources of financial distress are corporate illiquidity and insolvency. In his paper [19], Gryglewicz explains how changes in solvency affect liquidity and also how liquidity concerns affect solvency via capital structure choice. Corporate solvency is the ability to cover debt obligations in the long run. Uncertainty about average future profitability, with financial leverage, generates solvency concerns. Corporate insolvency may lead to corporate reorganization or to bankruptcy of the firm in the worst case.

Corporate bankruptcy is central to the theory of the firm. A firm is generally considered bankrupt when it cannot meet a current payment on a debt obligation. In this event the equity holders lose all claims on the firm, and the remaining loss which is the difference between the face value of the fixed claims and the market value of the firm, is supported by the debt holders. This is the definition of bankruptcy that we adopt in this thesis.

Loan guarantees have been proposed by several authors as a way to encourage new investments for companies when they become insolvent. In the contingent liabilities framework, loan guarantees and similar insurance schemes have previously been analyzed within a single maturity period and homogeneous class of debt. In their paper [52], Selby et al. then extend the analysis of the loan guarantees to a multi period and heterogeneous loan

capital structure. One of the objective of this thesis is to derive a fair formula for loan guarantees considering time delay in the firm value, within a heterogeneous loan capital structure. We can also compute the profitability index for new projects undertaken by the company.

The risk structure of interest rates on bonds with the same maturity is the degree of the likelihood of default on the payment of interest and the principal. Returns are measured by yields of the risky bonds to maturity of each bond. The difference between the yields of the risky bonds and that of the risk free bonds is called the yield spreads. This yield spread is sometimes called risk premium since it is supposed to measure the additional yield that risky bonds pay in order to motivate investors to buy risky bonds instead of the less risky ones.

There does not seem to be a consensus among researchers on what the determinants of the risk structure are. Different variables have been considered to represent a valid measure of risk depending on whether one uses the same maturity or different maturities, see ([38]). The question is: are we able to know what are the determinants of the risk structure in case the model includes information from the past?

The studies of stochastic models with memory including nonlinear equations do not allow for explicit solutions. Therefore numerical approximation methods of solution are needed. Fortunately, researchers have developed multiple numerical schemes and simulation methods over years for stochastic functionals differential equations (SFDE).

This research is the result of an attempt to obtain satisfactory answers to the questions raised above and to examine some solutions to corporate insolvency. Moreover, I analyze different situations to determine a measure of risk, corporate insolvency characteristics seen in practice, and to suggest possible extensions and improvements for further research.

In the corporate finance studies, delay equations have not been introduced. Because of the isomorphic relationship between a levered equity and a European call option (see Merton [41]) on the one hand, and the isomorphic correspondence between loan guarantees

and common stock put options (see Merton [39]), we can claim that results obtained in the theory of option pricing are feasible in corporate liabilities pricing.

The contribution of this thesis is the introduction of delay models in corporate claims pricing. In this work, we derive a formula for the price of an option used for the pricing of corporate defaultable bonds and adopt this approach to the valuation of loan guarantees for companies in financial distress. Moreover, we test our model against real data using the implicit Euler Maruyama scheme. We further analyze the risk structure of interest rates. From these analysis, we are able to distinguish what are the valid measure of risk of a firm. This contributes to the application of the model with delay on practical cases. We provide the implicit Euler-Maruyama scheme to approximate the company value. We give a scheme approximation to diligently solve the RPDE. For the discretization with respect to v, we made use of the combination difference and finite volume (methods most used are finite element and finite difference). For the discretization with respect to time, we use the exponential method on a practical case. Although the later method has been used in other application in flows, we applied it in financial problem. We perform simulations to test the models and compare with Merton's model. Overall, this work is a contribution to the area of stochastic analysis and its application to corporate claims pricing and to simulations of real world situation.

The outline of this thesis is as follows. In Chapter 1, some useful definitions and results are stated. It also presents some previous work in the field of corporate claims pricing and delay models by giving a survey of the financial markets, pricing theory in delay settings and numerical methods available to perform the valuation. In Chapter 2, we propose a generalized delay model for pricing corporate liabilities. In Chapter 3, we consider a single homogeneous class of debt to derive a formula for the fair value of a risky debt and that of a loan guarantee. Chapter 4 presents analysis of the risk structure of interest rate of a company. In Chapter 5, we derive a formula for loan guarantees in the case of more than one class of debt. Chapter 6 provides some numerical schemes used and testing our models

against real data and also for Merton's models. We also compare the graphs obtained for specific company values for both models.

CHAPTER 1

BACKGROUND AND PREVIOUS WORK

Let's start with some useful definitions and results. All the prices below are assumed to be random processes defined on some probability space $(\Omega, \mathcal{F}, P)$, where Ω is a sample space (set of all possible outcomes), $\mathcal{F}$ is the family of all events (subsets of Ω) and P is the probability measure which specifies each event's likelihood of happening. We provide some useful definitions:

Definition. (Self-financing Strategy) [42]

Let $V(t)$ be the value of a portfolio for all $t \geq 0$, $V^0(t)$ and $V^1(t)$ the values of the riskless asset and the risky asset respectively. Let $(\theta^0(t))_{t\geq 0}$ and $(\theta^1(t))_{t\geq 0}$ be some adapted processes representing the number of units of the riskless asset and that of the risky asset respectively. A *self financing strategy* is defined as the pair $(\theta^0(t), \theta^1(t))_{t\geq 0}$ that satisfies

- $\int_0^t |\theta^0(u)|du < +\infty$ a.s., $\int_0^t \left(\theta^1(u)V^1(u)\right)^2 du < +\infty$ a.s. ,
- $V(t) = \theta^0(t)V^0(t) + \theta^1(t)V^1(t) = \theta^0(0)V^0(0) + \theta^1(0)V^1(0) + \int_0^t \theta^0(u)dV^0(u) + \int_0^t \theta^1(u)dV^1(u)$ a.s.

Definition. (Admissible Self-financing Strategy) [16]

A self financing strategy is $\theta = (\theta^0, \theta^1)$ is admissible (or tame) if for some $k < \infty$,

$$\tilde{V}_\theta(t) \geq -k, \quad \text{for all } t \in [0, T] \text{ a.s.}$$

Definition. (Martingale, submartingale, supermartingale) [42]

An $\mathcal{F}_t$-adapted stochastic process $X : T \times \Omega \to \mathbb{R}$ is a *martingale, submartingale, supermartingale* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a probability measure P if it satisfies $E_P(X_t) < \infty$ and $X_s = E\left(X_t|\mathcal{F}_s\right)$, $E_P(X_t^+) < \infty$ and $X_s \leq E\left(X_t|\mathcal{F}_s\right)$ and $E_P(X_t^-) < \infty$

and $X_s \geq E\left(X_t|\mathcal{F}_s\right)$ for all $s < t \in P$-a.s., respectively. ($X^+ = X \vee 0 = 1_{X>0}X$ and $X^- = -X \wedge 0 = -1_{X>0}X$.)

Definition. (Local Martingale) [42]

An $\mathcal{F}_t$-adapted stochastic process $X : T \times \Omega \to \mathbb{R}$ is a *local martingale* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ and probability measure P if there exists a sequence T_n of stopping times such that

- $P(T_n < T_{n+1} = 1)$ and $T_n \to \infty$, P-a.s. for all n and
- the (stopped) process $X_t^{T_n} := X_{t\wedge T}$ is an $\mathcal{F}_{t\wedge T}$ martingale for each $n \geq 1$.

Definition. (Risk neutral probability) [16]

The *risk neutral probability* is a measure under which price process of any tradeable security which pays no coupons or dividends, becomes an $\mathcal{F}$-martingale.

Definition. (No arbitrage) [42]

The market consisting of $V^0(t), V^1(t)$ is said to satisfy the *no arbitrage property* if there does not exist any self-financing strategy θ such that the following relations hold:

- $V_\theta(0) = 0$ (no initial investment),
- $V_\theta(T) \geq 0$ (no risk),
- $P(V_\theta(T) > 0) > 0$ (possible profit),

where V_θ is the value of the portfolio with respect to the strategy θ.

In other words, a no arbitrage opportunity is a self strategy that consists of having a positive return at a later time if and only if an initial investment is made.

Theorem 1.0.1. *[23]*

Let $Q \in \mathcal{M}(P)$ where $\mathcal{M}(P)$ is the set of equivalent martingale to P such that the discounted

value of V_t^i is a Q-local martingale. For an admissible strategy $\theta = (\theta^0, \theta^1, ..., \theta^k)$,

$$Z(t) := \sum_{i=1}^{k} \int_0^t \theta^i(s) d\tilde{V}^i(s)$$

is a Q-local martingale and a Q-super martingale.

Thus $\mathcal{M}(P) \neq 0$ implies No Arbitrage

Proposition 1.0.2. (*First Fundamental Theorem of Asset Pricing*) [*35*]
The market model admits no arbitrage opportunities if and only if there exists a probability measure P^* equivalent to P such that the discounted value of a company is a martingale under P^*.

Theorem 1.0.3. (*Martingale Representation Theorem*) [*23*]
Suppose W is a Wiener process on a probability space $(\Omega, \mathcal{F}, P)$, where P is a probability measure. Let $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$ be the filtration generated by $\{W(t),\, t \in [0,T]\}$. If M is a $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$-martingale, then there exists an $\left(\mathcal{F}_t^W\right)_{t\in[0,T]}$-predictable process φ such that

$$\int_0^t |\varphi(s)|^2 ds < \infty \ a.s.$$

and

$$M(t) = E_P(M(0)) + \int_0^t \varphi(s) dW(s), \quad t \in [0,T].$$

Theorem 1.0.4. (*Girsanov Theorem*) [*23*]
Let $W(t)$, $t \in [0,T]$ be a standard Wiener process under the market probability measure P; and let $(\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by $\{W(t),\, t \in [0,T]\}$. Suppose that we are given a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$. Moreover, consider $\theta(t)$, $t \in [0,T]$ an $(\mathcal{F}_t)$-adapted integrable and predictable process such that $\int_0^T |\theta(s)|^2 ds < \infty$ a.s. Define a process

$$W^*(t) = -\int_0^t \theta(s) ds + W(t)$$

and

$$L_t = exp\Big(-\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t |\theta(s)|^2 ds\Big),\ t \in [0,T].$$

In addition, define the probability measure P^* on $(\Omega, \mathcal{F})$, equivalent to P, by $dP^* := L_T dP$. Then under P^* the process $W^*(t)$ is a standard Wiener process.

Proof. see [23].

1.1 FINANCE LITERATURE REVIEW

In the literature of corporate finance, Black and Scholes ([8], 1973) and Merton ([38], 1974) appear to be the main pioneers in the derivation of formulas for corporate claims. Merton further analyzed the risk structure of interest rates. More specifically, he found the relation between corporate bond spreads and government bond, and attempted to determine a valid measure of risk. Merton ([39], 1977) derived a fair value of loan guarantees for a single class of debt, and Selby ([52], 1988) studied the case of multiple debts.

Of course several features of individual firms and claims can affect their values and hence their yield spreads. In fact, the assumption of constant volatility in financial models such as in the original Black-Scholes model from which most claims derivations are inspired, is incompatible with derivatives prices observed in the market. See for example [5]. Indeed, many empirical tests have shown that although the Black-Scholes price is fairly close to the observed prices, there are well-known discrepancies such as the option smile. Moreover, Black, Scholes and Merton ignore the history in their models. Many researchers have considered various volatility models with the intent to match the real market data.

1.1.1 Volatility Function Models for Pricing

Volatility functions have been modeled and studied under different assumptions:

- Volatility is assumed to be constant ([8]) or more generally to be a deterministic function of time ([58]). In this approach, the volatility is independent of the current

price of the underlying asset.

- Volatility is assumed to be a function of time and current level of asset ([58]); because asset price and volatility are perfectly correlated, there is only one source of randomness. The time and level dependence of the volatility usually prevents the existence of a closed form solution. However, the no arbitrage argument and the completeness of the market remain unchanged. Several methods have been developed to derive the formula for a corresponding option price. Different methods were used such as stochastic calculus via Itô's lemma, solving a partial differential equation obtained via Green function. Also for the option price the formula maybe interpreted as a continuous time limit of a binomial random model.

- Volatility is assumed to involve a different source of randomness which may be perfectly correlated to the initial Wiener process in the underlying asset price. In case the sources of randomness are not correlated, one obtains a stochastic volatility model.

1.1.2 Model with Delay

The Efficient Markets hypothesis as described in ([8]) implies that all information available is already reflected in the present underlying asset price and that no information on its history can help in the prediction of the future performance. However, some statistical studies of stock prices indicate the dependence on past returns ([7], [51]). Bernard and Thomas in their paper ([7]) analyzed the drift of estimated cumulative abnormal returns after earnings are announced. They observed that the returns continue to drift up for good news firms and drift down for bad news firms. Their two explanations suggest that:

- at least a portion of the price response of new information is delayed. They explained that the delay might occur either because traders fail to assimilate available information, or because certain costs (such as transaction costs) exceed gains from immediate exploitation of information for a sufficiently large number of traders.

- because the capital asset pricing model used to calculate the abnormal returns is either incomplete or misestimated, researchers fail to adjust raw returns fully for risk.

They concluded that their results are consistent with a delayed response to information.

Kind, Liptser and Runggaldier in their paper ([29]) obtained a diffusion approximation result for processes satisfying equations with past dependent coefficients and their result was applied to a model of option pricing in which the underlying asset price volatility depends on the past evolution. It was shown that the volatility is a deterministic function of time, which is determined by the initial stock price path.

In their work ([13]), Chang and Yoree studied the pricing of a European contingent claim for the Black Scholes securities market with a hereditary price structure and showed that Black Scholes formula can be generalized to include the Black Scholes securities market with affine hereditary price structure.

Mohammed et al. ([2]) derived a delayed option price formula by solving a random pde similar to that of Black and Scholes. A closed form formula is obtained for the last delay period.

In all the above mentioned papers the authors show that the past dependence of the stock price process is an important feature to capture the better prediction of the future dynamic. Therefore it should not be ignored.

This is a motivation for introducing delay models in corporate finance pricing.

1.2 STOCHASTIC MODELS REVIEW

1.2.1 A General Delay Model

The content introduced in this section will be used throughout this chapter. Let $L > 0,\ T > 0,$ and n a positive integer. Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space

with the Euclidean norm

$$|x| := \sqrt{x_1^2 + ... + x_n^2}, \quad x = (x_1, ..., x_n) \in \mathbb{R}^n,$$

and $\mathbb{R}^{n\times l}$ the space of real $n \times l$ matrices with the Frobenius norm

$$|A|_{n\times l} := \sqrt{\text{trace}(A^T A)},$$

where A^T is the transpose of A.

For each $t \in [0, \infty)$ and each continuous path $x : [-L, \infty) \to \mathbb{R}^n$, define the segment $x_t : [-L, 0] \to \mathbb{R}^n$ by

$$x_t(u) := x(t + u) \ \text{ a.s., , } \ -L \leq u \leq 0.$$

Denote by $C_n := C([-L, 0]; \mathbb{R}^n)$ the Banach space of all continuous paths $\eta : [-L, 0] \to \mathbb{R}^n$ given the supremum norm

$$\|\eta\|_{C_n} := \sup_{s\in[-L,0]} |\eta(s)|, \ \eta \in C_n.$$

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ satisfying the following conditions: the filtration $(\mathcal{F}_{t\geq 0})$, is right-continuous, and each $\mathcal{F}_t, t \in [0, \infty)$, contains all P-null sets in $\mathcal{F}$.

Existence, Uniqueness and Stability

Let $H : [0, T] \times C_n \to \mathbb{R}^n$ and $G : [0, T] \times C_n \to \mathbb{R}^{n\times l}$ be jointly continuous and Lipschitz in the second variable, uniformly with respect to the first variable, viz.

$$\|H(t, \eta_1) - H(t, \eta_2)\|_{\mathbb{R}^n} + \|G(t, \eta_1) - G(t, \eta_2)\|_{\mathbb{R}^{n\times l}} \leq \mu\|\eta_1 - \eta_2\|_{C_n} \tag{1.1}$$

for all $t \in [0,T]$ and $\eta_1, \eta_2 \in C_n$. The Lipschitz constant μ is independent of $t \in [0,T]$. Because H and G are continuous, for each $(\mathcal{F}_t)_{0\le t\le T}$-adapted process $x : [0,T] \to C_n$, the processes $H(.,x(.))$ and $G(.,x(.))$ are also $(\mathcal{F}_t)_{0\le t\le T}$-adapted. Let W be an l-dimensional Brownian Motion on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\ge 0}, P)$ and let $\eta \in C_n$.

The functionals H and G also satisfy the linear growth property

$$\|H(t,\eta)\|_{\mathbb{R}^n} + \|G(t,\eta)\|_{\mathbb{R}^{n\times l}} \le C(1 + \|\eta\|_{C_n}), \tag{1.2}$$

where C is a positive constant independent of $t \in [0,T]$. To see this, set $\eta_1 = \eta$ and $\eta_2 = 0$ in the Lipschitz condition (1.1), and use joint continuity of H and G. Consider the stochastic functional differential equation (SFDE)

$$\left.\begin{aligned} dx(t) = H(t,x_t)dt + G(t,x_t)dW(t),\ \ t \in [0,T] \\ x(t) = \varphi(t),\ \ t \in [-L,0], \end{aligned}\right\} \tag{1.3}$$

where the segment $x_t \in C_n$ is as given before. A solution of (1.3) is a process $x \in C([-L,T],\mathbb{R}^n)$ adapted to $(\mathcal{F}_t)_{0\le t\le T}$, with initial process φ, which satisfies the Itô integral equation

$$x(t) = \begin{cases} \varphi(0) + \int_0^t H(s,x_s)ds + \int_0^t G(s,x_s)dW(s),\ \ t \in [0,T] \\ \varphi(t),\ \ t \in [-L,0], \end{cases} \tag{1.4}$$

almost surely.

Theorem 1.2.1. *Suppose H and G satisfy the hypotheses discussed above (i.e. Lipschitz and joint continuity conditions) and let $\varphi : \Omega \to C_n, \mathcal{F}_0$-adapted.*

Then the SDDE (1.3) has a unique $(\mathcal{F}_t)_{0\le t\le T}$-adapted solution $x^\varphi : [-L,T] \to \mathbb{R}^n$ with $x_t^\varphi \in C_n$ sample continuous for all $t \in [0,T]$ and $x^\varphi \in C([-L,T],\mathbb{R}^n)$ for all $T > 0$. For

a given φ, uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted processes in $C([-L,T],\mathbb{R}^n)$.

The following theorem shows the Liptchitz continuous dependence of the solution on the initial condition Let $t_1 < t \in [0.T]$. The SFDE

$$x(t) = \begin{cases} \varphi(0) + \int_{t_1}^{t} H(s,x_s)ds + \int_{t_1}^{t} G(s,x_s)dW(s), \;\; t \in [t_1,T] \\ \varphi(t-t_1), \;\; t \in [t_1-L,t_1], \end{cases} \tag{1.5}$$

can be solved for any $\mathcal{F}_t$-measurable initial process $\varphi : \Omega \to C_n$ at time t_1 with a unique solution $x \in C([t_1-L,t_1],\mathbb{R}^n)$. From the above, one can build the family of maps

$$D_t^{t_1}: \;\; L^2(\Omega,C_n) \to L^2(\Omega,C_n) \qquad t > t_1$$

$$\varphi \mapsto x_t$$

We define $D_t^0 = D_t, \; t > 0$ as

$$D_t : L^2(\Omega,C_n) \to L^2(\Omega,C_n).$$

Theorem 1.2.2. *Assume G and W in equation* (1.3) *satisfy the conditions of theorem 1.2.1. Then each map*

$$D_t : C([-L,0],\mathbb{R}^n) \to C([-L,0],\mathcal{F}_t), \;\; t \in [0,T],$$

is Lipschitz; indeed for all $t \in [0,T]$, $\varphi_1,\varphi_2 \in C([-L,0],\mathbb{R}^n), \mathcal{F}_0$-adapted, if μ is the Lipschitz constant of H and G which is independent of $t \in [0,T]$, and M is a martingale

constant then

$$\|D_t(\varphi_2) - D_t(\varphi_1)\|_{L^2(\Omega,C_n)} \leq \sqrt{2}\|\varphi_2 - \varphi_1\|_{L^2(\Omega,C_n)} e^{M\mu^2 t}.$$

1.2.2 Euler Maruyama Scheme for SFDEs

SFDE approximations have been studied by several authors. The strong (or almost sure) Euler scheme (order $\frac{1}{2}$) and the strong Milstein scheme (order 1) for SDDEs were developed by Ahmed, Elsanousi and Mohammed [1], Mohammed [43], Hu, Mohammed and Yan [21] and Baker and Buckwar [3], Küchler and Platen [33]. Weak approximations for SODEs (stochastic ordinary differential equations) are well-developed by Bally and Talay, Kloeden and Platen, Kohatsu-Higa in [4, 30, 32] respectively. : Buckwar and Shardlow ([9]) studied the weak Euler scheme of order 1 following the n-dimensional SFDE with linear smooth memory drift term and memoryless diffusion term:

$$x(t) = \begin{cases} v + \int_0^t \int_{-\tau}^0 x(u+s)\mu(ds)du + \int_0^t f(x(u))du + \int_0^t g(x(u))dW(u), t \geq 0 \\ \eta(t), -\tau < t < 0. \end{cases}$$

In the above equation, the initial condition $(v, \eta) \in M_2 := R^n \times C([-\tau, 0], R^n)$. Buckwar and Shardlow embeded the above SFDE in an infinite-dimensional nondelay stochastic equation in the Hilbert space M_2, then performed the weak numerical approximation on the induced non delay equation in M_2. The weak approximation in M_2 following duality methods for weak Euler scheme has been independently studied in [14]. In [10] the authors proved the weak convergence of order 1 of the Euler scheme for fully nonlinear SDDEs with multiple discrete delays and smooth memory. Further, in their paper [21] Mohammed et al. proved the strong convergence of the Euler-Maruyama scheme for SFDEs.

Strong discrete-time approximations for SFDEs have been derived in [21], including an Euler-Maruyama scheme for a class of SFDEs with mixed discrete and continuous delay.

The convergence of a Euler-Maruyama scheme for the SFDE (1.3), was given [21].

Define the projection $P : C_n \to \mathbb{R}^{nl}$ associated with $u_1, ..., u_l \in [-L, 0]$ by

$$P(\zeta) := (\zeta(u_1), ..., \zeta(u_l)) \in \mathbb{R}^{nl}, \text{ for all } \zeta \in C_n. \tag{1.6}$$

Definition. A function $Q \in C([0, T] \times C_n, \mathbb{R})$ is *tame* if there exist $q \in C([0, T] \times \mathbb{R}^{nl}, \mathbb{R})$ and a projection P such that

$$Q(t, \zeta) = q(t, P(\zeta)), \quad \text{ for all } t \in [0, T] \text{ and } \zeta \in C_n. \tag{1.7}$$

Assume that $G : [0, T] \times \mathbb{R}^{nl_1} \to L(\mathbb{R}^m, \mathbb{R}^n)$ and $H : [0, T] \times \mathbb{R}^{nl_2} \to \mathbb{R}^n$ are Lipschitz viz. there are two constants $\gamma_1, \gamma_2 > 0$ such that for all $t \in [0, T]$

$$|G(t, x_2) - G(t, x_1)| \le \gamma_1 |x_2 - x_1| \quad \text{ for all } x_1, x_2 \in \mathbb{R}^{nl_1}, \tag{1.8}$$

$$|H(t, y_2) - H(t, y_1)| \le \gamma_2 |y_2 - y_1| \quad \text{ for all } y_1, y_2 \in \mathbb{R}^{nl_2}. \tag{1.9}$$

Assume also

$$\sup_{0 \le t \le T} (|G(t, 0)| + |H(t, 0)|) < \infty. \tag{1.10}$$

We want to approximate the solution $X(t)$ to the SDDE (1.3). Let $(u_{1,1}, ..., u_{1,l_1}) \in [-L, 0]^{l_1}$ and $(u_{2,1}, ..., u_{2,l_2}) \in [-L, 0]^{l_2}$ be two sets of points associated with the projections P_1 and P_2 respectively. Let $W(t),\ t \in [0, \infty)$ be a m-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, and $\varphi : \Omega \to C_n$ be a random initial process independent of $W(t),\ t \in [0, T]$. Under the Lipschitz and boundedness conditions (1.8), (1.9) and (1.10),

Mohammed had proven in [45] the following class of SDDE's

$$X(t) = \begin{cases} \varphi(0) + \int_0^t H(s, P_2(X_s))ds + \int_0^t G(s, P_1(X_s))dW(s), & t \in [0, \infty) \\ \varphi(t), & t \in [-L, 0) \end{cases} \tag{1.11}$$

has a unique strong solution. Let $\pi : 0 = t_0 < t_1 < ... < t_{k-1} < t_k = T$ be a partition of $[0, T]$ with mesh

$$|\pi| := \max_{1 \le i \le n} [t_i - t_{i-1}].$$

Then define the Euler Maruyama approximation X^π for the solution X of (1.11) by

$$X^\pi(t) = \begin{cases} X^\pi(t_{i-1}) + H(t_{i-1}, P_2(X^\pi_{t_{i-1}}))(t - t_{i-1}) + G(t_{i-1}, P_1(X^\pi_{t_{i-1}}))(W(t) - W(t_{i-1})), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (t_{i-1}, t_i] \\ \varphi^\pi(t), \quad t \in [-L, 0] \end{cases} \tag{1.12}$$

where $X^\pi_t = X^\pi(t + u)$, $u \in [-L, 0]$, $t \ge 0$, φ^π is the piecewise linear approximation $\varphi^\pi(u) := \dfrac{(t_s - u)\varphi(t_{s-1}) + (u - t_{s-1})\varphi(t_s)}{t_s - t_{s-1}}, u \in [t_{s-1}, t_s]$ with the negative partition $\pi \equiv \pi^* : -L = t_{-i} < t_{-i+1} < t_{-i+2} < ... < t_{-i+(i-1)} < t_0 = 0$ i.e. $s \in [-i, 0]$. We consider the partition $\pi_p := \{t_s : s = -i, ..., 0, ...k\}$ of the interval $[-L, T]$.

Assume the regularity condition:

$$\|H(t_2, \varphi) - H(t_1, \varphi)\|_{\mathbb{R}^n} + \|G(t_2, \varphi) - G(t_1, \varphi)\|_{L(\mathbb{R}^m, \mathbb{R}^n)} \le \beta(1 + \|\varphi\|_{C_n})|t_2 - t_1|^{\frac{1}{2}}. \tag{1.13}$$

Mohammed et al. in [21] showed that under the regular condition (1.13) on the coefficients, one has the error estimate

$$E \sup_{0 \le t \le T} \|X^\pi_t - X_t\|^p_{C_n} \le K(p)|\pi|^{\frac{p}{2}}, \quad \text{for any } p \ge 1, \tag{1.14}$$

which shows that the Euler Maruyama scheme has a strong order of convergence 0.5.

We now give a proof of the estimate (1.14). Without loss of generality assume that T is a multiple of i. Define for any $u \in [-L, T]$

$$\lfloor u \rfloor := t_i, \text{ if } t_i \leq u < t_{i+1}, \quad -L \leq i \leq n-1.$$

With this notation, we can write the Euler-Maruyama approximation in the form

$$X^{\pi}(t) = \begin{cases} \varphi(0) + \int_0^t H(\lfloor s \rfloor, P_2(X^{\pi}_{\lfloor s \rfloor}))ds + \int_0^t G(\lfloor s \rfloor, P_1(X^{\pi}_{\lfloor s \rfloor}))dW(s), \ \ t \in [0, \infty) \\ \varphi(t), \ \ t \in [-L, 0). \end{cases} \tag{1.15}$$

We want to show that the Euler-Maruyama approximation $X^{\pi}(t)$ converges to the solution $X(t)$ of (1.11).

To show this, let $p \geq 1$ such that $|\pi| = \frac{1}{p}$. Consider the sequence $(X^p(t))_{-L \leq t \leq T}$ defined as

$$X^p(t) = \begin{cases} \varphi(0) + \int_0^t H(s - \frac{1}{p}, X^p_{s-\frac{1}{p}})ds + \int_0^t G(s - \frac{1}{p}, X^p_{s-\frac{1}{p}})dW(s), \ \ t \in [0, \infty) \\ \varphi(t), \ \ t \in [-L, 0). \end{cases} \tag{1.16}$$

The above SFDE is a general case of the SDDE (1.15) It easy to see that the above $X^{\pi}(t) = X^p(t)$ for all $t \in [-L, \infty)$. If we rewrite (1.3), (**??**) and (1.16) with no time dependency, we have

$$\left.\begin{aligned} dX(t) = H(X_t)dt + G(X_t)dW(t), \ \ t \in [0, T] \\ X(t) = \varphi(t), \ \ t \in [-L, 0], \end{aligned}\right\} \tag{1.17}$$

$$X^{\pi}(t) = \begin{cases} \varphi(0) + \int_0^t H(X^{\pi}_{\lfloor s \rfloor})ds + \int_0^t G(X^{\pi}_{\lfloor s \rfloor})dW(s), \ \ t \in [0, \infty) \\ \varphi(t), \ \ t \in [-L, 0) \end{cases} \tag{1.18}$$

$$X^{p}(t) = \begin{cases} \varphi(0) + \int_0^t H(X^{p}_{s-\frac{1}{p}})ds + \int_0^t G(X^{p}_{s-\frac{1}{p}})dW(s), \ \ t \in [0, \infty) \\ \varphi(t), \ \ t \in [-L, 0) \end{cases} . \tag{1.19}$$

Considering $\lfloor s \rfloor = s - \frac{1}{p}$ the two equations (1.17) and (1.18) are the same. We first state the following four results:

Theorem 1.2.3 (Jensen's inequality)**.** *Let $\theta : \mathbb{R} \to \mathbb{R}$ be a real convex function.*

- *For a sequence $(x_i)_{i=1,\dots k} \in \mathbb{R}$, the finite form of the Jensen's inequality is given by*

$$\theta\left(\sum_{i=1}^{k} x_i\right) \leq \frac{1}{k}\sum_{i=1}^{k} \theta(kx_i),$$

- *Let $f : [a, b] \to \mathbb{R}$ be a nonnegative function that is Lebesgue integrable. Then the integral form of the Jensen's inequality is given by*

$$\theta\left(\int_a^b f(x)dx\right) \leq \frac{1}{b-a}\int_a^b \theta((b-a)f(x))dx.$$

The above inequalities are proved in Rudin [50].

Theorem 1.2.4. *Let $W : [a, b] \times \Omega \to \mathbb{R}^l$ be an l-dimensional Brownian Motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{a \leq t \leq b}, P)$. Suppose $F : [a, b] \times \Omega \to L(\mathbb{R}^l, \mathbb{R}^n)$ is measurable, $(\mathcal{F}_t)_{a \leq t \leq b}$-adapted and $\int_a^b E|F(s, .)|^{2k}ds < \infty$, for an integer $k \geq 1$. Then*

$$E \sup_{a \leq t \leq b} \left| \int_a^t F(s, .)dW(s) \right|^{2k} \leq c_k(l, n)(b-a)^{k-1} \int_a^b E|F(s, .)|^{2k}ds$$

$$\textit{where } c_k(l,n) := n^{k-1}\left(\frac{4k^3l^2}{2k-1}\right)^k .$$

For a proof of the above result see [49] and [45] (Chap. 1 Theorem 8.5).

Lemma 1.2.5 (Grownwall's inequality). *Let $\theta, \varphi_1,$ and φ_2 be real-valued non-negative functions defined on $[a,b]$. Assume that θ and φ_2 are continuous and φ_1 is non-decreasing. If*

$$\theta(t) \le \varphi_1(t) + \int_a^t \varphi_2(s)\theta(s)ds, \;\; t \in [a,b],$$

then

$$\theta(t) \le \varphi_1(t)\exp\left(\int_a^t \varphi_2(s)ds\right), \;\; t \in [a,b].$$

The above result is proved in Willet [57].

Theorem 1.2.6 (Kolmogorov's continuity criterion). *Let $(X^n(t))_{n=1}^{\infty}, \; t \in [0,T]$, be a sequence of stochastic processes with values in a Banach space E. Assume that there exist positive constants $\alpha_1, \; \alpha_2, \; \gamma > 1$, all independent of n, such that*

$$E\left(\|X^n(t_2) - X^n(t_1)\|_E^{\alpha_1}\right) \le \gamma |t_2 - t_1|^{\alpha_2+1}, \quad \textit{for all } t_1, t_2 \in [0,T].$$

Then each X^n has a continuous modification $\tilde{X}^n$. Further, let β be an arbitrary positive number less than $\frac{\alpha_2}{\alpha_1}$. Then there exists a positive random variable η_n with $E\left(\eta_n^{\alpha_1}\right) < K_1$, where K_1 is a constant independent of n, such that

$$\left\|\tilde{X}^n(t_2) - \tilde{X}^n(t_1)\right\|_E \le \eta_n |t_2 - t_1|^{\beta}, \quad \textit{for all } t_1, t_2 \in [0,T] \textit{ a.s.}.$$

The proof can be found in Kunita [34].

We will proceed by integrating forward over steps of length $\frac{1}{p}$.

Lemma 1.2.7. *Let X^p be defined as in* (1.18) *and $t \in [-L,T]$. Then*

(i) $X_t^p, t \in [0,T]$ *and* $X^p(t), t \in [-L,T]$ *are well defined and* $\mathcal{F}_t$*-measurable.*

(ii) $(X^p(s))_{-L\le s\le t} \in L^2(\Omega, C([-L,t], \mathbb{R}^n))$ *and* $X_t^p \in L^2(\Omega, C_n)$*, such that*

$$E\left(\sup_{s\in[-L,t]} |X^p(s)|^2\right) + \|X_t^p\|^2_{L^2(\Omega,C_n)} \le c_1,$$

where c_1 *is a constant independent of* p*.*

(iii) Let $q \ge 1$ *be an integer. Each* X^p *satisfies*

$$E|X^p(t_2) - X^p(t_1)|^{2q} \le c_q |t_2 - t_1|^q \textit{ for all } t_1, t_2 \in [0,T],$$

where c_q *is a constant independent of* p*.*

Proof. We prove (i) by induction on m. Without loss of generality, consider T a positive integer. If $t \in [-L, -L + \frac{1}{p}]$ and $t \in [0, \frac{1}{p}]$ for $X(t)$ and X_t respectively, then (i) holds trivially. Let m be an integer such that $-Lp + 1 \le m \le pT - 2$. Assume that property (i) holds for all $t \in \left[\frac{m}{p}, \frac{m+1}{p}\right]$. Let $t \in \left[\frac{m+1}{p}, \frac{m+2}{p}\right]$ then

$$X^p(t) = X^p\left(\frac{m+1}{p}\right) + \int_{\frac{m+1}{p}}^t H\left(X^p_{s-\frac{1}{p}}\right) ds + \int_{\frac{m+1}{p}}^t G\left(X^p_{s-\frac{1}{p}}\right) dW(s). \tag{1.20}$$

By induction hypothesis, $(X^p(t))_{t\in[\frac{m}{p},\frac{m+1}{p}]}$ is well defined, continuous and $(\mathcal{F}_t)_{t\in[\frac{m}{p},\frac{m+1}{p}]}$, then by Lemma 2.1 in Mohammed [45] chap.2, $(X_t^p)_{t\in[\frac{m}{p},\frac{m+1}{p}]}$ is well defined, continuous and $(\mathcal{F}_t)_{t\in[\frac{m}{p},\frac{m+1}{p}]}$-adapted. Hence $X^p_{t-\frac{1}{p}}$ is $\mathcal{F}_{t-\frac{1}{p}}$-measurable for all $t \in \left[\frac{m+1}{p}, \frac{m+2}{p}\right]$. Thus for all $t \in \left[\frac{m+1}{p}, \frac{m+2}{p}\right]$ $H\left(X^p_{t-\frac{1}{p}}\right)$ and $G\left(X^p_{t-\frac{1}{p}}\right)$ are continuous and $(\mathcal{F}_t)_{t\in[\frac{m+1}{p},\frac{m+2}{p}]}$-adapted. Consequently by (1.20), $(x^p(t))_{t\in[\frac{m+1}{p},\frac{m+2}{p}]}$ is well defined, continuous semi-martingale and $(\mathcal{F}_t)_{t\in[\frac{m+1}{p},\frac{m+2}{p}]}$-adapted. Therefore (i) is true for all $t \in \left[\frac{m+1}{p}, \frac{m+2}{p}\right]$.

To complete the proof of (ii), we first write the following trivial inequalities for all

$t \in \left[0, \frac{m+2}{p}\right]$

$$E\left(\sup_{u\in[-L,0]} |X_t^p(u)|^2\right) \leq E\left(\sup_{u\in[-L,t]} |X^p(u)|^2\right) \leq \|\varphi\|^2_{L^2(\Omega,C_n)} + E\left(\sup_{u\in[0,t]} |X^p(u)|^2\right). \tag{1.21}$$

But for all $t \in \left[0, \frac{m+2}{p}\right]$,

$$\begin{aligned} & E\left(\sup_{u\in[0,t]} |X^p(u)|^2\right) \\ & \quad = E\left(\sup_{u\in[0,t]} \left|\varphi(0) + \int_0^u H\left(X^p_{s-\frac{1}{p}}\right) ds + \int_0^u G\left(X^p_{s-\frac{1}{p}}\right) dW(s)\right|^2\right) \\ & \quad \text{by Theorem 1.2.3} \\ & \quad \leq 3E\left(\sup_{u\in[0,t]}\left(|\varphi(0)|^2 + \left|\int_0^u H\left(X^p_{s-\frac{1}{p}}\right) ds\right|^2 + \left|\int_0^u G\left(X^p_{s-\frac{1}{p}}\right) dW(s)\right|^2\right)\right). \end{aligned} \tag{1.22}$$

Again from Theorem 1.2.3,

$$\left|\int_0^t H\left(X^p_{s-\frac{1}{p}}\right) ds\right|^2 \leq t \int_0^t \left\|H\left(X^p_{s-\frac{1}{p}}\right)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} ds, \quad \text{for all } t \in \left[0, \frac{m+2}{p}\right]. \tag{1.23}$$

Moreover by Theorem 1.2.4, for all $t \in \left[0, \frac{m+2}{p}\right]$

$$E \sup_{0\leq u\leq t} \left|\int_0^t G\left(X^p_{s-\frac{1}{p}}\right) dW(s)\right|^2 \leq 4l^2 \int_0^t E\left\|G\left(X^p_{s-\frac{1}{p}}\right)\right\|^2_{L^2(\Omega,\mathbb{R}^{n\times l})} ds. \tag{1.24}$$

Further, by the linear growth property of H and G, we have for all $t \in \left[0, \frac{m+2}{p}\right]$

$$\begin{aligned}
&t \int_0^t \left\| H\left(X^p_{s-\frac{1}{p}}\right)\right\|^2_{L^2(\Omega,\mathbb{R}^n)} ds + 4l^2 \int_0^t \left\| G\left(X^p_{s-\frac{1}{p}}\right)\right\|^2_{L^2(\Omega,\mathbb{R}^{n\times l})} ds \\
&\quad \le (t+4l^2)K^2 \int_0^t \left(1 + \|x^p_{s-\frac{1}{p}}\|_{L^2(\Omega,C_n)}\right)^2 ds \\
&\quad \le 2TK^2(T+4l^2)\left(1+\|\varphi\|^2_{L^2(\Omega,C_n)}\right) + 2K^2(T+4l^2)\int_0^t E\left(\sup_{0\le u\le s}|X^p(u)|^2\right) ds.
\end{aligned} \tag{1.25}$$

By (1.21), (1.22), (1.23) and (1.24) we have

$$E\left(\sup_{u\in[0,t]}|X^p(u)|^2\right) \le K_1 + K_2 \int_0^t E\left(\sup_{u\in[0,s]}|X^p(u)|^2\right) ds, \tag{1.26}$$

where $K_1 := 3\|\varphi\|^2_{L^2(\Omega,C_n)} + 6TK^2(T+4l^2)\left(1+\|\varphi\|^2_{L^2(\Omega,C_n)}\right)$ and $K_2 := 6K^2(T+4l^2)$. By Gronwall's Lemma, we obtain

$$E\left(\sup_{u\in[0,t]}|x^p(u)|^2\right) \le K_1 e^{K_2 T}. \tag{1.27}$$

Hence

$$E\left(\sup_{u\in[0,t]}|X^p(u)|^2\right) + \|X^p_t\|_{L^2(\Omega,C_n)} \le 2K_1 e^{K_2 T} + 2\|\varphi\|_{L^2(\Omega,C_n)}. \tag{1.28}$$

To prove (iii), we use Jensen's inequality, Theorem 1.2.4, (ii) and the linear growth property

of H and G to obtain for all $0 \le t_1 < t_2 \le T$

$$
\begin{aligned}
E\left|X^p(t_2) - x^p(t_1)\right|^{2q} &= E\left|\int_{t_1}^{t_2} H\left(X^p_{s-\frac{1}{p}}\right) ds + \int_{t_1}^{t_2} G\left(X^p_{s-\frac{1}{p}}\right) dW(s)\right|^{2q} \\
&\le 2^{2q-1}|t_2 - t_1|^{2q-1} \int_{t_1}^{t_2} E\left\|H\left(X^p_{s-\frac{1}{p}}\right)\right\|^{2q}_{L^2(\Omega,\mathbb{R}^n)} ds \\
&+2^{2q-1}(n|t_2 - t_1|)^{q-1}\left(\tfrac{4q^3l^2}{2q-1}\right)^q \int_{t_1}^{t_2} E\left\|G\left(X^p_{s-\frac{1}{p}}\right)\right\|^{2q}_{L^2(\Omega,\mathbb{R}^{n\times l})} ds \\
&\le 2^{2q-1}|t_2 - t_1|^{q-1}\left(T^q + n^{q-1}\left(\tfrac{4q^3l^2}{2q-1}\right)^q\right) K^{2q} \int_{t_1}^{t_2} E\left(1 + \left\|X^p_{s-\frac{1}{p}}\right\|_{L^2(\Omega,C_n)}\right)^{2q} ds \\
&\le 2^{2q-1}|t_2 - t_1|^q K^{2q}\left(T^q + n^{q-1}\left(\tfrac{4q^3l^2}{2q-1}\right)^q\right)(1+\sqrt{c_1})^{2q} \\
&= c_q|t_2 - t_1|^q
\end{aligned}
\tag{1.29}
$$

where $c_q := 2^{2q-1}K^{2q}\left(T^q + n^{q-1}\left(\frac{4q^3l^2}{2q-1}\right)^q\right)(1+\sqrt{c_1})^{2q}$.

Proposition 1.2.8. *Assume that the initial process φ in (1.19) is pathwise $\frac{1}{2}$-Hölder continuous. Let $\gamma \in (0, \frac{1}{2})$ be a fixed constant. X^p satisfies*

(i) $|X^p(t_2) - X^p(t_1)| \le c_p|t_2 - t_1|^\gamma$ *for all $t_1, t_2 \in [0, T]$ a.s.,*

(ii) $\left\|X^p_{t_2} - X^p_{t_1}\right\|^2_{L^2(\Omega,C_n)} \le 3C_2|t_2 - t_1|^{2\gamma} + 2E\sup_{u\in(-(t_2\wedge L)\wedge 0,-(t_1\wedge L)\wedge 0]}|\varphi(0) - \varphi(t_1+u)|^2$
$+ E\sup_{u\in[-L,-(t_2\wedge L)\wedge 0]}|\varphi(t_2+u) - \varphi(t_1+u)|^2$ *for all $0 \le t_1 < t_2 \le T$, a.s.,*

where C_2 is a constant independent of p and c_p is a positive random variable satisfying $E(c_p^\theta) \le C_2$ $(\theta > 1)$.

Proof. To prove (i), let $\zeta > \frac{1}{1-2\gamma}$ be an integer. By Lemma 1.2.7 (iii), $E|X^p(t_2) - X^p(t_1)|^{2\zeta} \le c_\zeta|t_2 - t_1|^\zeta$ for all $t_1, t_2 \in [0, T]$. Notice that $\gamma < \frac{\zeta-1}{2\zeta}$, then it follows from Kolmogorov's continuity criterion that there exists a positive random variable c_p such that $|X^p(t_2) - X^p(t_1)| \le |t_2 - t_1|^\gamma$ a.s., with $E(c_p^\theta) \le C_2$, C_2 is a constant independent of p and $\theta = 2\gamma$. To prove (ii), let's consider $0 \le t_1 < t_2 \le T$, then

$$
\begin{aligned}
\left\|X_{t_2}^p - X_{t_1}^p\right\|_{L^2(\Omega,C_n)}^2 &= E\sup\nolimits_{u\in[-L,0]}|X^p(t_2+u)-X^p(t_1+u)|^2 \\
&\leq E\sup\nolimits_{u\in(-(t_1\wedge L)\wedge 0,0]}|X^p(t_2+u)-X^p(t_1+u)|^2 \\
&+ E\sup\nolimits_{u\in(-(t_2\wedge L)\wedge 0,-(t_1\wedge L)\wedge 0]}|X^p(t_2+u)-X^p(t_1+u)|^2 \\
&+ E\sup\nolimits_{u\in[-L,-(t_2\wedge L)\wedge 0]}|X^p(t_2+u)-X^p(t_1+u)|^2.
\end{aligned}
\tag{1.30}
$$

Using (i), we obtain the following estimates:

$$
E\sup_{u\in(-(t_1\wedge L)\wedge 0,0]}|X^p(t_2+u)-X^p(t_1+u)|^2 \leq E(c_p^2)|t_2-t_1|^{2\gamma} \leq C_2|t_2-t_1|^{2\gamma}, \tag{a}
$$

$$
\begin{aligned}
&E\sup\nolimits_{u\in(-(t_2\wedge L)\wedge 0,-(t_1\wedge L)\wedge 0]}|X^p(t_2+u)-X^p(t_1+u)|^2 \\
&\qquad\leq 2C_2|t_2-t_1|^{2\gamma} + 2E\sup\nolimits_{u\in(-(t_2\wedge L)\wedge 0,-(t_1\wedge L)\wedge 0]}|\varphi(0)-\varphi(t_1+u)|^2
\end{aligned}
\tag{b}
$$

and

$$
E\sup_{u\in[-L,-(t_2\wedge L)\wedge 0]}|X^p(t_2+u)-X^p(t_1+u)|^2 = E\sup_{u\in[-L,-(t_2\wedge L)\wedge 0]}|\varphi(t_2+u)-\varphi(t_1+u)|^2 \tag{c}
$$

Plugging (a), (b) and (c) in (1.30), we obtain

$$
\begin{aligned}
\left\|X_{t_2}^p - X_{t_1}^p\right\|_{L^2(\Omega,C_n)}^2 &\leq 3C_2|t_2-t_1|^{2\gamma} + 2E\sup\nolimits_{u\in(-(t_2\wedge L)\wedge 0,-(t_1\wedge L)\wedge 0]}|\varphi(0)-\varphi(t_1+u)| \\
&+ E\sup\nolimits_{u\in[-L,-(t_2\wedge L)\wedge 0]}|\varphi(t_2+u)-\varphi(t_1+u)|^2 \quad \text{for all } -L\leq t_1<t_2\leq T.
\end{aligned}
\tag{1.31}
$$

Theorem 1.2.9. *Let the process $X^\pi(t)$ be defined as in (1.18). Assume that the initial process φ is pathwise $\frac{1}{2}$-Hölder continuous. Then under the Lipschitz and the joint continuity conditions on H and G, $X^\pi(t)$ converges in $L^2(\Omega, C([-L,T],\mathbb{R}^n))$ to $X(t)$ that satisfies*

(1.17). *The rate of convergence is given by*

$$E \sup_{u\in[0,t]} |X^{\pi}(u) - X(u)|^2 \le K^{'}|\pi|$$

where $K^{'}$ *is a positive constant independent of* π.

Proof. By Jensen's inequality and Theorem 1.2.4, we have

$$\begin{aligned}
&E\sup_{u\in[0,t]} |X^{\pi}(u) - X(u)|^2 = \\
&\quad E\sup_{u\in[0,t]} |\textstyle\int_0^u \left(H\left(X^{\pi}_{\lfloor s\rfloor}\right) - H\left(X_s\right)\right) ds + \int_0^u \left(G\left(X^{\pi}_{\lfloor s\rfloor}\right) - G\left(X_s\right)\right) dW(s)|^2 \\
&\quad\le 2t\textstyle\int_0^t \|H\left(X^{\pi}_{\lfloor s\rfloor}\right) - H\left(X_s\right)\|^2_{L^2(\Omega,\mathbb{R}^n)} ds \\
&\qquad\quad +2(4l^2)\textstyle\int_0^t \|G\left(X^{\pi}_{\lfloor s\rfloor}\right) - G\left(X_s\right)\|^2_{L^2(\Omega,\mathbb{R}^{n\times l})} ds \qquad \text{by Jensen's inequality} \\
&\quad\le 2(t+4l^2)\textstyle\int_0^t \left[\|H\left(X^{\pi}_{\lfloor s\rfloor}\right) - H\left(X_s\right)\|^2_{L^2(\Omega,\mathbb{R}^n)} + \|G\left(X^{\pi}_{\lfloor s\rfloor}\right) - G\left(X_s\right)\|^2_{L^2(\Omega,\mathbb{R}^{n\times l})}\right] ds \\
&\quad\le 2(t+4l^2)\mu^2\textstyle\int_0^t \|X^{\pi}_{\lfloor s\rfloor} - X_s\|^2_{L^2(\Omega,Cn)} ds \\
&\quad\le 4\mu^2(t+4l^2)\textstyle\int_0^t \left(\|X^{\pi}_{\lfloor s\rfloor} - X^{\pi}_s\|^2_{L^2(\Omega,C_n)} + \|X^{\pi}_{\lfloor s\rfloor} - X_s\|^2_{L^2(\Omega,C_n)}\right) ds \\
&\quad\le 4\mu^2(t+4l^2)3C_2|\pi|T + 4\mu^2(t+4l^2)\textstyle\int_0^t E\sup_{u\in[0,s]}|X^{\pi}(u) - X(u)|^2 ds.
\end{aligned} \tag{1.32}$$

The last inequality comes from Proposition 1.2.8 and the definition of the norm in $L^2(\Omega, Cn)$. Finally, using Gronwall's Lemma, we obtain

$$E \sup_{u\in[0,t]} |X^{\pi}(u) - X(u)|^2 \le K^{''}|\pi|$$

with $K^{''} := 12\mu^2(t+4l^2)C_2Te^{4\mu^2(1+4l^2)T}$. Hence x^{π} converges in $L^2(\Omega, C([-L,T],\mathbb{R}^n))$ as π goes to 0. The rate of convergence is $\frac{1}{2}$.

We present the numerical approximation of the solution for our model in Chapter 6, which is a particular case of (1.3).

CHAPTER 2

A GENERALIZED DELAY MODEL FOR PRICING CORPORATE LIABILITIES

We provide definition of some financial terms.

Definition. (Firm Value or Company value)

The Market value of the company's machines and commercial activities. This value is equal to the market value of the equityholders plus the market value of the net financial debt.

Definition. (Equity Value)

The total dollar market value of all of a company's outstanding shares. Market value of equity is calculated by multiplying the company's current stock price by its number of outstanding shares. It's the total value of the business after taking out the amount owed to debtholders.

Definition. (Trading Strategy)[42]

A dynamic portfolio which is a predictable $\mathbb{R}^2$-valued process $(x_t, y_t),\ t \in [0, T]$, satisfying $\int_0^t x_s^2 ds < \infty$ and $\int_0^t y_s^2 ds < \infty$, a.s. for all $t \in (0, \infty)$, respectively. This requirement insures that the stochastic differentials are defined.

Definition. (Trading Strategy)

A trading strategy is a set of objective rules that guide a trader in his trading decisions. It may design the conditions that must be met for trade entries and exits in order to avoid arbitrage or to hedge the risk.

Definition. (Corporate claim or corporate liability) [56]

An official request for money usually in the form of compensation, from a corporation.

Definition. (Contingent liability) [56]

A liability that may or may not occur, but for which provision is made in a company's

accounts, as opposed to “provisions”, for which money is set aside for an anticipated expenditure.

Definition. (Attainable Contingent Claim)[42]

A contingent claim X is said to be attainable if there exists a strategy ϕ such that $\tilde{Y}_\phi$ is a martingale under the risk neutral measure and $Y_\phi(t) = X$. Such a strategy is called a replicating strategy (or hedging strategy) for the contingent claim X.

Definition. (Debt security) [56]

A security issued by a company or government which represents money borrowed from the security's purchaser and which must be repaid at a specified maturity date, usually at a specified interest rate.

Definition. (Loan Guarantees) [56]

Loan on which a promise is made by a third party or guarantor that he or she will be liable if the creditor fails to fulfill their contractual obligations.

2.1 STOCHASTIC DELAY MODELS FOR A FIRM VALUE

In this chapter we propose a stochastic delay model for a firm value. We consider a firm whose value at time t is given by a stochastic process $V(t)$ on a complete probability space $(\Omega, \mathcal{F}, P)$ with a right continuous filtration $(\mathcal{F}_t)_{0\leq t\leq T}$. The firm value $V(t)$ satisfies the following nonlinear stochastic delay differential equation (SDDE)

$$\left.\begin{aligned} dV(t) = f(V_t, t)dt + g(V(t-L_2))V(t)dW(t),\ t \in [0,T] \\ V(t) = \varphi(t),\ t \in [-L, 0], \end{aligned}\right\} \tag{2.1}$$

where L, L_2 and T are positive constants with $L \geq L_2$.

The function $f : [0,T] \times C([-L,0],\mathbb{R}) \longrightarrow \mathbb{R}$ is a given continuous functional and $g : \mathbb{R} \to \mathbb{R}$ is continuous. The initial process $\varphi : \Omega \to C([-L,0],\mathbb{R})$ is $\mathcal{F}_0$-measurable with

respect to the Borel σ-algebra of $C([-L,0],\mathbb{R})$ which is the Banach space of all continuous functions $\eta:[-L,0]\longrightarrow\mathbb{R}$ with the supremum norm $\|\eta\|:=\sup_{s\in[-L,0]}|\eta(s)|$.

The process W is a one dimensional standard Brownian motion adapted to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$; and $V_t\in C([-L,0],\mathbb{R})$ is the segment $V_t(s):=V(t+s), s\in[-L,0], t\geq 0$. Let us consider the following two possible definitions of the continuous functional f as $f^i:[0,T]\times C([-L,0],\mathbb{R})\longrightarrow\mathbb{R},\ i=1,2,$ where

$$f^1(\varphi,t):=\alpha\varphi(-L_1)\varphi(0),\ \ f^2(\varphi,t):=\alpha\varphi(-L_1),$$

for all $(\varphi,t)\in C([-L,0],\mathbb{R})\times[0,T],\ \alpha,L_1$ are positive constant with $L=\max\{L_1,L_2\}$. If we replace f by $f^i, i=1,2$ in equation (2.1), we obtain the following two possible stochastic delay differential equations (SDDE):

$$\left.\begin{aligned} dV(t)=\alpha V(t)V(t-L_1)dt+g(V(t-L_2))V(t)dW(t),\ t\in[0,T] \\ V(t)=\varphi(t),\ t\in[-L,0], \end{aligned}\right\} \tag{2.2}$$

$$\left.\begin{aligned} dV(t)=\alpha V(t-L_1)dt+g(V(t-L_2))V(t)dW(t),\ t\in[0,T] \\ V(t)=\varphi(t),\ t\in[-L,0]. \end{aligned}\right\} \tag{2.3}$$

The SDDE (2.2) admits a pathwise unique solution V under the following hypotheses:

- (i) $g:\mathbb{R}\longrightarrow\mathbb{R}$ is continuous.
- (ii) L_1 and L_2 are positive constants.

The unique solution of equation (2.2) satisfies the following relation (see [2], Theorem 1)

$$V(t) = \varphi(0) \exp\left(\int_0^t g(V(s-L_2))dW(s) + \alpha \int_0^t V(s-L_1)ds - \frac{1}{2}\int_0^t g^2(V(s-L_2))ds \right), \quad \text{a.s. for } t \in [0,T] \tag{2.4}$$

for a given $\mathcal{F}_0$-measurable initial process $\varphi : \Omega \longrightarrow C([-L, 0], \mathbb{R})$. This implies that $V(t) \geq 0$ almost surely for all $t \geq 0$ whenever the initial path $\varphi(t) \geq 0$ for all $t \in [-L, 0]$, therefore $V(t) > 0$ for all $t \geq 0$ a.s. if $\varphi(0) > 0$.

2.2 PRICING EQUITY AND DEBT OF A LEVERED FIRM

Consider a portfolio consisting of a riskless asset (e.g., a bond or riskless debt) $B(t)$ with rate of return $r \geq 0$ (i.e., $B(t) = e^{rt}$) and a single firm value whose time evolution $V(t)$ at time t is modeled by the SDDE (2.2) where $\varphi(0) > 0$ a.s., the delays L_1, L_2 are positive and g is continuous. Consider the equity value with maturity at some future time $T > t$, with the face value of the debt $B(T)$. Also assume that there are no transaction costs and that there is no dividends payment on the firm value. We want to derive the fair price of the equity and debt values at time $t \in [0, T]$.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ be a probability space equipped with the natural filtration of a standard Wiener process $W(t)$, $t \in [0, T]$. We recall the Girsanov's theorem (see Theorem 4.2.2 in [35]), Theorem 1.0.4:

Theorem 2.2.1. *Let $(\Theta_t)_{0\leq t\leq T}$ be a left continuous adapted process satisfying $\int_0^T |\Theta(s)|^2 ds < \infty$ a.s., and such that the process $(\varrho_t)_{0\leq t\leq T}$ is defined by*

$$\varrho_t := exp\left(\int_0^t \Theta(s)dW(s) - \int_0^t |\Theta(s)|^2 ds \right), \ t \in [0,T].$$

Define the probability measure P^* on $(\Omega, \mathcal{F})$ by $dP^* := \varrho_T dP$, assuming that $E_P(\varrho_T) = 1$,

where E_P stands for the expectation with respect to the probability measure P. Then the process

$$W^*(t) := W(t) - \int_0^t \Theta(s)ds, \ t \in [0,T],$$

is a standard Wiener process under the measure P^*.

In the following discussion, we will obtain an equivalent martingale measure with the help of Girsanov's theorem above.

Let $\tilde{V}(t) := \dfrac{V(t)}{B(t)} = e^{-rt}V(t), t \in [0,T]$, be the discounted firm value process. Then by Itô's formula (the product rule) (see [49] Theorem 4.1.2), we obtain

$$\begin{aligned} d\tilde{V}(t) &= e^{-rt}dV(t) - re^{-rt}V(t)dt \\ &= \tilde{V}(t)\left[(\alpha V(t-L_1) - r)\, dt + g(V(t-L_2))dW(t)\right]. \end{aligned}$$

Define

$$V_1(t) := \int_0^t (\alpha V(s-L_1) - r)\, ds + \int_0^t g(V(s-L_2))dW(s), \ t \in [0,T]$$

therefore

$$d\tilde{V}(t) = \tilde{V}(t)dV_1(t), \ t \in [0,T]. \tag{2.5}$$

Hence

$$\tilde{V}(t) = \varphi(0) + \int_0^t \tilde{V}(s)dV_1(s), t \in [0,T], \tag{2.6}$$

where $\tilde{V}(0) = \varphi(0)$.

Let us apply Girsanov's theorem to the process

$$\Theta(s) := -\frac{\alpha V(s-L_1) - r}{g(V(s-L_2))}, s \in [0,T].$$

If we suppose the following

$$\text{If } x \neq 0 \text{ for all } x \in (0, \infty) \text{ then } g(x) \neq 0, \tag{2.7}$$

then Θ is well-defined since $V(t) > 0$, for all $t \in [-L, T]$ a.s..

Obviously, $\Theta(t)$, $t \in [0, T]$ is a predictable process. In addition, $\int_0^T |\Theta(s)|^2 ds < \infty$ a.s. This is because $V(t), t \in [0, T]$ is almost surely bounded and (2.7) implies that $\dfrac{1}{g(x)}$, $x \in (0, \infty)$, is bounded on bounded intervals. If $\mathcal{F}_t := \mathcal{F}_0$ for $t \leq 0$ then $\Theta(s), s \in [0, T]$, is measurable with respect to the σ-algebra $\mathcal{F}_{T-l}$, where $l := \min\{L_1, L_2\}$ Therefore, the stochastic integral $\int_{T-l}^T \Theta(s) dW(s)$ conditioned on $\mathcal{F}_{T-l}$ has a normal distribution with mean 0 and variance $\int_{T-l}^T \Theta^2(s) ds$. Thus

$$E_P \left[\exp \left(\int_{T-l}^T \Theta(s) dW(s) \right) \mid \mathcal{F}_{T-l} \right] = \exp \left(\frac{1}{2} \int_{T-l}^T |\Theta(s)|^2 ds \right) \text{ a.s..}$$

i.e.

$$E_P \left[\exp \left(\int_{T-l}^T \Theta(s) dW(s) - \frac{1}{2} \int_{T-l}^T |\Theta(s)|^2 ds \right) \mid \mathcal{F}_{T-l} \right] = 1 \text{ a.s.}$$

which implies

$$\begin{aligned} &E_P \left[\exp \left(\int_0^T \Theta(s) dW(s) - \frac{1}{2} \int_0^T |\Theta(s)|^2 ds \right) \mid \mathcal{F}_{T-l} \right] \\ &= \exp \left(\int_0^{T-l} \Theta(s) dW(s) - \frac{1}{2} \int_0^{T-l} |\Theta(s)|^2 ds \right) \text{ a.s.} \end{aligned}$$

Consider an integer $k > 0$ such that $0 \leq T - kl \leq l$. Using backward steps of length l, sequential conditioning and induction gives

$$\begin{aligned} &E_P \left[\exp \left(\int_0^T \Theta(s) dW(s) - \frac{1}{2} \int_0^T |\Theta(s)|^2 ds \right) \mid \mathcal{F}_{T-kl} \right] \\ &= \exp \left(\int_0^{T-kl} \Theta(s) dW(s) - \frac{1}{2} \int_0^{T-kl} |\Theta(s)|^2 ds \right) \text{ a.s.} \end{aligned}$$

Hence

$$E_P\left[\exp\left(\int_0^T \Theta(s)dW(s) - \frac{1}{2}\int_0^T |\Theta(s)|^2 ds\right) \mid \mathcal{F}_0\right]$$
$$= E_P\left[\exp\left(\int_0^{T-kl} \Theta(s)dW(s) - \frac{1}{2}\int_0^{T-kl} |\Theta(s)|^2 ds\right) \mid \mathcal{F}_0\right]$$
$$= 1 \quad \text{a.s.}$$

Therefore

$$E_P\left\{E_P\left[\exp\left(\int_0^T \Theta(s)dW(s) - \frac{1}{2}\int_0^T |\Theta(s)|^2 ds\right) \mid \mathcal{F}_0\right]\right\}$$
$$= E_P\left[\exp\left(\int_0^T \Theta(s)dW(s) - \frac{1}{2}\int_0^T |\Theta(s)|^2 ds\right)\right]$$
$$= E_P\left[\varrho_T\right]$$
$$= 1 \quad \text{a.s.}$$

where $\varrho_T := \exp\left(-\int_0^T \frac{\alpha V(s-L_1) - r}{g(V(s-L_2))} dW(s) - \frac{1}{2}\int_0^T \left|\frac{\alpha V(s-L_1) - r}{g(V(s-L_2))}\right|^2 ds\right)$

Hence by Girsanov's theorem, the process

$$W^*(t) := W(t) + \int_0^t \frac{\alpha V(s-L_1) - r}{g(V(s-L_2))} ds, \ t \in [0, T],$$

is a standard Wiener process under the measure P^* with $dP^* := \varrho_T dP$. Thus the process $V_1(t), \ t \in [0, T]$ defined as

$$V_1(t) := \int_0^t (\alpha V(s-L_1) - r)\, ds + \int_0^t g(V(s-L_2)) dW(s), \ t \in [0, T]$$

i.e.

$$V_1(t) := \int_0^t g(V(u-L_2)) dW^*(u), \ t \in [0, T] \tag{2.8}$$

is a continuous local martingale under the measure P^*. Also by (2.6) the discounted firm value process $\tilde{V}(t), \ t \in [0, T]$ is a continuous local martingale under the measure P^*. This

means that P^* is an equivalent local martingale measure. Hence by the theorem on trading strategies (see 1.0.1 and [23] Theorem 7.1), the market consisting of $(B(t), V(t))$, $t \in [0,T]$ satisfies the no-arbitrage property. We want to show that $(B(t), V(t))$, $t \in [0,T]$ is complete. From (2.4),

$$\tilde{V}(t) = \varphi(0) \exp\left(\int_0^t g(V(s-L_2)) dW^*(s) - \frac{1}{2} \int_0^t g^2(V(s-L_2)) ds \right), \quad \text{a.s. for } t \in [0,T]. \tag{2.9}$$

Indeed,

$$\begin{aligned}
\tilde{V}(t) &= e^{-rt} V(t) \\
&= e^{-rt} \varphi(0) \exp\left(\textstyle\int_0^t g(V(s-L)) dW(s) + \alpha \int_0^t V(s-L) ds - \frac{1}{2} \int_0^t g^2(V(s-L)) ds \right) \\
&= \varphi(0) \exp\left(-\textstyle\int_0^t r ds + \int_0^t g(V(s-L)) dW(s) + \alpha \int_0^t V(s-L) ds - \frac{1}{2} \int_0^t g^2(V(s-L)) ds \right) \\
&= \varphi(0) \exp\left(\textstyle\int_0^t g(V(s-L)) dW(s) + \alpha \int_0^t V(s-L) ds - \int_0^t r ds - \frac{1}{2} \int_0^t g^2(V(s-L)) ds \right) \\
&= \varphi(0) \exp\left(\textstyle\int_0^t g(V(s-L)) dW(s) + \int_0^t (\alpha V(s-L) - r)\, ds - \frac{1}{2} \int_0^t g^2(V(s-L)) ds \right) \\
&= \varphi(0) \exp\left(\textstyle\int_0^t g(V(s-L)) \left\{ dW(s) + \dfrac{(\alpha V(s-L) - r)}{g(V(s-L))} ds \right\} - \frac{1}{2} \int_0^t g^2(V(s-L)) ds \right) \\
&= \varphi(0) \exp\left(\textstyle\int_0^t g(V(s-L)) dW^*(s) - \frac{1}{2} \int_0^t g^2(V(s-L)) ds \right), \quad \text{a.s. for } t \in [0,T]
\end{aligned}$$

From the definitions of $\tilde{V}, W^*, V_1$, and equation (2.5), $\mathcal{F}_t^V = \mathcal{F}_t^{\tilde{V}} = \mathcal{F}_t^{W^*} = \mathcal{F}_t^W$ for all $t \geq 0$, where $\mathcal{F}_t^V, \mathcal{F}_t^{\tilde{V}}, \mathcal{F}_t^{W^*}, \mathcal{F}_t^W$ are the σ-algebras generated by $(V(s), s \leq t)$, $\left(\tilde{V}(s), s \leq t\right)$, $(W^*(s), s \leq t)$, $(W(s), s \leq t)$, respectively. Moreover $\mathcal{F}_t^W \subseteq \mathcal{F}_t$. Now, let X be an integrable non-negative $\mathcal{F}_T^V$-measurable random variable which is a contingent claim. Let $Z(t)$ be the martingale under the measure P^* defined as

$$Z(t) := E_{P^*}(e^{-rT} X | \mathcal{F}_t^V) = E_{P^*}(e^{-rT} X | \mathcal{F}_t^{W^*}), \; t \in [0,T].$$

From the martingale representation theorem (see Theorem 1.0.3,[23] Theorem 9.4), we can

find an $\mathcal{F}_t^{W^*}$-predictable process $k_0(t), t \in [0,T]$, such that $\int_0^T (k_0(s))^2 ds < \infty,$ a.s. and

$$Z(t) = E_{P^*}(e^{-rT}X) + \int_0^t k_0(s)dW^*(s),\ t \in [0,T].$$

From (2.5) and (2.8),

$$d\tilde{V}(t) = \tilde{V}(t)g(V(t-L_2))dW^*(t),\ t \in [0,T].$$

Define

$$n_V(t) := \frac{k_0(t)}{\tilde{V}(t)g(V(t-L_2))},\ \ n_B(t) := Z(t) - n_V(t)\tilde{V}(t),\ \ t \in [0,T].$$

Consider the market portfolio $(B(t), V(t)),\ t \in [0,T]$ whose strategy $(n_V(t), n_B(t)),\ t \in [0,T]$, consists of holding $n_V(t)$ units of the firm value $V(t)$ and $n_B(t)$ units of the debt $B(t)$ at time t. It corresponds to building a company portfolio $(V(t), Y(t), B(t)),\ t \in [0,T]$ where $Y(t)$ is the claim value at time t, with strategy $(n_V(t), n_Y(t), n_B(t)),\ t \in [0,T]$ where $n_Y(t) = |1|$. The value of the market portfolio at any time $t \in [0,T]$ is given by

$$Y(t) := n_V(t)V(t) + n_B(t)e^{rt};$$

we also have $Y(t) = e^{rt}Z(t)$. From the definition of the strategy $(n_V(t), n_B(t)),\ t \in [0,T]$ and the product rule, we have $dY(t) = re^{rt}Z(t) + e^{rt}dZ(t)$ and

$$dY(t) = n_V(t)dV(t) + n_B(t)re^{rt},\ t \in [0,T].$$

In fact

$$
\begin{aligned}
dY(t) &= e^{rt}dZ(t) + re^{rt}Z(t) \\
&= e^{rt}d\left(e^{-rt}(n_V(t)V(t) + n_B(t)re^{rt})\right) + r\left(n_V(t)V(t) + n_B(t)re^{rt}\right).
\end{aligned}
$$

Hence

$$dV(t) = n_Y(t)dY(t) + n_B(t)re^{rt},\ t \in [0,T],$$

therefore the firm strategy $(n_Y(t), n_B(t)),\ t \in [0,T]$ is self-financed. In addition, $Y(T) = e^{rT}Z(T) = X$ a.s. Thus the contingent claim X is attainable. This implies that the assets market $(B(t), V(t)),\ t \in [0,T]$ is complete, since every contingent claim is attainable. In order for the augmented market $(B(t), V(t), X),\ t \in [0,T]$ to satisfy the no-arbitrage condition, as shown in [23] (See Theorem 9.2), the value of the claim X at each time t must be

$$Y(t) = e^{-r(T-t)}E_{P^*}(X|\mathcal{F}_t^V) \text{ a.s.}$$

The following result gives a compact version of the details described above, presenting a formula for the fair value $Y(t)$ of any arbitrary claim of a company whose value's dynamic is described by the SDDE (2.2).

Theorem 2.2.2. *Let $V(t)$ be the value of a company whose dynamic follows the nonlinear SDDE* (2.2)*, where $\varphi(0) > 0$ and the g satisfies assumption* (2.7)*. Consider an $\mathcal{F}_T^V$-measurable non-negative integrable random variable X which represents the payoff function of the firm's claim value $Y(t)$ with maturity date T. The following formula gives the fair value $Y(t)$ of the claim*

$$Y(t) = e^{-r(T-t)}E_{P^*}(X|\mathcal{F}_t^V), t \in [0,T] \tag{2.10}$$

where P^* stands for the probability measure on $(\Omega, \mathcal{F})$ defined by $dP^* = \varrho_T dP$ with

$$\varrho_t := \exp\left(-\int_0^t \frac{\alpha V(s-L_1) - r}{g(V(s-L_2))} dW(s) - \frac{1}{2}\int_0^t \left| \frac{\alpha V(s-L_1) - r}{g(V(s-L_2))} \right|^2 ds \right),\ t \in [0,T].$$

The probability measure P^* is a local martingale measure and the market is complete. Moreover, there exists an adapted and square integrable process $k_0(s), s \in [0,T]$ such that

$$E_{P^*}(e^{-rT} X | \mathcal{F}_t^V) = E_{P^*}(e^{-rT} X) + \int_0^t k_0(s) dW^*(s),\ t \in [0,T],$$

where W^* is a standard P^*-Wiener process defined by

$$W^*(t) := W(t) + \int_0^t \frac{\alpha V(s-L_1) - r}{g(V(s-L_2))} ds,\ t \in [0,T],$$

and the hedging strategy $(n_V(t), n_B(t)),\ t \in [0,T]$ is given by

$$n_V(t) := \frac{k_0(t)}{\tilde{V}(t) g(V(t-L_2))},\ n_B(t) := Z(t) - n_V(t)\tilde{V}(t),\ t \in [0,T].$$

For $T > l$ and $t < T - l$, one can develop a recursive procedure to calculate (2.15) by taking backwards steps of length l from the maturity time T of the claim. In addition to numerical approximations, this recursive procedure can be used to compute the equity value at any time $t \in [0,T]$. Clearly,

$$V(t) = e^{rt} E_{P^*}\left(E_{P^*}(\max[\tilde{V}(T) - Be^{rT}, 0]) \mathcal{F}_{T-l} \right)$$

The measurability arguments in the proof of Theorem 2.2.3 lead to

$$
\begin{aligned}
&E_{P^*}(\max[\tilde{V}(T) - Be^{rT}, 0]|\mathcal{F}_{T-l}) \\
&\qquad = R\left(\tilde{V}(T-l), -\frac{1}{2}\int_{T-l}^{T} g^2(V(s-L_2))ds, \int_{T-l}^{T} g^2(V(s-L_2))ds\right).
\end{aligned}
$$

We will study the distribution of $\tilde{V}(T-l)$ and $V(s-L_2)$, $s \in [T-l, T]$ under the equivalent martingale measure P^* in order to compute the conditional expectation of $E_{P^*}(\max[\tilde{V}(T) - Be^{rT}, 0]|\mathcal{F}_{T-l})$ with respect to the σ-algebra $\mathcal{F}_t$. We will discuss only the conditional distribution of the solution process $V(s-L_2)$ for $s \in [T-l, T]$. Similar way can be applied to the corresponding distribution of $\tilde{V}(T-l)$.

In case $T > nl$ for some positive integer n and $t \in [T-(n+1)l, T-nl]$, then for all $u \in [T-l, T]$ from equation (2.9) we get

$$
V(s-L_2) = e^{r(s-L_2)}\tilde{V}(t)\prod_{i=1}^{n}\exp\left(\int_{t_{i-1}}^{t_i} g(V(u-L_2))dW^*(u) - \frac{1}{2}\int_{t_{i-1}}^{t_i} g^2(V(u-L_2))du\right) \tag{2.11}
$$

with

$$
t_i := \begin{cases} t & \text{if } i = 0. \\ T-(n-i+1)l & \text{if } i = 1, ..., n-1, \\ s - L_2 & \text{if } i = n. \end{cases}
$$

It's easy to see that $\tilde{V}(t)$ is $\mathcal{F}_t$-measurable. The first factor in the product in (2.11) is

$$
\exp\left(\int_{t}^{T-nl} g(V(u-L_2))dW^*(u) - \frac{1}{2}\int_{t}^{T-nl} g^2(V(u-L_2))du\right). \tag{2.12}
$$

In order to obtain an approximation of (2.12), we see that $-\frac{1}{2}\int_t^{T-nl} g^2(V(u-L_2))du$ is $\mathcal{F}_t$-measurable, the integrand $(g(V(u-L_2)),\ u \in [t, T-nl])$ is also $\mathcal{F}_t$-measurable, and W^* is a standard Wiener process under the measure P^*. According to the conditional distribution

under P^*, one can construct an approximation of the integral $\int_t^{T-nl} g(V(u-L_2))dW^*(u)$ (see e.g. [30] and [31]), which in turn yields an approximation of (2.12). The second factor in the product in equation (2.11) has the form

$$\exp\left(\int_{t-nl}^{T-(n-1)l} g(V(u-L_2))dW^*(u) - \frac{1}{2}\int_{t-nl}^{T-(n-1)l} g^2(V(u-L_2))du\right). \tag{2.13}$$

Using equation (2.9), the integrand $(g(V(u-L_2)),\ u \in [t-nl, T-(n-1)l])$ in the above expression can be rewritten using the relation:

$$V(u-L_2) = e^{r(u-L_2)}\tilde{V}(t)\exp\left(\int_t^{u-L_2} g(V(s-L_2))dW^*(s) - \frac{1}{2}\int_t^{u-L_2} g^2(V(s-L_2))ds\right).$$

An approximation of the integrals in the interval $[t-nl, T-(n-1)l]$ can be constructed in the same way as above. This construction yields an approximation of equation (2.13). A similar approach can be used on the other factors of the product in (2.11). Numerical approximation will be presented similar way as in [21] and [36] in chapter 6.

As a consequence of Theorem 2.2.2, a Black-Scholes-Merton type formula for the value of the firm's equity at any time prior to maturity will be given. In the following result, we assume the conditions of Theorem 2.2.2 hold.

Theorem 2.2.3. *Let $E(t)$ be the fair price of the equity on a firm value $V(t)$ with the face value of the debt B and maturity time T. Let Φ denote the standard normal distribution function defined as*

$$\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{u^2}{2}}du, \ \ x \in \mathbb{R}.$$

Then for all $t \in [T-l, T]$ (where $l > 0$), $E(t)$ is given by

$$E(t) = V(t)\Phi(x_1(t)) - Be^{-r(T-t)}\Phi(x_2(t)), \tag{2.14}$$

where

$$x_1(t) := \frac{\log \dfrac{V(t)}{B} + r(T-t) + \frac{1}{2}\int_t^T g^2(V(s-L_2))ds}{\sqrt{\int_t^T g^2(V(s-L_2))ds}}$$

$$\text{and} \;\; x_2(t) := x_1(t) - \sqrt{\int_t^T g^2(V(s-L_2))ds}.$$

If $T > l$ *and* $t < T - l$*, then*

$$E(t) = e^{rt} E_{P^*}\left[R\left(\tilde{V}(T-l), -\frac{1}{2}\int_{T-l}^T g^2(V(s-L_2))ds, \int_{T-l}^T g^2(V(s-L_2))ds\right) |\mathcal{F}_t\right], \tag{2.15}$$

where R *is defined by* $R(v, m, \sigma^2) := ve^{m+\frac{\sigma^2}{2}}\Phi(d_1(v, m, \sigma)) - Be^{-rT}\Phi(d_2(v, v, \sigma))$*, with*

$$d_1(v, m, \sigma) := \frac{1}{\sigma}\left(\log\left(\frac{v}{B}\right) + rT + m + \sigma^2\right), \;\; d_2(v, m, \sigma) := \frac{1}{\sigma}\left(\log\left(\frac{v}{B}\right) + rT + m\right)$$

for σ*,* $v \in \mathbb{R}^+$ *and* $m \in \mathbb{R}$*. The hedging strategy is given by*

$$n_V(t) = \Phi(x_1(t)), \quad n_B(t) = -Be^{-rT}\Phi(x_2(t)), \;\; t \in [T-l, T].$$

Proof. Consider the equity value in the above market $(B(t), V(t)), \; t \in [0, T]$ with the face value of the debt B and maturity time T. Taking $X = \max[V(T) - B, 0]$ in Theorem 3, the fair price $E(t)$ of the equity is given by

$$\begin{aligned} E(t) &= e^{-rT-t}E_{P^*}(\max[V(T) - B, 0]|\mathcal{F}_t) \\ &= e^{rt}E_{P^*}(\max[\tilde{V}(T) - e^{-rT}B, 0]|\mathcal{F}_t), \;\; t \in [0, T]. \end{aligned}$$

We now derive an explicit formula for the equity value $E(t)$ at any time $t \in [T-l, T]$. By

equation (2.9) which represents the discounted value $\tilde{V}(t)$ of the company we obtain

$$\begin{aligned}\tilde{V}(T) &= \varphi(0)\exp\left(\int_0^T g(V(s-L_2))dW^*(s) - \frac{1}{2}\int_0^T g^2(V(s-L_2))ds\right)\\ &= \tilde{V}(t)\exp\left(\int_t^T g(V(s-L_2))dW^*(s) - \frac{1}{2}\int_t^T g^2(V(s-L_2))ds\right), \quad \text{a.s. for } t\in[0,T].\end{aligned}$$

Obviously, $\tilde{V}(t)$ is $\mathcal{F}_t$-measurable. $-\frac{1}{2}\int_t^T g^2(V(s-L_2))ds$ is also $\mathcal{F}_t$-measurable if $t\in[T-l,T]$. Further, when conditioned on $\mathcal{F}_t$, the distribution of $\int_t^T g(V(s-L_2))dW^*(s)$ under P^* is similar to that of a normal distributed random variable σZ with mean 0 and variance $\sigma^2 = \int_t^T g^2(V(s-L_2))ds$. Thus, the fair equity value at time t is given by

$$E(t) = e^{rt}R\left(\tilde{V}(t), -\frac{1}{2}\int_t^T g^2(V(s-L_2))ds, \int_t^T g^2(V(s-L_2))ds\right).$$

with R defined as

$$R(v,m,\sigma^2) := E_{P^*}(\max[ve^{m+\sigma Z} - Be^{-rT}, 0]), \quad \sigma, v\in\mathbb{R}^+,\ m\in\mathbb{R}.$$

Hence

$$R(v,m,\sigma^2) := ve^{m+\frac{\sigma^2}{2}}\Phi(d_1(v,m,\sigma)) - Be^{-rT}\Phi(d_2(v,v,\sigma)),$$

after simple computation. Finally, $E(t)$ is of the form

$$E(t) = V(t)\Phi(x_1(t)) - Be^{-r(T-t)}\Phi(x_2(t)),$$

with

$$x_1(t) := \frac{\log\dfrac{V(t)}{B} + r(T-t) + \frac{1}{2}\int_t^T g^2(V(s-L_2))ds}{\sqrt{\int_t^T g^2(V(s-L_2))ds}}$$

$$\text{and} \quad x_2(t) := x_1(t) - \sqrt{\int_t^T g^2(V(s-L_2))ds}.$$

For $T > l$ and $t < T - l$, by equation (2.9) of the discounted price $\tilde{V}(t)$ of the firm, we have

$$\tilde{V}(T) = \tilde{V}(T-l) \exp\left(\int_{T-l}^T g(V(s-L_2))dW^*(s) - \frac{1}{2} \int_{T-l}^T g^2(V(s-L_2))ds \right).$$

Therefore the fair equity value at time $t \in [-l, T-l]$ can be written as

$$E(t) = e^{rt} E_{P^*} \left(R\left(\tilde{V}(T-l), -\frac{1}{2} \int_{T-l}^T g^2(V(s-L_2))ds, \int_{T-l}^T g^2(V(s-L_2))ds \right) |\mathcal{F}_t \right).$$

To calculate the replicating strategy for $t \in [T-l, T]$, it suffices to use an idea from [6], pages 95–96. This completes the proof of the theorem. $\square$

We notice that equation (2.14) is a generalization of Black-Scholes-Merton formula which reduces to the classical Black-Scholes-Merton formula derived in [38] whenever $g(x) = \sigma$ with σ constant for all $x \in \mathbb{R}^+$. Note that, in contrast with the classical (non-delayed) Black-Scholes-Merton formula, the fair equity price $E(t)$ in a general delayed model considered in Theorem 2.2.3 depends not only on the company value $V(t)$ at the present time t, but also on the whole segment $\{V(u),\ u \in [t-L_2, T-L_2]\}$. (Of course $[t-L_2, T-L_2] \subset [0, t]$ since $t \geq T-l$.)

In the last delay period $[T-l, T]$, we can derive the formula $E(t) = f(V(t), t)$, $t \in [T-l, T]$ for the equity value f is the solution of the following random Black-Scholes-Merton partial differential equation

$$\begin{aligned} \frac{1}{2} g^2(V(t-L_2))v^2 f_{vv} + rvf_v + f_t - rf = 0, 0 < t < T \\ f(v, T) = \max[v - B, 0], v > 0 \end{aligned} \tag{2.16}$$

where the subscripts on f denote the partial derivatives with respect to the variables

v and t. The formula will be derived in Chapter 3. The above time-dependent random final-value problem admits an $(\mathcal{F}_t)_{t\geq 0}$-adapted random field $f(t,v)$. Using the classical Itô-Ventzell formula [34] and equation (2.10) of Theorem 2.2.2, it follows that

$$E(t) = e^{-r(T-t)} f(V(t), t), \;\; t \in [T-L, T].$$

Note that if $t < T - L$, the solution f of the final-value problem (2.16) is anticipating with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

CHAPTER 3

EVALUATION OF CORPORATE CLAIMS IN A SINGLE PERIOD AND A HOMOGENEOUS CLASS OF DEBT

Definition. (European Call (resp. put) Option, Strike Price) [56]

A *European call (resp. put) option* is an option that gives its holder the right (but not the obligation) to purchase (resp. sell) a specified number of shares of the underlying asset at an agreed-upon price (*strike or exercise price*) on the expiration date of the contract, regardless of the prevailing market price of the underlying asset.

Definition. (Yield to Maturity)

The expected rate of return on a bond if it is held until the maturity date.

Definition. (Risk Premium) [37]

The *risk premium* is the spread between the risk-free rate on treasuries and the rate (yield) on any other risky bond.

Definition. (Risk Structure of Interest Rates) [37]

The *risk structure of interest rates* is the analysis of why interest rates on bonds with the same maturity will vary, due to differences in risk.

3.1 EVALUATION OF RISKY DEBT

We continue to assume that the company value satisfies a nonlinear stochastic delay differential equation. The value of a particular issue of defaultable bonds depends essentially on three items:

- The required rate of return on riskless debt (in terms of default).
- The various provisions and restrictions contained in the indenture (maturity date, coupon rate, etc)

- The probability of default.

In this chapter, we develop the basic equation for the pricing of corporate liabilities. We will assume the following:

1. the value of the company is unaffected by how it is financed (the capital structure irrelevance principle).
2. the value $V(t)$ of the firm at time t, follows the nonlinear stochastic delay differential equation :

$$\left.\begin{aligned} dV(t) = (\alpha V(t)V(t-L_1) - C)dt + g(V(t-L_2))V(t)dW(t),\ t \in [0,T] \\ V(t) = \varphi(t),\ t \in [-L,0], \end{aligned}\right\} \quad (3.1)$$

where α is the constant riskless interest rate of return on the firm per unit time, C is the total amount payout by the firm per unit time to either the shareholders or claims-holders (e.g., dividends or interest payments) if positive, and it is the net amount received by the firm from new financing if negative; $g : \mathbb{R} \to \mathbb{R}$ is a continuous function representing the volatility function on the firm value per unit time; the initial process $\varphi : \Omega \to C([-L,0],\mathbb{R})$ is $\mathcal{F}_0$-measurable with respect to the Borel σ-algebra of $C([-L,0],\mathbb{R})$, where $L = \max(L_1, L_2)$ is a positive constant; the process W is a one dimensional standard Brownian motion adapted to the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$.

We recall that the process given by the formula

$$V(t) = \varphi(0)exp\left(\left((\alpha \int_0^t V(s-L_1)ds - \frac{1}{2}\int_0^t (g(V(s-L_2)))^2)ds\right) + \int_0^t g(V(s-L_2))dW(s)\right),$$

$\forall t \in [0,T]$ is the unique solution of the stochastic delay differential equation (3.1). (See (2.4) in Chapter 2.)

Proposition 3.1.1. *Assume there exists a claim with market value, $Y(t)$, at any point t in time, where $Y(t) = F(V(t), t)$ follows the dynamics of this claim's value in stochastic differential equation form as below*

$$\left.\begin{aligned} dY(t) = (\alpha_y Y(t) - C_y)dt + g_y(Y(t - L_2))Y(t)dW_y(t),\ t \in [0, T] \\ Y(t) = \varphi_y(t),\ t \in [-L, 0], \end{aligned}\right\} \tag{3.2}$$

where α_y is the constant riskless interest rate of return per unit time on this claim; C_y is the amount of payout per unit time on this claim; $g_y : \mathbb{R} \to \mathbb{R}$ is a continuous function representing the volatility function of the return on this claim per unit time; the initial process $\varphi_y : \Omega \to C([-L, 0], \mathbb{R})$ is $\mathcal{F}_0$-measurable with respect to the Borel σ-algebra of $C([-L, 0], \mathbb{R})$, where $L = \max(L_1, L_2)$ is a positive constant; the process W_y is a one dimensional standard Brownian motion adapted to the filtration $(\mathcal{F}_t)_{0 \le t \le T}$. Assume that the debt accumulates interest compounded continuously at a rate of r, that is $B(t) = B(0)e^{rt}$. Assume that $Y(t)$ can be replicated using self-financed strategy.

Then a random partial differential equation (RPDE) for F is given by

$$\left.\begin{aligned} \frac{1}{2}g^2(V(t - L_2))v^2 F_{vv} + (rv - C)F_v + F_t - rF + C_y = 0, 0 < t < T \\ F(v, T) = \max(v - B(T)), v > 0 \end{aligned}\right\} \tag{3.3}$$

Proof. Note that, for a given $Y(t) = F(V(t), t)$, there are similarities between the α_y, g_y, dW_y and the corresponding α, g, dW in SDDE (3.1). Knowing that $V(t)$ is an Itô process and assuming that F is twice continuously differentiable with respect to v and once

differentiable with respect to t, Itô's formula allows us to write the following

$$dF(V(t),t) = F_t(V(t),t)dt + F_v\left[(\alpha V(t)V(t-L_1) - C)dt + g(V(t-L_2))V(t)dW(t)\right]$$
$$+\frac{1}{2}F_{vv}\left[g^2(V(t-L_2))V^2(t)\right]dt$$

Hence,

$$dY(t) = \left[\frac{1}{2}F_{vv}g^2(V(t-L_2))V^2(t) + F_v(\alpha V(t)V(t-L_1) - C) + F_t(V(t),t)\right]dt \tag{3.4}$$

$$+F_v g(V(t-L_2))V(t)dW(t) \tag{3.5}$$

where $F_t = \frac{\sigma F}{\sigma t}, F_v = \frac{\sigma F}{\sigma v}, F_{vv} = \frac{\sigma^2 F}{\sigma v^2}$.

From equating the coefficients of the equations (3.2) and (3.4), we have the equality almost surely

$$\left.\begin{aligned} &\alpha_y Y(t) - C_y = \alpha_y F(Y(t),t) - C_y \\ &\equiv \frac{1}{2}g^2(V(t-L_2))V^2(t)F_{vv} + (\alpha V(t)V(t-L_1) - C)F_v + F_t \end{aligned}\right\} \tag{3.6}$$

$$g_y(Y(t-L_2))Y(t) = g_y(F(V(t-L_2),t-L_2))F(V(t),t) \equiv g(V(t-L_2))V(t)F_v(V(t),t) \tag{3.7}$$

$$dW_y(t) \equiv dW(t). \tag{3.8}$$

Following the self-financing and replication strategy ([2]), let z_1 be the amount invested in the firm, z_2 be the amount invested in the security and z_3 be the amount invested in riskless debt. Consider dx the instantaneous return to the portfolio and assume the total

investment in the portfolio is zero, we may write $z_1 + z_2 + z_3 = 0$ and then

$$\begin{aligned} dx &= z_1 \frac{dV(t) + Cdt}{V(t)} + z_2 \frac{dY(t) + C_y dt}{Y(t)} + z_3 r dt \\ &= \frac{z_1 \left[(\alpha V(t) V(t - L_1) - C)dt + g(V(t - L_2))V(t)dW(t) \right] + C z_1 dt}{V(t)} \\ &\quad + \frac{z_2 \left[(\alpha_y Y(t) - C_y)dt + g_y(Y(t - L_2))Y(t)dW_y(t) \right] + C_y z_2 dt}{Y(t)} + z_3 r dt \\ &= z_1 \alpha V(t - L_1)dt + z_1 g(V(t - L_2))dW(t) + z_2 \alpha_y dt \\ &\quad + z_2 g_y(Y(t - L_2))dW_y(t) - (z_1 + z_2) r dt \end{aligned}$$

Hence from the equivalence (3.8), we have:

$$dx = \left[z_1(\alpha V(t - L_1) - r) + z_2(\alpha_y - r) \right] dt + \left[z_1 g(V(t - L_2)) + z_2 g_y(Y(t - L_2)) \right] dW(t)$$

Since the return on the portfolio is non stochastic and there is no arbitrage condition we have: $z_1 g(V(t - L_2)) + z_2 g_y(Y(t - L_2)) = 0$ and $z_1(\alpha V(t - L_1) - r) + z_2(\alpha_y - r) = 0$ leading to the following system:

$$\begin{cases} z_1 g(V(t - L_2)) + z_2 g_y(Y(t - L_2)) & = 0 \\ z_1(\alpha V(t - L_1) - r) + z_2(\alpha_y - r) & = 0. \end{cases}$$

A non trivial solution ($z_i \neq 0$) to this system exists if and only if

$$\left(\frac{\alpha V(t - L_1) - r}{g(V(t - L_2))} \right) = \left(\frac{\alpha_y - r}{g_y(F(V(t - L_2), t - L_2))} \right). \tag{3.9}$$

But from (3.6) and (3.7) substituting for α_y and $g_y(F(V(t - L_2), t - L_2))$, we get

$$\alpha_y = \frac{\frac{1}{2} g^2(V(t - L_2))V^2(t)F_{vv} + (\alpha V(t)V(t - L_1) - C)F_v + F_t + C_y}{F(V(t), t)} \quad \text{and}$$

$$g_y(F(V(t-L_2),t-L_2)) = \frac{g(V(t-L_2))V(t)F_v}{F(V(t),t)}.$$

Replacing α_y and $g_y(F(V(t-L_2),t-L_2))$ in (3.9), we obtain

$$\frac{\alpha V(t-L_1)-r}{g(V(t-L_2))} = \frac{\frac{1}{2}g^2(V(t-L_2))V^2(t)F_{vv} + (\alpha V(t)V(t-L_1)-C)F_v + F_t + C_y - rF(V(t),t)}{g(V(t-L_2))V(t)F_v}.$$

By rearranging terms and simplifying, we get

$$\begin{aligned} &\alpha V(t)V(t-L_1)F_v - rV(t)F_v \\ &\quad = \frac{1}{2}g^2(V(t-L_2))V^2(t)F_{vv} + (\alpha V(t)V(t-L_1)-C)F_v + F_t + C_y - rF(V(t),t). \end{aligned} \tag{3.10}$$

Therefore, we can rewrite equation (3.10) as the following parabolic partial differential equation

$$\frac{1}{2}g^2(V(t-L_2))V^2(t)F_{vv} + (rV(t)-C)F_v + F_t + C_y - rF(V(t),t) = 0.$$

□

For any claim whose value depends on the value of the firm and time, the above equation must be satisfied under some specific boundary conditions and initial condition. From these boundary conditions, we will be able to distinguish the debt of a firm from its equity. This is the subject of our next section.

3.2 EVALUATION OF DEBT IN A LEVERED FIRM

In this section we are going to consider a claim market value as the simplest case of corporate debt. Assume the company is financed by:

a) a single class of debt

b) the equity.

Furthermore, assume the following restrictions and provisions are stipulated in the contract according to the bond issue:

1. the firm must pay an amount $B(T)$ to the debtholders at the maturity date T;

2. in case the firm cannot make the payment, the debtholders take over the company and the equityholders lose their investments;

3. the firm is not allowed to issue a new senior claim on the firm nor to pay cash dividend during the option life. In other words, there is no coupon payment nor dividends prior to the maturity of the debt.

Since there are no coupon payments, the values of C_y in equation (3.2) and C in equation (3.1) are zero. Equation (3.3) becomes

$$\frac{1}{2}g^2(V(t-L_2))v^2 f_{vv} + rvf_v + f_t - rf = 0,\ 0 < t < T \tag{3.11}$$

$$f(v,T) = \max(v - B(T), 0),\ v > 0 \tag{3.12}$$

with the boundary conditions

$$f(0,t) = 0 \text{ and } f(v,t) \sim v - B(T)e^{-r(T-t)},\ \text{as } v \to \infty. \tag{3.13}$$

We shall derive a formula for the above parabolic partial differential equation (3.11) using Green function.

Lemma 3.2.1. *The well known heat equation in physics over the domain* $x \in (-\infty, \infty),\ \tau > 0$ *which is a forward parabolic equation in the form*

$$h_\tau = h_{xx}, \quad h = h(x,\tau), \tag{3.14}$$

with initial condition

$$h(x,0) = e^{-rT}\max[e^x - 1, 0] \tag{3.15}$$

and boundary conditions

$$h(x,\tau) \sim 0 \ \text{as } x \to -\infty \quad \text{and} \quad h(x,\tau) \sim \frac{1}{e^{rt}}\left(e^x - e^{-r(T-t)}\right) \equiv e^x \ \text{as } x \to \infty, \tag{3.16}$$

can be extended to the backward parabolic random partial differential equation (RPDE) of the form (3.11) *with terminal condition* $f(v,T) = \phi(v) = \max[v - B(T), 0]$.

The reverse is also possible i.e. the RPDE (3.11) *with final and boundary conditions* (3.19) *and* (3.20) *can be reduced to equation* (3.14) *with a suitable initial and boundary conditions.*

Proof. Assume h satisfies the initial boundary value problem (3.14),(3.15) and (3.16). Let us make the change of variables (x,τ) and (v,t) which will transform the problem in terms of $h(x,\tau)$ with initial condition in $\tau = 0$ into $f(v,t)$ with terminal condition $t = T$. Consider the following $v = B(T)e^x$ and $\tau = \frac{1}{2}\int_t^T g^2(V(s - L_2))ds$ and let us define the mapping of the function $f(v,t)$ to the function $h(x,\tau)$ by the rule

$$f(v,t) = B(T)e^{rt}h(x,\tau) = B(T)e^{rt}h\left(x - \frac{1}{2}\int_t^T g^2(V(s-L_2))ds - r(T-t), \frac{1}{2}\int_t^T g^2(V(s-L_2))ds\right).$$

By the chain rule, the corresponding partial derivatives are given by

$$h_x = \frac{e^{-rt}}{B(T)}\frac{\partial v}{\partial x}f_v = \frac{e^{-rt}}{B(T)}v f_v,$$

$$f_t = B(T)re^{rt}h\left(x - \frac{1}{2}\int_t^T g^2(V(s-L_2))ds - r(T-t), \frac{1}{2}\int_t^T g^2(V(s-L_2))ds\right)$$
$$+B(T)e^{rt}\left[\left(\frac{1}{2}g^2(V(t-L_2)) - r\right)h_x - \frac{1}{2}g^2(V(t-L_2))h_\tau\right]$$

hence

$$h_\tau = \frac{-f_t + B(T)re^{rt}h\left(x - \frac{1}{2}\int_t^T g^2(V(s-L_2))ds - r(T-t), \frac{1}{2}\int_t^T g^2(V(s-L_2))ds\right)}{\frac{1}{2}B(T)e^{rt}g^2(V(t-L_2))}$$
$$+\frac{\frac{1}{2}B(T)e^{rt}g^2(V(t-L_2))h_x - rB(T)e^{rt}h_x}{\frac{1}{2}B(T)e^{rt}g^2(V(t-L_2))}.$$

Applying the change of variables using the definition of f, we get

$$h_\tau = \frac{-f_t + rf(v,t) + \frac{1}{2}g^2(V(t-L_2))vf_v - rvf_v}{\frac{1}{2}B(T)e^{rt}g^2(V(t-L_2))}, \tag{3.17}$$

$$h_{xx} = \frac{e^{-rt}}{B(T)}\frac{\partial(vf_v)}{\partial v}\frac{\partial v}{\partial x} = \frac{e^{-rt}}{B(T)}v(f_v + vf_{vv}). \tag{3.18}$$

Plugging (3.17) and (3.18) into (3.14), we get

$$\frac{1}{2}g^2(V(t-L_2))v^2f_{vv} + rvf_v + f_t - rf = 0,$$

corresponding to equation (3.14) with initial condition

$$h(x,0) = e^{-rT}\max[e^x - 1, 0] \text{ (when } t = T),$$

and boundary condition

$$h\left(x-\frac{1}{2}\int_t^T g^2(V(s-L_2))ds+r(T-t),\frac{1}{2}\int_t^T g^2(V(s-L_2))ds\right)\equiv 0 \text{ as } x\to-\infty.$$

$$h\left(x-\frac{1}{2}\int_t^T g^2(V(s-L_2))ds+r(T-t),\frac{1}{2}\int_t^T g^2(V(s-L_2))ds\right)$$
$$\to\frac{1}{e^{rt}}\left(e^x-e^{-r(T-t)}\right)\equiv e^x \text{ as } x\to\infty.$$

Hence final and boundary conditions for (3.11) conditions can be retrieved from those of the heat equation as follow

$$\begin{aligned}
f(v,T) &= B(T)e^{rT}h(x,0)\\
&= B(T)\max[\frac{v}{B(T)}-1,0],\\
f(0,t) &= B(T)e^{rt}h(x,\tau)\\
&= 0 \text{ as } x\to-\infty,\\
f(v,t) &= B(T)e^{rt}h(x,\tau)\\
&= B(T)e^{rt}\frac{1}{e^{rt}}\left(e^x-e^{-r(T-t)}\right) \text{ as } x\to\infty,\\
&= v-B(T)e^{-r(T-t)} \text{ as } v\to\infty.
\end{aligned}$$

□

Theorem 3.2.2. *If f is solution to the parabolic RPDE (3.11) with final and boundary conditions*

$$f(v,T)=\max(v-B(T),0), v>0 \tag{3.19}$$

$$f(0,t)=0, f(v,t)\sim v-B(T)e^{-r(T-t)} \text{ as } v\to\infty, \tag{3.20}$$

then the equity value of the company is given by

$$f(v,t) = v\Phi(x_1) - Be^{-r(T-t)}\Phi(x_2), \tag{3.21}$$

where

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy,$$

$$x_1 = \frac{\log \dfrac{v}{B} + r(T-t) + \frac{1}{2}\int_t^T g^2(V(s-L_2))ds}{\sqrt{\int_t^T g^2(V(s-L_2))ds}}$$

$$\textit{and} \qquad x_2 = x_1 - \sqrt{\int_t^T g^2(V(s-L_2))ds}.$$

Proof. From Lemma 3.2.1, we found that the parabolic partial differential equation of the form (3.11) can be reduced to the heat equation (3.14). We can then recover the previous function f by:

$$f(v,t) = B(T)e^{rt}h(x,\tau),$$

$$f(v,t) = B(T)e^{rt}h(x - \frac{1}{2}\int_t^T g^2(V(s-L_2))ds - r(T-t)), \frac{1}{2}\int_t^T g^2(V(s-L_2))ds). \tag{3.22}$$

But, the fundamental solution to the diffusion equation (3.14) is given by the Green's function

$$G(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{x^2}{4\tau}}.$$

Furthermore, the general solution h with initial condition $h(x,0) = \phi(x)$ is given by the convolution

$$\begin{aligned} h(x,\tau) &= h(x,0) * G(x,\tau) \\ &= \int_{-\infty}^{\infty} G(x-\eta,\tau)\phi(\eta)d\eta \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}}\phi(\eta)exp\left[-\frac{(x-\eta)^2}{4\tau}\right] d\eta. \end{aligned}$$

Now,

$$
\begin{aligned}
h(x,\tau) &= \frac{e^{-rT}}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} \max[e^{\eta}-1,0] \exp\left[-\frac{(x-\tau-rt+rT-\eta)^2}{4\tau}\right] d\eta \\
&= \frac{e^{-rT}}{\sqrt{4\pi\tau}} \int_{0}^{\infty} \exp\left[\frac{4\eta\tau-(x-\tau-rt+rT-\eta)^2}{4\tau}\right] d\eta \\
&\qquad - \frac{e^{-rT}}{\sqrt{4\pi\tau}} \int_{0}^{\infty} \exp\left[-\frac{(x-\tau-rt+rT-\eta)^2}{4\tau}\right] d\eta \\
&= I_1 - I_2,
\end{aligned} \tag{3.23}
$$

where

$$I_1 = \frac{e^{-rT}}{\sqrt{4\pi\tau}} \int_{0}^{\infty} \exp\left[\frac{4\eta\tau-(x-\tau-rt+rT-\eta)^2}{4\tau}\right] d\eta$$

and

$$I_2 = \frac{e^{-rT}}{\sqrt{4\pi\tau}} \int_{0}^{\infty} \exp\left[-\frac{(x-\tau-rt+rT-\eta)^2}{4\tau}\right] d\eta.$$

We first solve I_1. We make the following change of variable

$$z = \frac{4\eta\tau-(x-\tau-rt+rT-\eta)^2}{\sqrt{2\tau}} \rightarrow d\eta = \sqrt{2\tau}dz.$$

Completing the perfect square in the exponential of the integrand, we have

$$
\begin{aligned}
&\frac{4\eta\tau-(x-\tau-rt+rT-\eta)^2}{4\tau} \\
&\qquad = -((x+\tau-rt+rT)-\eta)^2 + (x+\tau-rt+rT)^2 - (x-\tau-rt+rT)^2 \\
&\qquad = -((x+\tau-rt+rT)-\eta)^2 + 4\tau(x-rt+rT).
\end{aligned}
$$

Moreover, we define the lower limit of the integration as

$$-d_1 = -\frac{(x+\tau-rt+rT)}{\sqrt{2\tau}}.$$

Hence we can write

$$I_1 = \exp\left[-rT + \frac{4\tau(x - rt + rT)}{4\tau}\right] \int_{-d_1}^{\infty} \frac{\exp\left[-\frac{z}{2}\right]}{\sqrt{2\pi}} dz$$
$$= e^{x-rt}\Phi(d_1).$$

We will compute I_2 in a similar way. Let us make the following change of variable

$$y = \frac{(-x + \tau + rt - rT + \eta)}{\sqrt{2\tau}} \rightarrow \eta = \sqrt{2\tau}dy.$$

As before we define the lower limit of the integration as

$$-d_2 = -\frac{(x - \tau - rt + rT)}{\sqrt{2\tau}}$$

$$I_2 = \frac{e^{-rT}}{\sqrt{4\pi\tau}} \int_0^{\infty} \exp\left[-\frac{(x - \tau - rt + rT - \eta)^2}{4\tau}\right] d\eta$$
$$= e^{-rT} \int_{-d_2}^{\infty} \frac{\exp[-\frac{y^2}{2}]}{\sqrt{2\pi}} \quad = e^{-rT}\Phi(d_2).$$

Using the results above, we obtain the solution of the heat equation

$$h(x, \tau) = e^{x-rt}\Phi(d_1) - e^{-rT}\Phi(d_2). \tag{3.24}$$

Now, we want to proceed backward using the relation between h, x and τ and f, v and t, respectively to get the solution for the RPDE. From relation (3.22), we can write the solution of the RPDE (3.11) as follow

$$f(v, t) = v\Phi(d_1) - B(T)e^{-r(T-t)}\Phi(d_2).$$

Finally, if we replace v by the value of the company $V(t)$ in the above equation, we obtain the formula for the equity value

$$f(V(t),t) = V(t)\Phi(d_1) - B(T)e^{-r(T-t)}\Phi(d_2), \tag{3.25}$$

where $d_1 = x_1$ and $d_2 = x_2$ as defined in the statement of the theorem. $\square$

We will derive a solution of equation (3.11) for debt value of a levered company using probabilistic methods. From this method, we can clearly see the similarities between equity value $f(V(t),t)$ and a European call option within Black-Scholes theory. We will state and prove some useful lemmas.

Lemma 3.2.3. *Let $Y(t) = f(V(t),t)$ be a security value that represents an asset composed of the firm values (risky) and risk free bonds. Assume the firm value follows the nonlinear SDDE*

$$dV(t) = \alpha V(t-L_1)V(t)dt + g(V(t-L_2))V(t)dW(t),\ t \in [0,T] \tag{3.26}$$

$$V(t) = \varphi(t),\ t \in [-L,0]. \tag{3.27}$$

and the bond accumulates interest compounded continuously at a rate r i.e. $B(t) = B(0)e^{rt}$. Assume this portfolio is replicable and self-financed. The following results hold

1. *the discounted value of $Y(t)$ is a martingale under the risk neutral measure,*

2. *$Y(t)$ satisfies equation*

$$\frac{Y(t)}{B(t)} = \frac{Y(0)}{B(0)} + \int_0^t \theta(s)dW^*(s),\ 0 \le t \le T \tag{3.28}$$

where $\{\theta(t)\}$ is an $(\mathcal{F}_t)$-adapted integrable process.

Proof. From the fact that $Y(t)$ is a replication portfolio, assume we have $x(t)$ units of firm values $V(t)$ and $y(t)$ units of bonds (debt) $B(t)$. We have

$$Y(t) = x(t)V(t) + y(t)B(t), \tag{3.29}$$

where $Y(t)$ is the value of this portfolio at time t. Since the portfolio is self-financed, then we have $dY(t) = x(t)dV(t) + y(t)dB(t)$. But, the bond compounds interest continuously at a rate of r, that is $B(t) = B(0)e^{rt}$. We can now write $Y(t)$ as follow

$$dY(t) = x(t)dV(t) + y(t)rB(t)dt. \tag{3.30}$$

By (3.29),

$$\begin{aligned} dY(t) &= x(t)dV(t) + r[Y(t) - x(t)V(t)]dt \\ &= x(t)\left[\alpha V(t)V(t-L_1)dt + g(V(t-L_2))V(t)dW(t)\right] + r[Y(t) - x(t)V(t)]dt \\ &= rY(t)dt + x(t)V(t)\left[(\alpha V(t-L_1) - r)dt + g(V(t-L_2))dW(t)\right] \end{aligned} \tag{3.31}$$

From Girsanov theorem, we have a probability measure P^* equivalent to P such that $W^*(t) := \int_0^t \dfrac{\alpha V(u-L_1) - r}{g(V(u-L_2))}du + W(t)$ with $g(t) \neq 0$. Hence

$$dW(t) = dW^*(t) - \frac{\alpha V(t-L_1) - r}{g(V(t-L_2))}dt \tag{3.32}$$

Using (3.32) in (3.31) gives

$$\begin{aligned} dY(t) &= rY(t)dt + x(t)V(t)\left[(\alpha V(t-L_1)-r)dt + g(V(t-L_2))\left(dW^*(t) - \frac{\alpha V(t-L_1)-r}{g(V(t-L_2))}dt\right)\right] \\ &= rY(t)dt + x(t)V(t)g(V(t-L_2))dW^*(t) \end{aligned} \tag{3.33}$$

Differentiating the discounted value of the portfolio $\dfrac{Y(t)}{B(t)}$ by the product rule for Itô differentials, we obtain:

$$d\left(\frac{Y(t)}{B(t)}\right) = dY(t)\frac{1}{B(t)} + Y(t)d\left(\frac{1}{B(t)}\right) \tag{3.34}$$

Substituting (3.33) into (3.34) we have

$$d\left(\frac{Y(t)}{B(t)}\right) = x(t)g(V(t-L_2))\frac{V(t)}{B(t)}dW^*(t). \tag{3.35}$$

Therefore,

$$\frac{Y(t)}{B(t)} = \frac{Y(0)}{B(0)} + \int_0^t x(s)g(V(s-L_2))\frac{V(s)}{B(s)}dW^*(s), \ 0 \le t \le T.$$

Assuming that $x(t)$ is a $\mathcal{F}_{t\in[0,T]}$-adapted process, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by $W(t)$ or $W^*(t)$, the Itô integral (3.28) is a martingale on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P^*)$. □

Lemma 3.2.4. *The solution of equation* (3.30) *for the portfolio value process* $Y(t)$ *given in the form*

$$Y(t) = e^{-r(T-t)}\mathbb{E}^*[Y(T) \mid \mathcal{F}_t]$$

exists.

Proof. From result (1) of the previous Lemma, $\dfrac{Y(t)}{B(t)}$ is a $\mathcal{F}_t$-martingale under P^* then there

exists a $\mathcal{F}_t$-measurable process $\theta(t)$ such that

$$\frac{Y(T)}{B(T)} = \frac{Y(t)}{B(t)} + \int_0^T \theta(s)dW^*(s),\ 0 \leq t \leq T.$$

By taking $\theta(s) = x(s)g(V(t-L_2))\dfrac{V(s)}{B(s)}$ and $y(t) = \dfrac{Y(t) - x(t)V(t)}{B(t)}$, we see that (3.28) satisfies $Y(t) = x(t)V(t) + y(t)B(t)$. Moreover, from Itô's product rule, we have

$$dY(t) = d\left(\frac{Y(t)}{B(t)}B(t)\right) = B(t)d\left(\frac{Y(t)}{B(t)}\right) + rB(t)\frac{Y(t)}{B(t)}dt,$$

by simplifying and replacing $d\left(\dfrac{Y(t)}{B(t)}\right)$ by its value from (3.35), we have

$$dY(t) = x(t)g(V(t-L_2))V(t)dW^*(t) + rY(t)dt,$$

by adding and subtracting the term $rx(t)V(t)dt$, we have

$$dY(t) = x(t)V(t)\left[g(V(t-L_2))dW^*(t) + rdt\right] + r\left[Y(t) - x(t)V(t)\right]dt,$$

and finally obtain

$$dY(t) = x(t)dV(t) + ry(t)B(t)dt, \quad \text{where} \;\; dV(t) = V(t)\left[g(V(t-L_2))dW^*(t) + rdt\right]$$

$$\text{and} \;\; y(t)B(t) = Y(t) - x(t)V(t)$$

which proves that (3.28) also satisfies $dY(t) = x(t)dV(t) + ry(t)B(t)dt$.

In addition,

$$\frac{Y(t)}{B(t)} = \mathbb{E}^*\left[\frac{Y(T)}{B(T)} \mid \mathcal{F}_t\right],\ 0 \leq t \leq T,$$

where $\mathbb{E}^*[. \mid \mathcal{F}_t]$ is the conditional expectation on the probability space $(\Omega, \mathcal{F}_T, P^*)$ under. Therefore, since for all t, $B(t) = B_0 e^{rt}$ we have

$$Y(t) = \frac{e^{rt}}{e^{rT}} \mathbb{E}^*[Y(T) \mid \mathcal{F}_t],$$

Hence

$$Y(t) = e^{-r(T-t)} \mathbb{E}^*[Y(T) \mid \mathcal{F}_t]. \tag{3.36}$$

□

Two boundary conditions and a terminal condition are needed to solve equation (3.11) for debt value of a firm. These boundary conditions are derived from the provisions stipulated in the contract and the capital structure.

Remark. Consider a firm whose value is given by $V(t), t \in [-L, T]$. Let $f(V(t), t)$ and $F(V(t), t)$ represent the equity value and the debt value of this firm, respectively. By the structural model, we have

$$V(t) = f(V(t), t) + F(V(t), t). \tag{3.37}$$

We have proved in 3.2.2 that f is solution to the RPDE (3.11) under the terminal

$$f(V(T), T) = \max[V(T) - B, 0] \tag{3.38}$$

and the boundary condition

$$f(0, t) = 0. \tag{3.39}$$

From relation (3.37), we have $f(V(t), t) = V(t) - F(V(t), t)$, and substitute for F in (3.11),

(3.39),and (3.38), to get

$$-\frac{1}{2}g^2(V(t-L_2))v^2F_{vv}+rv(1-F_v)-F_t-r(v-F)=0$$

from which we deduce the following partial differential equation for F

$$\frac{1}{2}g^2(V(t-L_2))v^2F_{vv}+rvF_v+F_t-rF=0, \tag{3.40}$$

under the boundary condition

$$F(0,t)=0,$$

and the terminal condition for the debt

$$F(V(t),t)=\min[B,V(T)]. \tag{3.41}$$

The last equation comes from the combination of equations (3.37) and (3.38) that is $F(V(T),T)=V(T)-\max[V(T)-B,0]=\min[B,V(T)]$.

Theorem 3.2.5. *Assume a levered firm are under the restrictions and provisions 1,2,3 stated at the begining of this Section 3.2. The value V of the company can be written as*

$$V(t)=F(V(t),t)+f(V(t),t),$$

where $f(V(t),t)$ is the value of the equity, $F(V(t),t)$ the value of debt a any time t before the maturity. Moreover, f and F are solutions to the RPDE with the boundary and terminal conditions:

- *the boundary condition*

$$F(0,t)=f(0,t)=0,$$

- *the terminal condition for the debt and for the equity at $t = T$ given respectively as*

$$F(V(T), T) = \min[V(T), B] \quad \textit{and} \quad f(V(T), T) = \max[V(T) - B, 0],$$

where B is the face value of the debt.

Then the debt value for $t \in [T - L, T]$ is given by

$$F(V(t), t) = Be^{-r(T-t)} \left\{ \Phi \left[N_2(d, \int_t^T g^2(V(s - L_2))ds) \right] + \frac{1}{d} \Phi \left[N_1(d, \int_t^T g^2(V(s - L_2))ds) \right] \right\}, \tag{3.42}$$

where

$$N_1 \left(d, \int_t^T g^2(V(s - L_2))ds \right) \equiv - \frac{\left(\frac{1}{2} \int_t^T g^2(V(s - L_2))ds - \log(d) \right)}{\sqrt{\int_t^T g^2(V(s - L_2))ds}},$$

$$N_2 \left(d, \int_t^T g^2(V(s - L_2))ds \right) \equiv - \frac{\left(\frac{1}{2} \int_t^T g^2(V(s - L_2))ds + \log(d) \right)}{\sqrt{\int_t^T g^2(V(s - L_2))ds}} \quad \textit{and} \quad d \equiv \frac{Be^{-r(T-t)}}{V(t)}.$$

Proof. We have learned from Black-Scholes [9] that there is a unique formula $f(V(t), t)$ that satisfies the differential equation (3.40) subject to the terminal condition (3.41). Furthermore, (3.40) and (3.41) are analogous to the equations for a European call option on a stock with no dividend, where the firm value V corresponds to the stock price and the face value of the debt B to the strike price. We will use this correspondence and the Black-Scholes formula derivation to derive the solution in the present context of a levered firm. Since there is no arbitrage opportunities, from Proposition 1.0.2 there exists P^* equivalent to P such that the discounted value of $f(V(t), t)$ is a martingale under P^*. Lemmas 3.2.4 and 3.2.3 show that the fair price $f(V(t), t)$ at time t exists and is given by

$$f(V(t), t) = e^{-r(T-t)} \mathbb{E}^{P^*} \left(f(V(T), T) | \mathcal{F}_t \right). \tag{3.43}$$

From (2.4)

$V(t) = \varphi(0)exp\left(\alpha\int_0^t V(s-L_1)ds - \frac{1}{2}\int_0^t g^2(V(s-L_2))ds + \int_0^t g(V(s-L_2))dW(s)\right)$,

$\forall t \in [0,T]$. Moreover, $W(t)$ is a normal distribution with mean 0 and variance t. Therefore, $V(t)$ has a lognormal distribution. Hence

$$V(T) = \varphi(0)exp\left(\alpha\int_0^T V(s-L_1)ds - \frac{1}{2}\int_0^T g^2(V(s-L_2))ds + \int_0^T g(V(s-L_2))dW(s)\right).$$

Let $W^*(t) := W(t) + \int_0^t \frac{\alpha V(s-L_1)-r}{g(V(s-L_2))}ds$, $t \in [0,T]$. Therefore,

$$\tilde{V}(t) = \varphi(0)exp\left(-\frac{1}{2}\int_0^t g^2(V(s-L_2))ds + \int_0^t g(V(s-L_2))dW^*(s)\right).$$

and

$$\begin{aligned}\tilde{V}(T) &= \varphi(0)\exp\left(-\tfrac{1}{2}\int_0^T g^2(V(s-L_2))ds + \int_0^T g(V(s-L_2))dW^*(s)\right)\\ &= \tilde{V}(t)\exp\left(-\tfrac{1}{2}\int_t^T g^2(V(s-L_2))ds + \int_t^T g(V(s-L_2))dW^*(s)\right).\end{aligned}$$

But $f(V(T),T) = \max[V(T)-B,0]$, then for $t \in [T-L,T]$,

$$\begin{aligned}&f(V(t),t) = e^{-r(T-t)}\mathbb{E}^{P^*}\left(f(V(T),T)|\mathcal{F}_t\right)\\ &= e^{-r(T-t)}\mathbb{E}^{P^*}\left(\max\left[V(T)-B,0\right]|\mathcal{F}_t\right)\\ &= e^{rt}\mathbb{E}^{P^*}\left(\max\left[\tilde{V}(T)-Be^{-rT},0\right]|\mathcal{F}_t\right)\\ &= e^{rt}\mathbb{E}^{P^*}\left(\max\left[\tilde{V}(t)\exp\left(-\tfrac{1}{2}\int_t^T g^2(V(s-L_2))ds + \int_t^T g(V(s-L_2))dW^*(s) - Be^{-rT}\right),0\right]|\mathcal{F}_t\right)\\ &= e^{rt}\mathbb{E}^{P^*}\left(\max\left[\tilde{V}(t)\exp\left(-\tfrac{1}{2}\int_t^T g^2(V(s-L_2))ds + \int_t^T g(V(s-L_2))dW^*(s)\right) - Be^{-rT},0\right]\right)\\ &= e^{rt}\int_{-\infty}^{\infty}\max\left[\tilde{V}(t)\exp\left(-\tfrac{1}{2}\int_t^T g^2(V(s-L_2))ds + x\right) - Be^{-rT},0\right]\frac{e^{-\frac{x^2}{2\int_t^T g^2(V(s-L_2))ds}}}{\sqrt{2\pi\int_t^T g^2(V(s-L_2))ds}}dx.\end{aligned}$$

Assuming that

$$\exp\left(-\frac{1}{2}\int_t^T g^2(V(s-L_2))ds + x\right) > \frac{Be^{-rT}}{\tilde{V}(t)},$$

we have the following inequality

$$x > \log\frac{Be^{-rT}}{\tilde{V}(t)} + \frac{1}{2}\int_t^T g^2(V(s-L_2))ds = \log\frac{Be^{-r(T-t)}}{V(t)} + \frac{1}{2}\int_t^T g^2(V(s-L_2))ds.$$

So equation (3.2) becomes

$$f(V(t),t) = \frac{V(t)}{\sqrt{2\pi\int_t^T g^2(V(s-L_2))ds}} \int_D^\infty e^{\left(-\frac{1}{2}\int_t^T g^2(V(s-L_2))ds + x - \frac{x^2}{2\int_t^T g^2(V(s-L_2))ds}\right)} dx$$
$$- Be^{-r(T-t)} \int_D^\infty \frac{e^{-\frac{x^2}{2\int_t^T g^2(V(s-L_2))ds}}}{\sqrt{2\pi\int_t^T g^2(V(s-L_2))ds}} dx,$$

where $D = \log\dfrac{B}{V(t)} - r(T-t) + \frac{1}{2}\int_t^T g^2(V(s-L_2))ds$.

Making the change of variable $x^2 = y^2\int_t^T g^2(V(s-L_2))ds$, we get

$$f(V(t),t) = \frac{V(t)}{\sqrt{2\pi}}\int_{d_1}^\infty \exp\left(-\frac{1}{2}\left(y^2 - 2y\sqrt{\int_t^T g^2(V(s-L_2))ds} + \int_t^T g^2(V(s-L_2))ds\right)\right)dy$$
$$- \frac{Be^{-r(T-t)}}{\sqrt{2\pi}}\int_{d_1}^\infty e^{-\frac{y^2}{2}}dy$$
$$= V(t)\int_{d_1}^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(y - \sqrt{\int_t^T g^2(V(s-L_2))ds}\right)^2\right)dy - Be^{-r(T-t)}\left(1 - \Phi\left(d_1\right)\right).$$

where $d_1 = \dfrac{\log\dfrac{B}{V(t)} - r(T-t) + \frac{1}{2}\int_t^T g^2(V(s-L_2))ds}{\sqrt{\int_t^T g^2(V(s-L_2))ds}}$.

Again changing the variable

$$z = y - \sqrt{\int_t^T g^2(V(s-L_2))ds},$$

we have

$$\begin{aligned} f(V(t),t) &= V(t)\int_{d_1-\sqrt{\int_t^T g^2(V(s-L_2))ds}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - Be^{-r(T-t)}\left(1-\Phi\left(d_1\right)\right) \\ &= V(t)\left(1-\Phi\left(d_1-\sqrt{\int_t^T g^2(V(s-L_2))ds}\right)\right) - Be^{-r(T-t)}\Phi\left(-d_1\right), \end{aligned}$$

therefore

$$f(V(t),t) = V(t)\left(\Phi\left(\sqrt{\int_t^T g^2(V(s-L_2))ds} - d_1\right)\right) - Be^{-r(T-t)}\Phi\left(-d_1\right).$$

Thus

$$f(V(t),t) = V(t)\Phi(x_1) - Be^{-r(T-t)}\Phi(x_2), \tag{3.44}$$

where

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy,$$

$$x_1 = \frac{\log\dfrac{V(t)}{B} + r(T-t) + \frac{1}{2}\int_t^T g^2(V(s-L_2))ds}{\sqrt{\int_t^T g^2(V(s-L_2))ds}}$$

$$\text{and} \quad x_2 = x_1 - \sqrt{\int_t^T g^2(V(s-L_2))ds}.$$

From equation (3.44) and $F(V(t),t) = V(t) - f(V(t),t)$, we can write the value of the debt as

$$F(V(t),t) = Be^{-r(T-t)}\left\{\Phi\left[x_2(d,\int_t^T g^2(V(s-L_2))ds)\right] + \frac{1}{d}\Phi\left[-x_1(d,\int_t^T g^2(V(s-L_2))ds)\right]\right\},$$

where $x_1\left(d, \int_t^T g^2(V(s-L_2))ds\right)$ and $x_2\left(d, \int_t^T g^2(V(s-L_2))ds\right)$ are defined as

above and, $d \equiv \dfrac{Be^{-r(T-t)}}{V(t)}$ is the debt to firm value ratio. If we replace in the above formula $-x_1\left(d, \int_t^T g^2(V(s-L_2))ds\right)$ and $x_2\left(d, \int_t^T g^2(V(s-L_2))ds\right)$ by $N_1\left(d, \int_t^T g^2(V(s-L_2))ds\right)$ and $N_2\left(d, \int_t^T g^2(V(s-L_2))ds\right)$, respectively we obtain

$$F(V(t),t) = Be^{-r(T-t)}\left\{\Phi\left[N_2(d, \int_t^T g^2(V(s-L_2))ds)\right] + \frac{1}{d}\Phi\left[N_1(d, \int_t^T g^2(V(s-L_2))ds)\right]\right\}, \tag{3.45}$$

and the proof is complete. $\square$

Because it is common in discussion of bond pricing to talk in terms of yields rather than prices, from (3.45)

$$\begin{aligned}\log \frac{F(V(t),T-t)}{B} \\ &= -r(T-t) + \log\left\{\Phi\left[N_2(d, \textstyle\int_t^T g^2(V(s-L_2))ds)\right] + \frac{1}{d}\Phi\left[N_1(d, \textstyle\int_t^T g^2(V(s-L_2))ds)\right]\right\}\end{aligned} \tag{3.46}$$

we can rewrite equation (3.45) as

$$\begin{aligned}R(T-t) - r \\ &= -\frac{1}{T-t}\log\left\{\Phi\left[N_2(d, \textstyle\int_t^T g^2(V(s-L_2))ds)\right] + \frac{1}{d}\Phi\left[N_1(d, \textstyle\int_t^T g^2(V(s-L_2))ds)\right]\right\}\end{aligned} \tag{3.47}$$

where $e^{-R(T-t)(T-t)} = \dfrac{F(V(t),T-t)}{B}$, and $R(T-t)$ is the yield to the maturity on the risky debt provided that the firm does not default. It seems reasonable to call $R(T-t)-r$ a risk premium in which case equation (3.47) defines a risk structure of interest rates.
As in Merton [38] case, the risk premium is a function of the volatility function g on the firm and the (biased upward estimate) debt to firm value ratio d.

3.3 EVALUATION OF LOAN GUARANTEES

Now let us examine the impact of a guarantor, that is a government or an institution insuring payment to the bondholders in any case. We are going to consider a claim market value as the simplest case of corporate debt. Assume the company is financed by:

a) a single class of debt,

b) the equity,

c) the guarantee on the debt.

Furthermore, assume the following restrictions and provisions are stipulated in the contract according to the loan guarantees issue: the contract stipulates that

1. in case the management on the maturity date is unable to make the payment promised, the government will meet these payments with no uncertainty;

2. the firm is expected to pay an amount at least equal to its actuarial cost for the guarantee, so that in case this happens, the firm is required to default all its assets to the guarantor;

3. the firm is not allowed to issue a new senior claim on the firm nor to pay cash dividend during the option's life.

Notice that the presence of a guarantor transforms the debt which was a risky asset to a riskless asset. If the firm value is less than the promised payment, then the debtholders receive the amount B, and the equity holders receive nothing. Therefore, the guarantor loses the amount $B - V(T)$. However, if the firm value is greater than the promised payment, then the debtholders receive B and the equity holders receive $V(T) - B$ as without the guarantee. In other words, the guarantor has no impact on the equity value ($\max[V(T) - B, 0]$) at the maturity date, but the debt value is riskless and always equal to B. However, the value of the guarantee is the nonpositive value $\min[V(T) - B, 0]$. In

effect, the result of the guarantee is to create an additional cash inflow to the firm of the amount $-\min[V(T)-B,0]$. But, $-\min[V(T)-B,0]=\max[B-V(T),0]$. Therefore, if $G(T)$ is the cost we are looking for, where the length of time until the maturity date of the bond is T, then

$$G(T)=G(V(T),T)=\max[B-V(T),0].$$

So, the face value of the debt B can be taken to be analogous to the strike price and $V(T)$ to the stock price in option pricing theory. These similarities between the evaluation of the company loan guarantees $G(T)$ and the evaluation of a European put option allow us to say that loan guarantees work as European put options on the firm value giving to the management the right but not an obligation to sell the amount B to a guarantor.

Let us apply the theory of contingent claims to loan guarantees pricing.

Theorem 3.3.1. *Assume a levered firm is under the restrictions and provisions 1, 2, 3 in Section 3.2. Moreover, assume the following conditions:*

- *the boundary condition*

$$G(0,t)=0, \tag{3.48}$$

- *the final condition for the loan guarantees at $t=T$ given as*

$$G(V(T),T)=\max[B-V(T),0], \tag{3.49}$$

where B is the face value the debt.

Then the loan guarantee at $t\in[T-L,T]$ is given by

$$G(V(t),t)=Be^{-r(T-t)}\Phi\left(d_{11}(d,\int_t^T g^2(V(s-L_2))ds)\right)+V(t)\Phi\left(d_{12}(d,\int_t^T g^2(V(s-L_2))ds)\right), \tag{3.50}$$

where $d \equiv \dfrac{Be^{-r(T-t)}}{V(t)}$, $\qquad d_{11}(d, \int_t^T g^2(V(s-L_2))ds) \equiv \dfrac{\left(\frac{1}{2}\int_t^T g^2(V(s-L_2))ds + \log(d)\right)}{\sqrt{\int_t^T g^2(V(s-L_2))ds}}$

and $d_{12}(d, \int_t^T g^2(V(s-L_2))ds) \equiv \dfrac{\left(\log(d) - \frac{1}{2}\int_t^T g^2(V(s-L_2))ds\right)}{\sqrt{\int_t^T g^2(V(s-L_2))ds}}$.

Proof. Substituting for G in (3.11), (3.48), and (3.49), we get

$$-\frac{1}{2}g^2(V(t-L_2))v^2G_{vv} + rv(1-G_v) - G_t - r(v-G) = 0$$

from which we deduce the following partial differential equation for G

$$\frac{1}{2}g^2(V(t-L_2))v^2G_{vv} + rvG_v + G_t - rG = 0. \tag{3.51}$$

We also have

$$G(V(T), T] = \max[B - V(T), 0]. \tag{3.52}$$

Moreover, $G(V(t), t) \leq F(V(t), t) - V(t)$. From Black-Scholes-Merton [9] there is a unique formula $G(v, t)$ that satisfies the differential equation (3.51) subject to the terminal condition (3.52) and boundary condition $G(0, t) = 0$. Furthermore, (3.51) and (3.52) are identical to the equations for an European put option on a stock with no dividend, where the firm value $V(t)$ at time t corresponds to the stock price and the face value of the debt B to the strike price. As before, we use this equivalent relation between levered equity of the firm and a European put option to derive the solution (3.50). Since there is no arbitrage opportunities, from Proposition 1.0.2 there exists a probability measure P^* equivalent to P such that the discounted value of $G(V(t), t)$ is a martingale under P^*. Lemmas 3.2.4

and 3.2.3 show that the fair price $G(V(t),t)$ at time t exists and is given by

$$G(V(t),t) = e^{-r(T-t)}\mathbb{E}^{P^*}\left(G(V(T),T)|\mathcal{F}_t\right). \tag{3.53}$$

But $G(V(T),T) = \max[B - V(T), 0]$, then for $t \in [T-L, T]$, therefore

$$\begin{aligned}
&\mathbb{E}^{P^*}\left(G(V(T),T)|\mathcal{F}_t\right) = \mathbb{E}^{P^*}\left(\max\left[B - V(T), 0\right]|\mathcal{F}_t\right) \\
&= \mathbb{E}^{P^*}\left(\max\left[Be^{-rT} - \tilde{V}(T), 0\right]|\mathcal{F}_t\right) \\
&= \mathbb{E}^{P^*}\left(\max\left[Be^{-rT} - \tilde{V}(t)exp\left(-\tfrac{1}{2}\int_t^T g^2(V(s-L))ds + \int_t^T g(V(s-L))dW^*(s)\right), 0\right]|\mathcal{F}_t\right) \\
&= \mathbb{E}^{P^*}\left(\max\left[Be^{-rT} - \tilde{V}(t)exp\left(-\tfrac{1}{2}\int_t^T g^2(V(s-L))ds + \int_t^T g(V(s-L))dW^*(s)\right), 0\right]\right) \\
&= \int_{-\infty}^{\infty} \max\left[Be^{-rT} - \tilde{V}(t)exp\left(-\tfrac{1}{2}\int_t^T g^2(V(s-L))ds + x\right), 0\right] \frac{e^{-\frac{x^2}{2\int_t^T g^2(V(s-L))ds}}}{\sqrt{2\pi\int_t^T g^2(V(s-L))ds}}dx,
\end{aligned}$$

hence

$$\begin{aligned}
&G(V(t),t) \\
&= e^{rt}\int_{-\infty}^{\infty} \max\left[Be^{-rT} - \tilde{V}(t)exp\left(-\tfrac{1}{2}\int_t^T g^2(V(s-L))ds + x\right), 0\right] \frac{e^{-\frac{x^2}{2\int_t^T g^2(V(s-L))ds}}}{\sqrt{2\pi\int_t^T g^2(V(s-L))ds}}dx.
\end{aligned} \tag{3.54}$$

Assuming that

$$exp\left(-\frac{1}{2}\int_t^T g^2(V(s-L))ds + x\right) < \frac{Be^{-rT}}{\tilde{V}(t)},$$

we have the following inequality

$$x < \log\,\frac{Be^{-rT}}{\tilde{V}(t)} + \frac{1}{2}\int_t^T g^2(V(s-L))ds = \log\,d + \frac{1}{2}\int_t^T g^2(V(s-L))ds,$$

where d is defined as in the statement of the theorem. So equation (3.54) becomes

$$G(V(t),t) = \frac{Be^{-r(T-t)}}{\sqrt{2\pi \int_t^T g^2(V(s-L))ds}} \int_{\infty}^{D} e^{-\frac{x^2}{2\int_t^T g^2(V(s-L))ds}} dx$$
$$-\frac{V(t)}{\sqrt{2\pi \int_t^T g^2(V(s-L))ds}} \int_{\infty}^{D} e^{\left(-\frac{1}{2}\int_t^T g^2(V(s-L))ds + x - \frac{x^2}{2\int_t^T g^2(V(s-L))ds}\right)} dx,$$

where $D = \log\, d + \frac{1}{2}\int_t^T g^2(V(s-L))ds$.

Making the change of variable $x^2 = y^2 \int_t^T g^2(V(s-L))ds$, we get

$$G(V(t),t) = \frac{Be^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{d_{11}} e^{-\frac{y^2}{2}} dy$$
$$-\frac{V(t)}{\sqrt{2\pi}} \int_{-\infty}^{d_{11}} exp\left(-\frac{1}{2}\left(y^2 - 2y\sqrt{\int_t^T g^2(V(s-L))ds} + \int_t^T g^2(V(s-L))ds\right)\right) dy$$
$$= Be^{-r(T-t)}\Phi\left(d_{11}\right) - V(t)\int_{-\infty}^{d_{11}} \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(y - \sqrt{\int_t^T g^2(V(s-L))ds}\right)^2\right) dy.$$

where $d_{11} = \dfrac{\log\, d + \frac{1}{2}\int_t^T g^2(V(s-L))ds}{\sqrt{\int_t^T g^2(V(s-L))ds}}$.

Making another change of variable $z = y - \sqrt{\int_t^T g^2(V(s-L))ds}$, we have

$$\begin{aligned} G(V(t),t) &= Be^{-r(T-t)}\Phi\left(d_{11}\right) - V(t)\int_{-\infty}^{d_{11}-\sqrt{\int_t^T g^2(V(s-L))ds}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= Be^{-r(T-t)}\Phi\left(d_{11}\right) - V(t)\Phi\left(d_{11} - \sqrt{\int_t^T g^2(V(s-L))ds}\right). \end{aligned}$$

Therefore

$$G(V(t),t) = Be^{-r(T-t)}\Phi(d_{11}) - V(t)\Phi(d_{12}), \tag{3.55}$$

where

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy,$$

$$d_{11} = \frac{\log\ d + \frac{1}{2}\int_t^T g^2(V(s-L))ds}{\sqrt{\int_t^T g^2(V(s-L))ds}} \qquad \text{and} \qquad d_{12} = d_{11} - \sqrt{\int_t^T g^2(V(s-L))ds}.$$

$\square$

CHAPTER 4

RISK STRUCTURE OF THE FIRM

The probability of bankruptcy is the likelihood that equityholders will walk away with nothing and the equity will lose all its value. In this chapter, we present a theory of the risk structure of interest rates. Several authors define the term ”risk” as the possible gains or losses to the debtholders as a result of changes in inflation and interest in general. In this work, we restrict our definition to the possible gains or losses to the debtholders as a consequence of changes in the probability of default. We study the monotonicity of the risk premium and the standard deviation of the debt (the business risk of the debt) with respect to the same variables. The main objective is to find out what can be considered as a valid measure of risk. In order to analyze the risk structure, we shall consider all the important features which enable to measure the firm risk viz. the risky debt, the cross section of debt prices at time $T-t$, the relative riskiness of the debt in terms of the riskiness of the firm at time $T-t$, the risk premium.

Definition. (Profitabilty index)

Ratio of the present value of a project's cash flows to the initial investment. A profitability index number greater than 1 indicates an acceptable project, and is consistent with a net present value greater than 0.

4.1 A COMPARATIVE ANALYSIS OF THE RISK STRUCTURE FOR AN HOMOGENEOUS CLASS OF DEBT

Equation (3.45) in Chapter 3 shows the functional dependence of the debt value as a function $F^{'}(V(t), t, B, g^2(V(t-L)), r)$. If we assume the distribution of the returns per dollar invested in the common firm value is independent of the level of the company value, Merton in his paper ([40]) has shown that $F^{'}$ is a first degree homogeneous convex function

of B and $V(t)$

$$F_v^{'} = 1 - f_v^{'} \geq 0.$$

Moreover, his proofs of Theorems 14 and 15 in the same paper allows us to consider that for a given firm value, $F^{'}$ is an increasing function of B (the price of a riskless (in terms of default) discounted debt which pays one dollar, T years from now), and hence a decreasing function of the $T-$year interest rate

$$F_B^{'} = -f_B^{'} > 0,\ F_{T-t}^{'} = -f_{T-t}^{'} < 0;$$

also that $F^{'}$ is a nondecreasing function of $g^2(V(t-L))$

$$F^{'}_{\int_t^T g^2(V(s-L))ds} = -f^{'}_{\int_t^T g^2(V(s-L))ds} < 0,\ F_r^{'} = -f_r^{'} < 0$$

where subscripts are partial derivatives.

The above results correspond to the what should be expected for a discount debt which is an increasing function of the current market value of the firm and the promised payment B at maturity, and a decreasing function of the time to maturity, the business risk of the firm, and the riskless rate of interest. As the risk structure describes the relationship between interest rates on bonds with the same term to maturity, it seems more reasonable to work with $P = \dfrac{F[V(t),t]}{Be^{-r(t)}}$ (the price today of a risky dollar promised at time t in the future in terms of a dollar delivered at that date with certainty), which is always less than or equal to one. We have from equation (3.45)

$$P(d, \int_t^T g^2(V(s-L))ds) = \Phi\left[N_2(d, \int_t^T g^2(V(s-L))ds)\right] + \frac{1}{d}\Phi\left[N_1(d, \int_t^T g^2(V(s-L))ds)\right] \tag{4.1}$$

P is a function of the "quasi" debt-to-firm value ratio d and of $\int_t^T g^2(V(s-L))ds$, which is a measure of the volatility of the firm's value over $[T-t, T]$ of the bond. Is it a decreasing function of both i.e. we have $P_d < 0$ and $P_{\int_t^T g^2(V(s-L))ds} < 0$ where

$$P_d = -\frac{\Phi\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]}{d^2} \tag{4.2}$$

and

$$P_{\int_t^T g^2(V(s-L))ds} = -\frac{\Phi'\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]}{2d\sqrt{\int_t^T g^2(V(s-L))ds}} \tag{4.3}$$

where $\Phi'(x) \equiv \dfrac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ is the standard normal density function. To further analyze the risk structure, let $q\left(d, \int_t^T g^2(V(s-L))ds\right)$ be a measure of the relative riskiness of the bond in terms of the riskiness of the firm at a given point in time. We have:

$$\begin{aligned}
q\left(d, \int_t^T g^2(V(s-L))ds\right) &= \sqrt{\frac{\int_t^T g_y^2(V(s-L))ds}{\int_t^T g^2(V(s-L))ds}} \\
&= \frac{V(t)F_v}{F(V(t),t)} \\
&= \frac{V(t)F_v}{Be^{-r(T-t)}\left\{\Phi\left[N_2\left(d, \int_t^T g^2(V(s-L))ds\right)\right] + \frac{1}{d}\Phi\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]\right\}} \\
&= \frac{\Phi\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]}{dP\left(d, \int_t^T g^2(V(s-L))ds\right)}
\end{aligned}$$

where $F_v = \Phi\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]$, $\sqrt{\int_t^T g_y^2(V(s-L))ds}$ is the instantaneous standard deviation of the return on the debt and $\sqrt{\int_t^T g^2(V(s-L))ds}$ is the instantaneous standard deviation of the return on the firm. Since the market is complete, and the payoff of debt and firm are driven by the same source of risk, these two returns are instantaneously perfectly correlated. Therefore, q is a measure of the relative riskiness of the debt in terms

of the riskiness of the firm at a given point in time. In addition, since there is no arbitrage, from (3.9) in Chapter 3 we have:

$$q\left(d, \int_t^T g^2(V(s-L))ds\right) = \frac{\alpha_y - r}{\alpha V(t-L) - r}, \tag{4.4}$$

where $\alpha_y - r$ is the expected excess return on the debt and $\alpha V(t-L) - r$ is the expected excess return on the firm as a whole. We can rewrite (4.2) and (4.3) in terms of q as:

$$\frac{dP_d}{P} = -q\left(d, \int_t^T g_y^2(V(s-L))ds\right)$$

and

$$\frac{\int_t^T g_y^2(V(s-L))dsP_{\int_t^T g_y^2(V(s-L))ds}}{P}$$

$$= -q\left(d, \int_t^T g^2(V(s-L))ds\right) \frac{\sqrt{\int_t^T g^2(V(s-L))ds\Phi'\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]}}{2\Phi\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]}$$

Further analysis will show that q is a valid measure of the risk when we are dealing with a single portfolio.

We now want to study the monotonicity of the risk premium H and that of the standard deviation of the return on the debt G. From equation (3.47), if we denote the risk premium on the debt as

$$R(T-t) - r \equiv H\left(d, t, \int_t^T g^2(V(s-L))ds\right),$$

then we have:

$$H_d = \frac{1}{dt} q\left(d, \int_t^T g^2(V(s-L))ds\right) > 0, \tag{4.5}$$

$$H_{\int_t^T g^2(V(s-L))ds} = \frac{1}{2\sqrt{\int_t^T g^2(V(s-L))ds}} \frac{\Phi'\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]}{\Phi\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]} > 0, \tag{4.6}$$

$$H_t = \frac{\left[\log P + \frac{\sqrt{\int_t^T g^2(V(s-L))ds}}{2} q\left(d, \int_t^T g^2(V(s-L))ds\right) \frac{\Phi'\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]}{\Phi\left[N_1\left(d, \int_t^T g^2(V(s-L))ds\right)\right]}\right]}{t^2} \gtrless 0 \tag{4.7}$$

$$H_r = H_d \frac{\partial d}{\partial r} = -q\left(d, \int_t^T g^2(V(s-L))ds\right) < 0 \tag{4.8}$$

While from Equation (4.7), the change in the premium with respect to a change in the length to the maturity (or the time) can have either positive or negative. We would like to see what simulations show depending on the value of d (whether $d \geq 1$ or $d < 1$). Equation (4.8) shows that the premium is a decreasing function of the riskless rate of interest.

Now, in order to verify the minimum necessary condition for H to be a valid measure of risk for debt, we will investigate the monotonicity of the standard deviation of the return on the debt

$$\begin{aligned} G\left(d, \sqrt{\int_t^T g^2(V(s-L))ds}, t\right) &\equiv \sqrt{\int_t^T g_y^2(V(s-L))ds} \\ &= \sqrt{\int_t^T g^2(V(s-L))ds}\, q\left(d, \int_t^T g^2(V(s-L))ds\right) \end{aligned}$$

as a function of the same variables $\left(d, \sqrt{\int_t^T g^2(V(s-L))ds}, t, r\right)$ as for H.

$$G_d = \frac{\sqrt{\int_t^T g^2(V(s-L))ds q^2}}{\sqrt{\int_t^T g^2(V(s-L))ds}} \frac{\Phi\left[N_2\right]}{\Phi\left[N_1\right]} \left[\frac{\Phi'\left[N_2\right]}{\Phi\left[N_2\right]} + \frac{\Phi'\left[N_1\right]}{\Phi\left[N_1\right]} + N_1 + N_2\right] > 0,$$

$$G_{\sqrt{\int_t^T g^2(V(s-L))ds}} = q \frac{\left[\Phi\left[N_1\right] - \Phi'\left[N_1\right]\left[\frac{1}{2}(1-2q) + \frac{\log d}{\int_t^T g^2(V(s-L))ds}\right]\right]}{\Phi\left[N_1\right]} > 0,$$

$$G_t = -\frac{\int_t^T g^2(V(s-L))dsG}{\sqrt{\int_t^T g^2(V(s-L))ds}} \frac{\Phi'\left[N_1\right]}{\Phi\left[N_1\right]} \left[\frac{1}{2}(1-2q) + \frac{\log d}{\int_t^T g^2(V(s-L))ds}\right] \geq 0$$

as

$d \leq 1,$

$$G_t = -\frac{\int_t^T g^2(V(s-L))dsG}{\sqrt{\int_t^T g^2(V(s-L))ds}} \frac{\Phi'\left[N_1\right]}{\Phi\left[N_1\right]} \left[\frac{1}{2}(1-2q) + \frac{\log d}{\int_t^T g^2(V(s-L))ds}\right] = 0 \tag{4.9}$$

as

$d = 1,$

$$G_t = -\frac{\int_t^T g^2(V(s-L))dsG}{\sqrt{\int_t^T g^2(V(s-L))ds}} \frac{\Phi'\left[N_1\right]}{\Phi\left[N_1\right]} \left[\frac{1}{2}(1-2q) + \frac{\log d}{\int_t^T g^2(V(s-L))ds}\right] \leq 0$$

as

$d \geq 1,$

$$G_r = G_d \frac{\partial d}{\partial r} = -(T-t)dG_d < 0,$$

where $N_i \equiv N_i\left(d, \int_t^T g^2(V(s-L))ds\right),\ i = 1, 2.$

The standard deviation of the return on the debt G measure the uncertainty of the rate of return on the debt over the next trading interval. In comparing the riskiness of the debt of different companies (or portfolios), G may not be sufficient to measure risk.

This is because the correlations of the returns of the portfolios with other assets may be different. However, the computation of the risk premium H do not require the knowledge of the correlations. Hence, as a necessary condition for H to be a valid measure of risk, H and G should coincide on the monotonicity(should move in the same direction) with respect to the same variables.

From the studies of the monotonicity of H and G, it's clear that they do not often change in the same direction with respect to the maturity. One can conclude that if we fix the maturity date T, the risk premium H is a valid measure of risk. Unfortunately, when the maturities are different, we cannot insure that H and G will change in the same direction with respect to the maturity.

4.2 IMPACT OF AN ADDITIONAL DEBT ON THE FIRM'S RISK STRUCTURE

Let us study the impact of the guarantee on the value of the company. Consider a levered company financed by equity and debt. Let $E(t)$, $B(t)$ and $V(t)$ denote the value of the equity, the value of the debt and that of the firm at any time $t \in [0, T]$, respectively. Assume the face value of the debt is B. Let us compute the probability of default of this company given by $P(V(T) < B)$. We know that $P(V(T) < B) = \Phi(d_{11})$ where $d_{11} = \dfrac{\log\, d + \frac{1}{2}\int_t^T g^2(V(s-L))ds}{\sqrt{\int_t^T g^2(V(s-L))ds}}$. We have the, $E(T) = \max[V(T) - B, 0]$ and

$B(T) = \min[V(T), B]$. Indeed,

$$
\begin{aligned}
P(V(T) < B) &= P\left(\tilde{V}(T)e^{rT} < B\right) \\
&= P\left(\tilde{V}(t)\exp\left(-\tfrac{1}{2}\int_t^T g^2(V(s-L))ds + \int_t^T g(V(s-L))dW^*(s)\right) < Be^{-rT}\right) \\
&= P\left(\exp\left(-\tfrac{1}{2}\int_t^T g^2(V(s-L))ds + \int_t^T g(V(s-L))dW^*(s)\right) < \frac{Be^{-r(T-t)}}{V(t)}\right) \\
&= P\left(\int_t^T g(V(s-L))dW^*(s) < \log\, d + \tfrac{1}{2}\int_t^T g^2(V(s-L))ds\right) \\
&= P\left(\frac{\int_t^T g(V(s-L))dW^*(s)}{\sqrt{\int_t^T g^2(V(s-L))ds}} < \frac{\log\, d + \frac{1}{2}\int_t^T g^2(V(s-L))ds}{\sqrt{\int_t^T g^2(V(s-L))ds}}\right) \\
&= \Phi\left(\frac{\log\, d + \frac{1}{2}\int_t^T g^2(V(s-L))ds}{\sqrt{\int_t^T g^2(V(s-L))ds}}\right).
\end{aligned}
$$

The last equality comes from the fact that $\dfrac{\int_t^T g(V(s-L))dW^*(s)}{\sqrt{\int_t^T g^2(V(s-L))ds}}$ is normally distributed with mean 0 variance 1, because $\int_t^T g(V(s-L))dW^*(s)$ is normally distributed with mean 0 and variance $\int_t^T g^2(V(s-L))ds$.

Now let us consider that a different debt is added to the value of the company $V(t)$ and compare the probability of default with the previous one. An additional debt of face value B' will increase the total face value to $B + B'$. If $V(t) > B$, then $\dfrac{V(t)}{B} > \dfrac{V(t) + B'}{B + B'} = \dfrac{V'(t)}{B + B'}$ where $V'(t) = V(t) + B'$. Since the logarithm is an increasing function, we can write $\log\left(\dfrac{B}{V(t)}\right) < \log\left(\dfrac{B + B'}{V'(t)}\right)$. So that, $d_{11} < d'_{11}$ where $d'_{11} = \dfrac{\log\left(\frac{B+B'}{V'(t)}\right) - r(T-t) + \frac{1}{2}\int_t^T g^2(V(s-L))ds}{\sqrt{\int_t^T g^2(V(s-L))ds}}$ and therefore $\Phi(d_{11}) < \Phi(d'_{11})$. From the previous analysis, we can say that loan guarantees do not prevent bankruptcy. They mainly care about debtholders investments.

Now, suppose $V(t) < B$ then $\Phi(d_{11}) > \Phi(d'_{11})$. This means, if the firm value is already less than the face value of the debt, they may be a chance that an additional debt may decrease the probability of default. But the question is whether this additional debt enable the firm to avoid bankruptcy? For this reason, we need to compute the profitability index for a new project that we want to invest in before making a decision.

CHAPTER 5

EVALUATION OF LOAN GUARANTEES WITHIN AN HETEROGENEOUS CLASS OF DEBT

5.1 EVALUATION OF A GUARANTEED LOAN FOR A LEVERED COMPANY

We start with some financial terms definitions.

Definition. (Senior Loan)
A *senior loan* is a class of debt whose terms in the event of bankruptcy, require it to be repaid before any other class of debt (by the same issuer) receives any payment.

Definition. (Subordinated Loan)
A *subordinated loan* is a class of debt that have a lower priority than other debts of the same issuer in case of liquidation.

Definition. (No Loss No Gain)
A *no loss no gain* is a concept requiring that aside from the actual changes in plan provisions, employees will not gain or lose any earned rights due to situations which would not have occurred without a plan modification.

In the event of bankruptcy or liquidation, there is a hierarchy of creditors. In case of multiple classes of debt, the priority is given to the debtholders who hold what is called senior debt viz. the senior debtholders have to be paid before any other debtholder. In this chapter, we suppose that a company is financed by the equity and a class of debt. At some point during the contract life (for whatever reason), the management decides to undertake a new project that will be financed by a new debt. This new debt is either equally senior or subordinated to the existing debt.

From the random partial differential equation (3.11) which any security's pricing func-

tion (of time and firm value) must satisfy, we observed a relationship between the instantaneous volatility function of the returns on the firm $g \equiv g(V(t-L_2))$ and the instantaneous standard deviation function of the returns on the equity $g_y \equiv g(Y(t-L_2))$. From the no dividends assumption and, assuming the total face value of the debt is $B(T)$ at the maturity date T, we obtained the value of the equity as a function of $(V(t), B, t)$ that is $E(t) = f_1(V(t), B, t)$ at any time prior to the maturity as a European call option for Black-Scholes formula

$$f_1(V(t), B, t) = V(t)\Phi(x_1) - Be^{-r(T-t)}\Phi(x_2), \tag{5.1}$$

where V is the value of the firm, r is the continuously compounded riskfree interest rate,

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy,$$

$$x_1 = \frac{\log \dfrac{V(t)}{B} + r(T-t) + \frac{1}{2}\int_t^T g^2(V(s-L_2))ds}{\sqrt{\int_t^T g^2(V(s-L_2))ds}}$$

$$\text{and} \quad x_2 = x_1 - \sqrt{\int_t^T g^2(V(s-L))ds}.$$

Now consider a levered company with an additional guaranteed debt which will be used to fund a new project which costs I. Assume the face value of the guaranteed debt is Ie^{rT}. Assume $E(t)$, $B(t)$, $V(t)$ are the value of the equity, the value of the debt and that of the firm without the additional debt, respectively. We can write $V(t) = E(t) + B(t)$.

Moreover, assume $\tilde{E}(t)$, $\tilde{B}(t)$, $\tilde{V}(t)$ are the value of the equity, the value of the debt and that of the firm with the additional debt, respectively. From the above we have

$$\tilde{V}(t) = V(t) + \eta I, \tag{5.2}$$

where η is the new project's profitability index. Assume the no-loss no-gain condition that is $\tilde{E}(t) = E(t)$. Then we may write

$$f_1\left(V(t),\, B,\, t\right) = f_1\left(\tilde{V}(t),\, B + Ie^{rT},\, t\right) \tag{5.3}$$

Let us investigate how to use the above condition for the evaluation of the loan guarantee and the profitability index. In the next two subsections, we will use the above condition to evaluate the loan guarantee value and that of the profitability index depending on whether the loans are equally senior or subordinated.

5.2 SUBORDINATED LOAN GUARANTEES

Let $\tilde{B}$ and $\tilde{E}$ be the total amount of the risky debt and the equity value of the company respectively after a new debt D (liability claim) is added. $\tilde{B}$, which is the sum of the existing debt and the new debt, has face value $B + Ie^{rT}$. Consider B_1 and E_1 the value of the existing risky debt and that of the equity after the liability claim D is added, respectively. B_1 has face value B. Let $\tilde{V}$ be the total value of the firm after the risky debt D is added. From (5.1), we have

$$\tilde{E}(t) = f_1(\tilde{V}(t), B + Ie^{rT}, t) = \tilde{V}(t)\Phi(\tilde{x}_1') - (B + Ie^{rT})e^{-r(T-t)}\Phi(\tilde{x}_2') \tag{5.4}$$

$$E_1(t) = f_1(\tilde{V}(t), B, t) = \tilde{V}(t)\Phi(\tilde{x}_1) - Be^{-r(T-t)}\Phi(\tilde{x}_2) \tag{5.5}$$

where $\tilde{V}$ is the value of the firm, r is the continuously compounded riskfree interest rate,

$$\Phi(\tilde{x}) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tilde{x}} e^{-\frac{1}{2}y^2} dy,$$

$$\tilde{x}_1 = \frac{\log \dfrac{\tilde{V}(t)}{B} + r(T-t) + \frac{1}{2}\int_t^T g^2(\tilde{V}(s-L_2))ds}{\sqrt{\int_t^T g^2(\tilde{V}(s-L_2))ds}}$$

$$\text{and} \quad \tilde{x}_2 = \tilde{x}_1 - \sqrt{\int_t^T g^2(V(s-L_2))ds};$$

and

$$\tilde{x}_1' = \frac{\log \dfrac{\tilde{V}(t)}{B+Ie^{rT}} + r(T-t) + \frac{1}{2}\int_t^T g^2(\tilde{V}(s-L_2))ds}{\sqrt{\int_t^T g^2(\tilde{V}(s-L_2))ds}}$$

$$\text{and} \quad \tilde{x}_2' = \tilde{x}_1 - \sqrt{\int_t^T g^2(V(s-L_2))ds}.$$

To find a formula for the liability claim D, we have to assume that there is no guarantee on it. Actually, if we consider the guarantee on D, it becomes riskless and the amount is known. We get

$$D = \tilde{B} - B_1$$

We will define the structure as

$$\tilde{B} = \tilde{V} - \tilde{E}$$

and

$$B_1 = \tilde{V} - E_1$$

Using (5.4), we shall compute $D(t)$ as

$$\begin{aligned} D(t) &= f(\tilde{V}(t), B, t) - f(\tilde{V}(t), B + Ie^{rT}, t) \\ &= \tilde{V}(t)\Phi(\tilde{x}_1) - Be^{-r(T-t)}\Phi(\tilde{x}_2) - \tilde{V}(t)\Phi(\tilde{x}_1') + (B + Ie^{rT})e^{-r(T-t)}\Phi(\tilde{x}_2') \\ &= \tilde{V}(t)\left[\Phi(\tilde{x}_1) - \Phi(\tilde{x}_1')\right] - Be^{-r(T-t)}\left[\Phi(\tilde{x}_2) - \Phi(\tilde{x}_2')\right] + Ie^{rt}\Phi(\tilde{x}_2'), \end{aligned}$$

for all $t \in [T-l, T]$.

Considering the loan is guaranteed, we need to add the guarantee cost $\tilde{G}$ to the debtholders' liability value D to obtain the bond proceeds' amount I which is the cost of the new project. If t_0 represents the time that the new debt is added then the value of the loan guarantee at any time $t_0 \leq t < T$ is given by

$$\tilde{G(t)} = I - D(t),$$

hence

$$G(t) = I(1 - e^{rt}\Phi(\tilde{x}_2^{'})) - \tilde{V}(t)\left[\Phi(\tilde{x}_1 - \Phi(\tilde{x}_1^{'})\right] + Be^{-r(T-t)}\left[\Phi(\tilde{x}_2) - \Phi(\tilde{x}_2^{'})\right], \; t_0 \leq t < T. \tag{5.6}$$

5.3 EQUALLY SENIOR LOAN GUARANTEES

Let $D = B_1 + D_1$ be the total amount of the liability claim after the new debt is added, where $B_1 = \dfrac{B}{B + Ie^{rT}}\left(\tilde{V} - \tilde{E}\right)$ and $D_1 = \dfrac{Ie^{rT}}{B + Ie^{rT}}\left(\tilde{V} - \tilde{E}\right)$ are the proportion of the total face value of the debt corresponding to the existing debtholders and the new debtholders respectively. Again using relation (5.4), the new debtholders' claim value at time $t \in [T - l, T]$ can be written as follow

$$\begin{aligned}
D_1(t) &= \frac{Ie^{rT}}{B + Ie^{rT}}\left(\tilde{V} - f(\tilde{V}(t), B + Ie^{rT}, t)\right) \\
&= \frac{Ie^{rT}}{B + Ie^{rT}}\left(\tilde{V}(t) - \tilde{V}(t)\Phi(\tilde{x}_1^{'}) + (B + Ie^{rT})e^{-r(T-t)}\Phi(\tilde{x}_2^{'})\right) \\
&= \frac{Ie^{rT}}{B + Ie^{rT}}\tilde{V}(t)\left(1 - \Phi(\tilde{x}_1^{'})\right) + Ie^{rt)}\Phi(\tilde{x}_2^{'}) \\
&= \frac{Ie^{rT}}{B + Ie^{rT}}\tilde{V}(t)\Phi(-\tilde{x}_1^{'}) + Ie^{rt)}\Phi(\tilde{x}_2^{'}) \quad t \in [T - l, T].
\end{aligned}$$

CHAPTER 6

SIMULATIONS

In this Chapter, we want to approximate numerically the solution of the SDDE below for the value of the company. This SDDE is similar to (3.1) presented in Chapter 3 (when $L_1 = L_2 = L$). We also test our model against real data for some companies.

$$\left.\begin{aligned} dV(t) = (\alpha V(t)V(t-L) - C)dt + g(V(t-L))V(t)dW(t),\ t \in [0,T] \\ V(t) = \varphi(t),\ t \in [-L,0], \end{aligned}\right\} \tag{6.1}$$

6.1 PRESENTATION OF THE DATA SET AND VOLATILITY ESTIMATION

The data on stock returns are from the Center for Research in Securty Prices (CRSP) database: http://www.crsp.com/ and those on debt amounts are from the Research Insight/Compustat database: http://www.compustat.com/. All data include firms that had valid data for all 20 years from 1991–2010 and including:

1. The risk free rate r is the average monthly yield on US T-Bills for that year (the same for all firms each year).

2. *Sdret*: the standard deviation of daily returns each year for each firm.

3. *Ndr*: the number of daily returns used to compute sdret for each firm each year (this is set to be at least 150).

4. B: the total book value of debt (in \$1,000,000).

5. V: the total value of the firm's assets (in \$1,000,000).

In fact the data set include all the parameters that we need to solve the stochastic equation (3.1) and the Random partial differential equation (3.3). All the simulations are performed

in Matlab 7.7. In most of our simulations, the data between 1991–2000 are used as memory data while those between 2001–2010 are used as the future data i.e. the data that we want our model to predict.

To estimate the volatility function g, we use the quadratic or linear interpolation of the memory part of data $Sdret$. We also estimate the volatility function g by using the splines interpolation of the memory part of the data $Sdret$.

As we only have yearly data set, we use also the interpolation to increase the amount of data if needed as the numerical schemes usually need small time steps (thus need more data) to ensure their stabilities.

6.2 NUMERICAL EVALUATION OF CORPORATE CLAIMS

Consider the SDDE (6.1) in the time interval $[0, T]$, where the value of T is 10 and the value of the delay L is 10. The time unit being the year. Indeed the values of α and C are time dependent, and therefore we consider those values as two time dependent functions, which are constant within the year interval. We solve numerically equation (6.1) by using the θ-semi implicit Euler-Maruyama scheme by

$$\begin{aligned} V_{n+1} &= V_n + \Delta T \left[\theta(\alpha_{n+1} V_{n+1} V_{n-m+1} - C_{n+1}) + (1-\theta)(\alpha_n V_n V_{n-m} - C_n)\right] \\ &\quad + g(V_{n-m}) V_n \Delta W_n \quad n = 1......., M, \;\; 0 \leq \theta \leq 1, \quad L = m\Delta T, \end{aligned} \tag{6.2}$$

where $\Delta T = T/M$ is the time step size, M the total number of time subdivisions, V_n is the approximation of $V(t_n)$, $t_n = n\Delta T$, $\alpha_n = \alpha(t_n)$, $C_n = C(t_n)$, and

$$\Delta W_n = W(t_{n+1}) - W(t_n)$$

are standard Brownian increments, independent identically distributed $\sqrt{\Delta T}\mathcal{N}(0, 1)$ random variables.

For $\theta = 0$ we have the classical Euler-Maruyama scheme which is less numerical stable than the semi implicit Euler-Maruyama with $\theta = 1$.

To ensure the convergence of the numerical equation (6.2) toward the unique solution of (6.1), the volatility function g needs to be globally Lipschitz, or locally Lipschitz and bounded [36]. These conditions are sufficient conditions for the convergence and not necessary conditions since the scheme can converge for some functions not verifying these conditions.

To approximate the expected value of the solution V, we use Monte Carlo to compute the mean of the numerical solution sample from (6.2). Monte Carlo can also be used to approximate any moment of V.

In the simulation, we test the delay model and Merton's model against real data for the following the companies:

A **Great Northern Iron Ore Pptys** (Figure 6.4(a) and Figure 6.4(b)),

B **Tor Minerals Intl Inc** (6.5(a) and 6.5(b)),

C **Magna International Inc** (6.1(a) and 6.1(b)),

D **Rentech Inc** (6.2(a) and 6.2(b)),

E **South Jersey Inds Inc** (6.3(a) and 6.3(b)).

In all our graphs, the time origin corresponds to the year (2000+1/2), the data before (2000+1/2) are memory data and we want to predict the data after (2000+1/2). We plot 400 samples of the numerical solution for our delay model and Merton model along with the means of the numerical solution (green curves), as the origin is year (2000+1/2), the part of the mean curves before the origin are just the curves of the data (V) in that interval. The curves of the data (V) as a function of time are in black (black thick curves).

In Figure 6.1, Figure 6.2 and Figure 6.3 we take $L = T = 10$, the graphs at the top (Figure 6.1(a), Figure 6.2(a) and Figure 6.3(a)) correspond to the delay model while

the graphs at the bottom (Figure 6.1(b), Figure 6.2(b) and Figure 6.3(b)) correspond to Merton's model. The function g is the quadratic interpolation of the standard deviation of daily returns $Sdret$ in the memory part while the volatility in the Merton's model is just the mean. Indeed if the volatility function g is the linear interpolation, the graphs have the same shape and we cannot establish the difference with the quadratic interpolation.

For firm C: Figure 6.1(a) shows the good prediction with reasonable standard deviation of the delay model while Figure 6.1(b) shows the early good prediction of the Merton model but the prediction has failed just after the year 2005. For firm D: Figure 6.2(a) shows the good prediction with reasonable standard deviation of the delay model while Figure 6.2(b) shows the early good prediction of the Merton model but starts moving out of the set of sample solutions after the year 2007, i.e. the prediction has failed just after the year 2007. For firm E: Figure 6.3(a) shows the early good prediction with reasonable standard deviation of the delay model but, although stays pretty close to the set of sample solutions, the prediction has failed after 2003 while Figure 6.3(b) shows the early good prediction of the Merton model but the prediction has completely failed just after the year 2002.

In Figure 6.4 and Figure 6.5 we take $L = 10,\ T = 5$, the graphs at the top (Figure 6.4(a) and Figure 6.5(a)) correspond to the delay model while the graphs at the bottom (Figure 6.4(b) and Figure 6.5(b)) correspond to Merton's model. The function g is the linear interpolation of the standard deviation of daily returns $Sdret$ in the memory part.

For firm A: 6.4(a) shows the good prediction with reasonable standard deviation of the delay model but the prediction has failed after 2004 while 6.4(b) shows the early good prediction of the Merton model but the prediction has failed just after the year 2002.

For firm B: 6.5(a) shows the good prediction with reasonable standard deviation of the delay model but the prediction has failed after 2005 while 6.5(b) shows the medium good prediction of the Merton model but the prediction has failed just after the year 2004.

6.3 NUMERICAL EVALUATION OF DEBT IN A LEVERED FIRM

We consider here the following random partial differential equation

$$\left.\begin{aligned} \frac{1}{2}g^2(V(t-L))v^2F_{vv}+(rv-C)F_v+F_t-rF+C_y=0, 0<t<T \\ F(v,T)=\max(v-B(T)), v>0, \end{aligned}\right\} \tag{6.3}$$

where C, C_y and r are time dependent functions. In our simulation we consider those values as time dependent functions, which are constant within the year interval as we have in our data set. Indeed to solve numerically this equation the domain of v needs to be truncated. Coupling (6.3) with final and boundary conditions therefore yields

$$\begin{cases} \frac{1}{2}\,g^2(V(t-L))v^2\,f_{vv}+(r(t)v-C(t))f_v+f_t-r(t)(f+C_y(t)=0, \\ f(v,T)=\max(v-B,0), \qquad v\in[0,V_{max}] \\ f(0,t)=0, \qquad\qquad t\in[T-L,T] \\ f(V_{max},t)=V_{max}-Be^{-\int_t^T r(s)ds}, \qquad\qquad t\in[T-L,T] \end{cases} \tag{6.4}$$

This equation is similar to Europeans call options prices, we can therefore take V_{max} three or four times B according to [58]. In our simulation we take $V_{max}=4B$, where B is the face value of the debt i.e. the amount that the firm must pay to the debtholders at the maturity date T (like the strike price for options prices). The equation (6.4) is a backward equation. To transform it to the forward one, we use the transformation $\tau=T-t$, and

the corresponding equation is given by

$$\begin{cases} \dfrac{1}{2}\, g^2(V(T-\tau-L))v^2\, f_{vv} + (r(\tau)v - C(\tau))f_v - r(\tau)f + C_y(\tau) = f_\tau, \\ f(v,0) = \max(v-B,0), \qquad v \in [0, V_{max}] \\ f(0,\tau) = 0, \qquad\qquad \tau \in [0,L] \\ f(V_{max},\tau) = V_{max} - Be^{-\int_{T-\tau}^{T} r(s)ds}, \qquad\qquad \tau \in [0,L] \end{cases} \tag{6.5}$$

To simulate the convection term (the term with f_v) and avoid numerical instabilities, let us put this term in the so called conservation form. In fact

$$(r(\tau)v - C(\tau))f_v = ((r(\tau)v - C(\tau))f)_v - fr(\tau).$$

Using this relation, equation (6.5) becomes

$$\begin{cases} \dfrac{1}{2}\, g^2(V(T-\tau-L))v^2\, f_{vv} + ((r(\tau)v - C(\tau))f)_v - 2r(\tau)f + C_y(\tau) = f_\tau, \\ f(v,0) = \max(v-B,0), \qquad v \in [0, V_{max}] \\ f(0,\tau) = 0, \qquad\qquad \tau \in [0,L] \\ f(V_{max},\tau) = V_{max} - Be^{-\int_{T-\tau}^{T} r(s)ds}, \qquad\qquad \tau \in [0,L]. \end{cases} \tag{6.6}$$

To solve equation (6.6) two cases can be considered:

1. The case where $T - \tau - L \leq 0, \quad \forall \tau \in [0, L]$, then $T \leq L$.
2. The case where $T > L$.

For the first case ($T \leq L$) the random partial differential equation (6.6) becomes a deterministic partial differential equation since $V(t) = \varphi(t) \quad$ for $t \in [-L, 0]$ as given in (6.1).

For the second case ($T > L$), to solve equation (6.6) the following step should be followed

1. Solve the stochastic equation (6.1) to have a sample of the numerical solution of V as we did in the previous section.
2. Use the numerical sample solution of V to build the diffusion coefficient (the coefficient of f_{vv}) in the random PDE (6.6), which therefore becomes a deterministic PDE for this fixed numerical sample of V.
3. Solve the deterministic PDE for the fixed numerical sample of V.
4. Repeat step 1, step 2 and step 3, M times (relatively large) and use the Monte Carlo technique to estimate the expectation value of f and also any moment of the stochastic process f if needed.

As the two cases require the solution of the deterministic PDE, in the sequel we will consider the first case ($T \leq L$), and the corresponding deterministic PDE is given by

$$\begin{cases} \frac{1}{2}\, g^2(\varphi(T-\tau-L))v^2 f_{vv} + ((r(\tau)v - C(\tau))f)_v - 2r(\tau)f + C_y(\tau) = f_\tau, \\ f(v,0) = \max(v-B,0), \qquad v \in [0, V_{max}] \\ f(0,\tau) = 0, \qquad \tau \in [0,L] \\ f(V_{max},\tau) = V_{max} - Be^{-\int_{T-\tau}^{T} r(s)ds}, \qquad \tau \in [0,L] \end{cases} \tag{6.7}$$

For the discretization in the direction of v we used the combined finite difference–finite volume method. The interval $[0, V_{max}]$ is subdivised into N parts that we assume equal without loss of generality. As in center finite volume method, we approximate f at the center of each interval. The diffusion part of the equation is approximated using the finite difference while the convection term is approximated using the standard upwinding usual

use in porous media flow problems [54, 55, 17, 18]. Let

$$v_i = (2i-1)h/2, \quad h = \frac{V_{max}}{N}, \quad i = 1, 2, ..., N$$

be the center of each subdivision. In the next approximations we denote

$$f_i(\tau) \approx f(v_i, \tau).$$

We approximate the diffusion term at each center by

$$\begin{aligned}
\frac{1}{2}\, g^2(\varphi(T-\tau-L))v_i^2\, f_{vv}(v_i) &\approx \frac{1}{2h^2}\, g^2(\varphi(T-\tau-L))v_i^2\, (f_{i+1}(\tau) - 2f_i(\tau) + f_{i-1}(\tau)) \\
& \qquad\qquad\qquad\qquad i = 2, ..., N-1. \\
\frac{1}{2}\, g^2(\varphi(T-\tau-L))v_1^2\, f_{vv}(v_1) &\approx \frac{2}{3h}\, g^2(\varphi(T-\tau-L))v_1^2 \left(\frac{f_2(\tau) - f_1(\tau)}{h} - 2\frac{f_1(\tau)}{h} \right) \\
\frac{1}{2}\, g^2(\varphi(T-\tau-L))v_1^2\, f_{vv}(v_N) &\approx \frac{2}{3h}\, g^2(\varphi(T-\tau-L))v_N^2 \times \\
& \left(\frac{f(V_{max}, \tau) - f_N(\tau)}{h/2} - \frac{f_N(\tau) - f_{N-1}(\tau)}{h} \right)
\end{aligned}$$

This approximation is similar to the one in [11], with central difference on non uniform grid. We approximate the convection term using the standard upwinding technique as following

$$((r(\tau)v - C(\tau))f)_v\,(v_i) \approx \frac{(r(\tau)v_{i+1/2} - C(\tau))f_i^+(\tau) - (r(\tau)v_{i-1/2} - C(\tau))f_{i-1}^+(\tau)}{h} \tag{6.8}$$

where

$$
\begin{aligned}
f_i^+(\tau) &= \begin{cases} f_i(\tau) & \text{if} \quad r(\tau)v_{i+1/2} - C(\tau) \geqslant 0 \\ f_{i-1}(\tau) & \text{if} \quad r(\tau)v_{i+1/2} - C(\tau) < 0 \end{cases} && (6.9) \\
v_{i+1/2} &= v_i + h/2, \quad v_{i-1/2} = v_i - h/2 = v_{i-1} + h/2. && (6.10)
\end{aligned}
$$

Reorganizing all approximations leads to the following initial value problem

$$
\begin{cases} \dfrac{d\mathbf{f}}{d\tau} = \mathbf{A}(\tau)\mathbf{f} + \mathbf{b}(\tau), \qquad \tau \in [0, L] \\ \mathbf{f}(0) = (\max(v_1 - B, 0), ..., \max(v_N - B, 0))^T \end{cases} \tag{6.11}
$$

where $\mathbf{A}(\tau)$ is a tridiagonal matrix and

$$
\mathbf{f}(\tau) = (f_i(\tau))_{1 \leq i \leq N}, \mathbf{b}(\tau) = (C_y(\tau)) + \mathbf{k}(\tau) \tag{6.12}
$$

where $\mathbf{k}$ is the contribution from boundary conditions.

The function $x \mapsto \max(x, 0)$ is not smooth, it is important to approximate it by a smooth function. The approximation in [11] is a fourth-order smooth function denoted π_ϵ and defined by

$$
\pi_\epsilon(x) = \begin{cases} x & \text{if} \quad x \geqslant \epsilon \\ c_0 + c_1 x + ... + c_9 x^9 & \text{if} \quad -\epsilon < x < \epsilon \\ 0 & \text{if} \quad x \leq -\epsilon \end{cases} \tag{6.13}
$$

where $0 < \epsilon \ll 1$ is the transition parameter and

$$\begin{aligned} c_0 = \frac{35}{256}\epsilon, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{35}{64\epsilon}, \quad c_4 = -\frac{35}{128\epsilon^3}, \\ c_6 = \frac{7}{64\epsilon^5}, \; c_8 = -\frac{5}{256\epsilon^7}, \; c_3 = c_5 = c_7 = c_9 = 0. \end{aligned}$$

This approximation allow us to write

$$\mathbf{f}(0) = \pi_\epsilon(\mathbf{v} - B), \quad \mathbf{v} = (v_i)_{1 \le i \le N}\,. \tag{6.14}$$

Let us introduce the time stepping discretization for the ODE (6.11) based on exponential integrators. Classical numerical methods use often Implicit Euler scheme, Crank–Nicolson scheme [15]. Using variation of parameters the exact solution of (6.11) (see [11, 54, 55])is given by

$$\begin{aligned} \mathbf{f}(\tau_n + \Delta\tau) &= e^{\int_{\tau_n}^{\tau_n+\Delta\tau} \mathbf{A}(s)ds}\left[\mathbf{f}(\tau_n) + \int_{\tau_n}^{\tau_n+\Delta\tau} e^{-\int_{\tau_n}^{s}\mathbf{A}(y)dy}\mathbf{b}(s)ds\right] && (6.15) \\ & \tau_n = n\,\Delta\tau, \qquad n = 0, ..., M, \qquad \Delta\tau > 0. && (6.16) \end{aligned}$$

In order to construct the numerical schemes, approximations are needed. The first approximations are:

$$\int_{t_n}^{t_n+\Delta\tau} \mathbf{A}(s)ds \approx \Delta\tau \mathbf{A}(\tau_n) \qquad \int_{\tau_n}^{s} \mathbf{A}(y)dy \approx (s - \tau_n)\mathbf{A}(\tau_n) \tag{6.17}$$

Using these approximations we therefore have

$$\mathbf{f}(\tau_n + \Delta\tau) \approx e^{\Delta\tau \mathbf{A}(\tau_n)}\left[\mathbf{f}(\tau_n) + \int_{\tau_n}^{\tau_n+\Delta\tau} e^{-(s-\tau_n)\mathbf{A}(\tau_n)}\mathbf{b}(s)ds\right] \tag{6.18}$$

The simple scheme called Exponential Differential scheme of order 1 (ETD1) is obtained by approximating the integral in the same way as (6.17) and is given by

$$\mathbf{f}_{n+1} = \mathbf{f}_n + (\Delta\tau \mathbf{A}(\tau_n))^{-1} [\mathbf{A}(\tau_n)\mathbf{f}_n + \mathbf{b}(\tau_n)] . \tag{6.19}$$

A second order scheme is given in [11].

Following the work in [47, Lemma 4.1], if the function $\mathbf{b}$ can be well approximated by the polynomial of degree p (which is the case here since we have the exponential decay at the boundary $v = V_{max}$), a high order scheme applying in the approximated version of the ODE (6.11) is given by

$$\mathbf{f}_{n+1} = \varphi_0(\Delta\tau \mathbf{A}(\tau_n))\mathbf{f}_n + \sum_{j=0}^{p-1}\sum_{l=0}^{j} \frac{\tau_n^{j-l}}{(j-l)!} \Delta\tau^{l+1} \varphi_{l+1}(\Delta\tau \mathbf{A}(\tau_n))\mathbf{b}_{j+1}, \tag{6.20}$$

where

$$\begin{aligned} \mathbf{b}(\tau) &\approx \sum_{j=0}^{p-1} \frac{\tau^j}{j!} \mathbf{b}_{j+1}, \\ \varphi_0(x) &= e^x, \quad \varphi_l(x) = x\varphi_{l+1}(x) + \frac{1}{l!}, \quad l = 0, 1, 2, ... \end{aligned}$$

All schemes here can be implemented using Krylov subspace technique in the computation of the exponential functions by Matlab functions 'expmvp.m' or 'phipm.m' from [47, 48]. The Krylov subspace dimension we use is $m = 10$ and the tolerance used in the computation of the expomential functions φ_i is $tol = 1e - 6$.

The companies we use in our simulation are those with C_y in their data set, namely:

A **Great Northern Iron Ore Pptys** (6.6(a) and 6.6(b)),

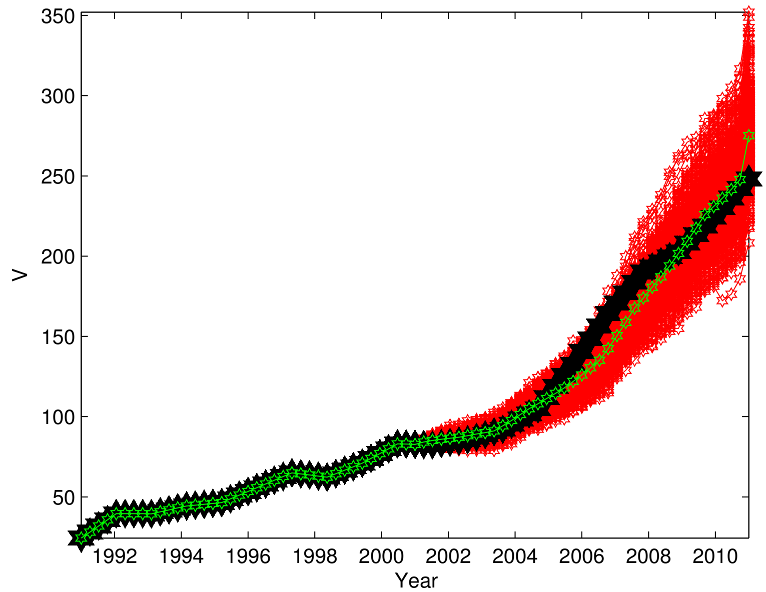
B **Tor Minerals Intl Inc** (6.7(a) and 6.7(b)),

C **Magna International Inc** (6.6(a) and 6.6(b)).

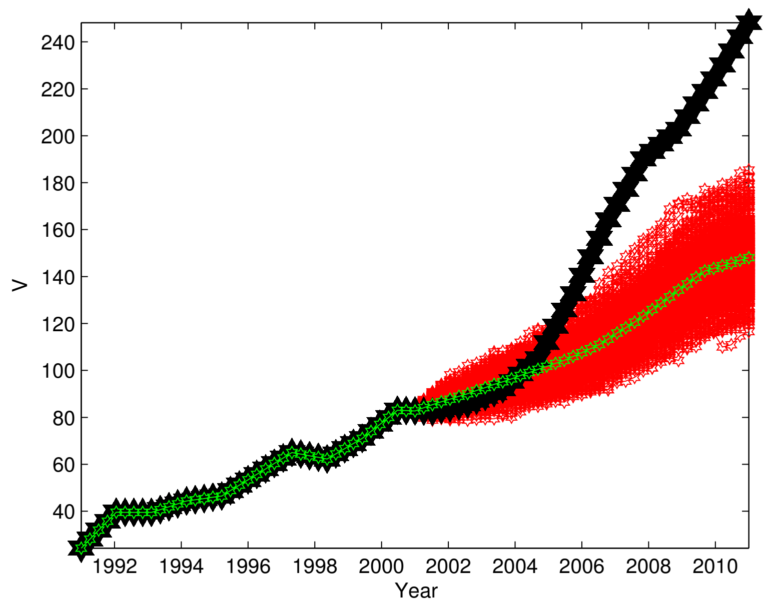
The goal here is to show the difference between delay model and Merton's model. We show the surface plot of the debt in function of the time and V for our delay model (6.6(a), 6.7(a) and 6.8(a)) along with the surface of the absolute value of the difference of the debt between the delay model and Merton's model (6.6(b), 6.7(b) and 6.8(b)). We take $T = L = 10$ and the function g to be the quadratic interpolation of the standard deviation of daily returns $Sdret$ in the memory part. For the Merton model, the volatility used is the mean of the memory part of $Sdret$.

For the company A (6.8(a) and 6.8(b)), the difference between the debt in the two model is high at early time and small at the final time. The difference between the two models is well noticeable.

For the company B (6.7(a) and 6.7(b)), the difference between the two model is noticeable at the final time while for company C (6.6(a) and 6.6(b)) the difference between the two models is noticeable at the early time. In summary, the delay model gives better prediction in all the testing.

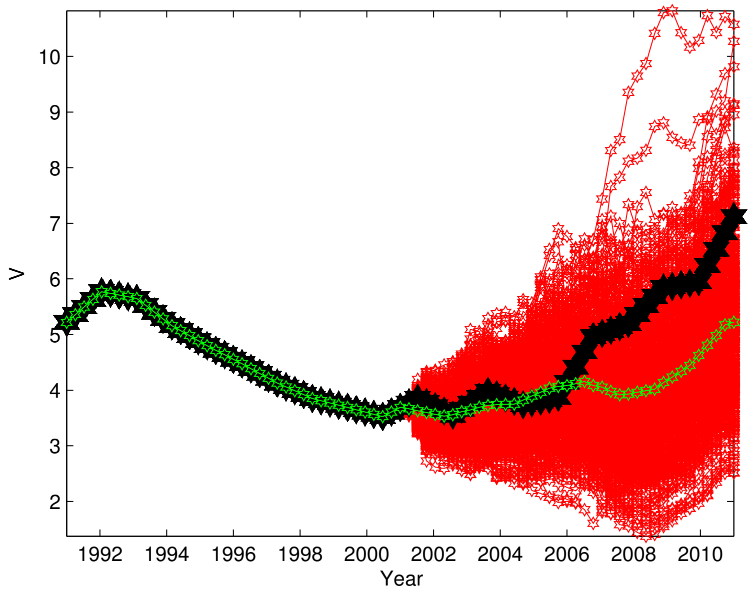


(a)

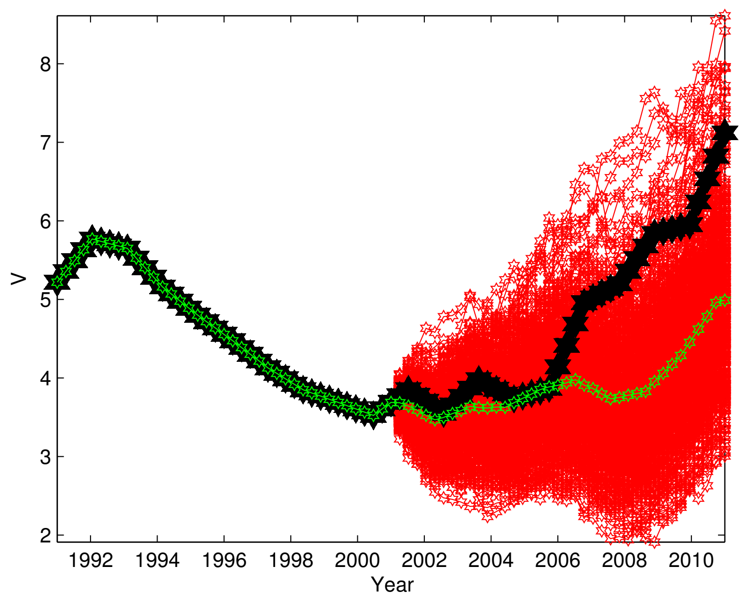


(b)

Figure 6.1: Firm C (Magna International Inc): The graph at the top (a) corresponds to the delay model while the graph at the bottom (b) corresponds to Merton's model. We take $T = L = 10$ and the function g is the linear interpolation of the standard deviation of daily returns $Sdret$ in the memory part. We plot 400 samples of the numerical solution along with the expectation (the means) of the numerical solution (green curves). The curves of the data V as a function of time are in black (black thick curves).

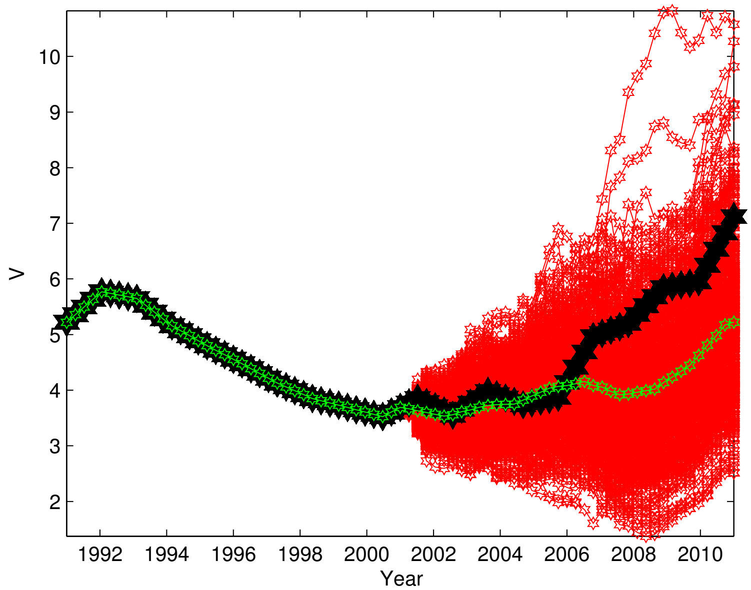


(a)

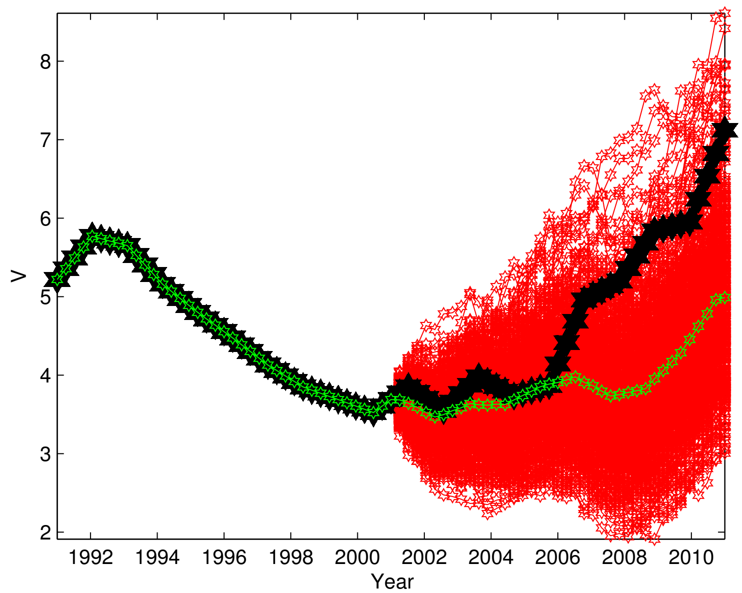


(b)

Figure 6.2: Firm D: The graph at the top (a) corresponds to the delay model while the graph at the bottom (b) corresponds to Merton's model. We take $T = L = 10$ and the function g is the linear interpolation of the standard deviation of daily returns $Sdret$ in the memory part. We plot 400 samples of the numerical solution along with the expectation (the means) of the numerical solution (green curves). The curves of the data V as a function of time are in black (black thick curves).

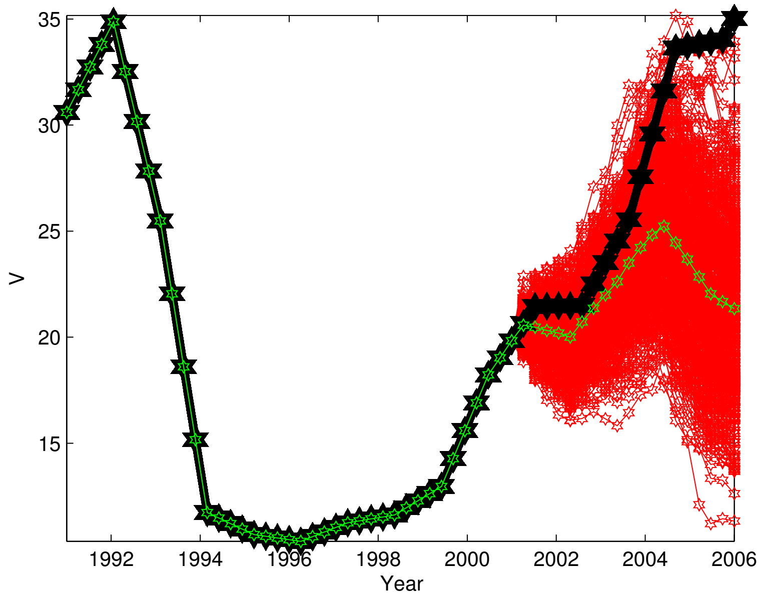


(a)

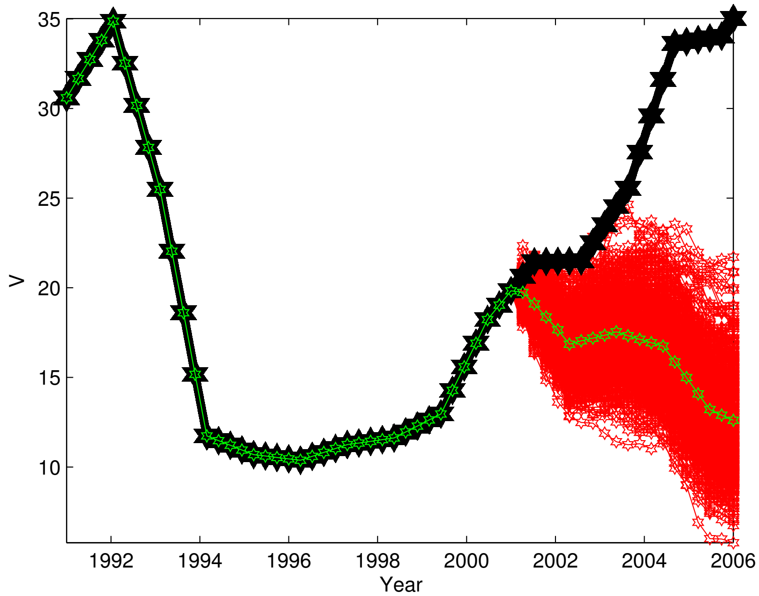


(b)

Figure 6.3: Firm E: The graph at the top (a) corresponds to the delay model while the graph at the bottom (b) corresponds to Merton's model. We take $T = L = 10$ and the function g is the linear interpolation of the standard deviation of daily returns $Sdret$ in the memory part. We plot 400 samples of the numerical solution along with the expectation (the means) of the numerical solution (green curves). The curves of the data V as a function of time are in black (black thick curves).

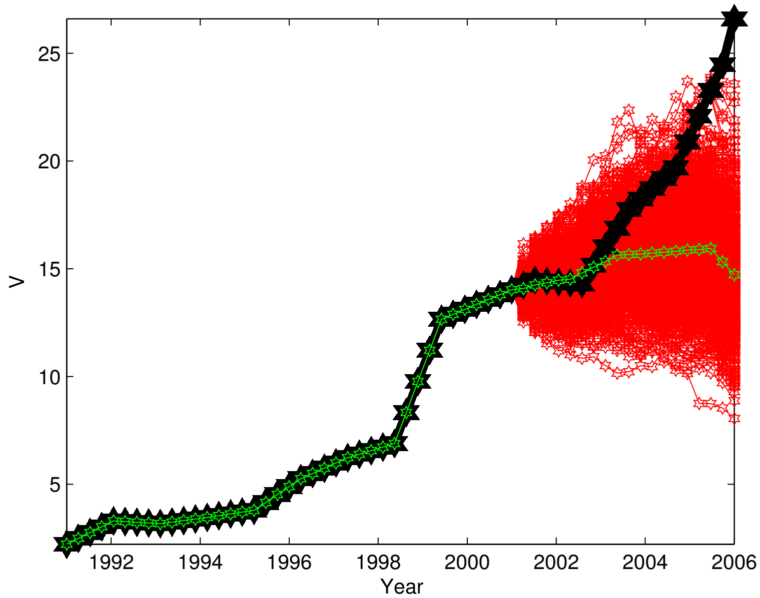


(a)

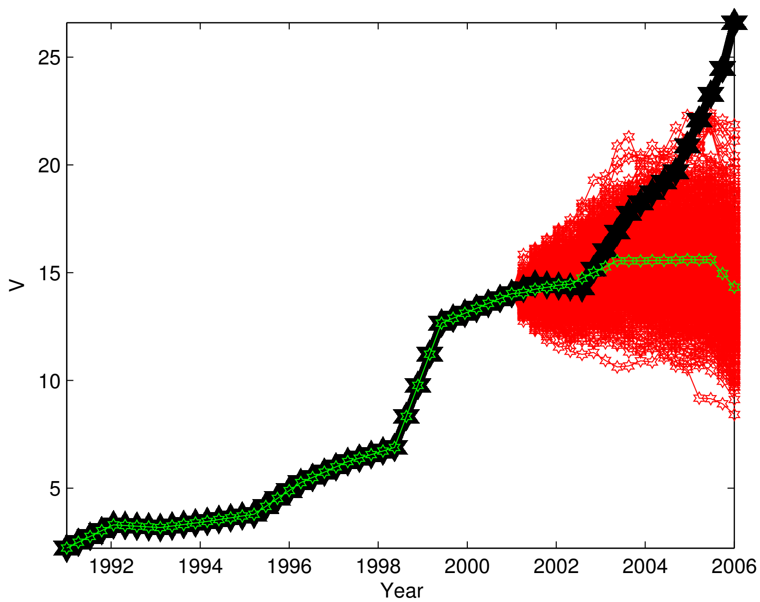


(b)

Figure 6.4: Firm A: The graph at the top (a) corresponds to the delay model while the graph at the bottom (b) corresponds to Merton's model. We take $T = 5$ $L = 10$ and the function g is the linear interpolation of the standard deviation of daily returns $Sdret$ in the memory part. We plot 400 samples of the numerical solution along with the expectation (the means) of the numerical solution (green curves). The curves of the data V as a function of time are in black (black thick curves).

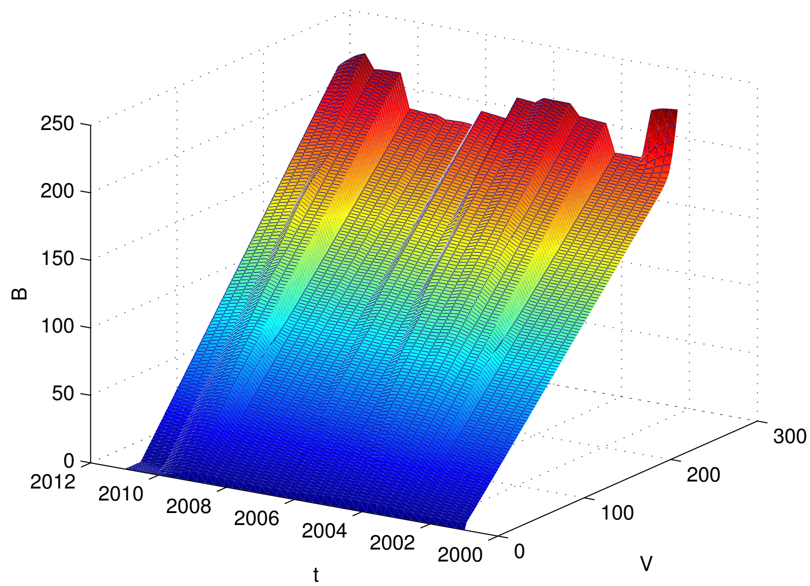


(a)

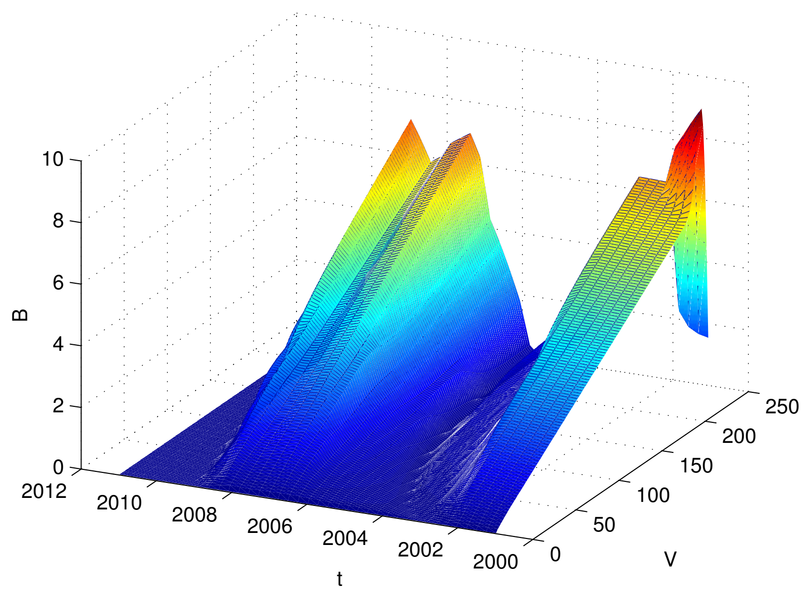


(b)

Figure 6.5: Firm B: The graph at the top (a) corresponds to the delay model while the graph at the bottom (b) corresponds to Merton's model. We take $T = 5$ $L = 10$ and the function g is the linear interpolation of the standard deviation of daily returns $Sdret$ in the memory part. We plot 400 samples of the numerical solution along with the expectation (the means) of the numerical solution (green curves). The curves of the data V as a function of time are in black (black thick curves).

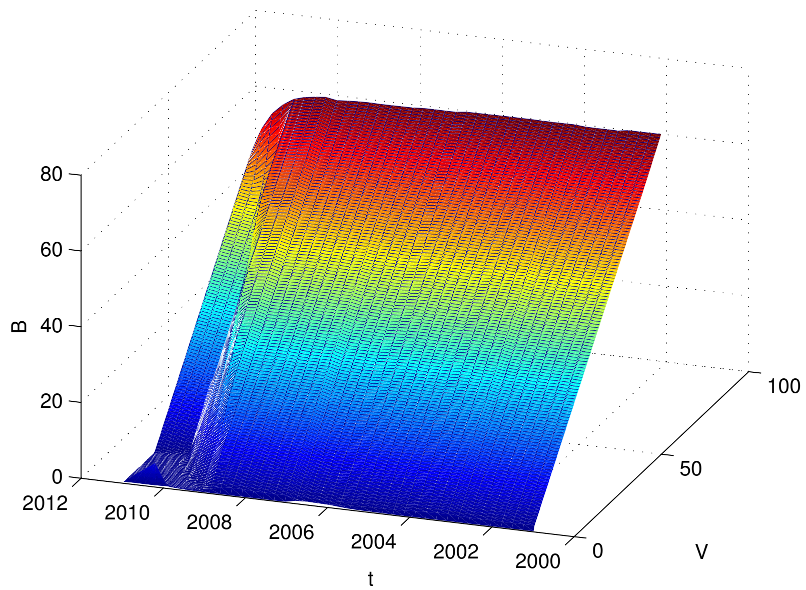


(a)

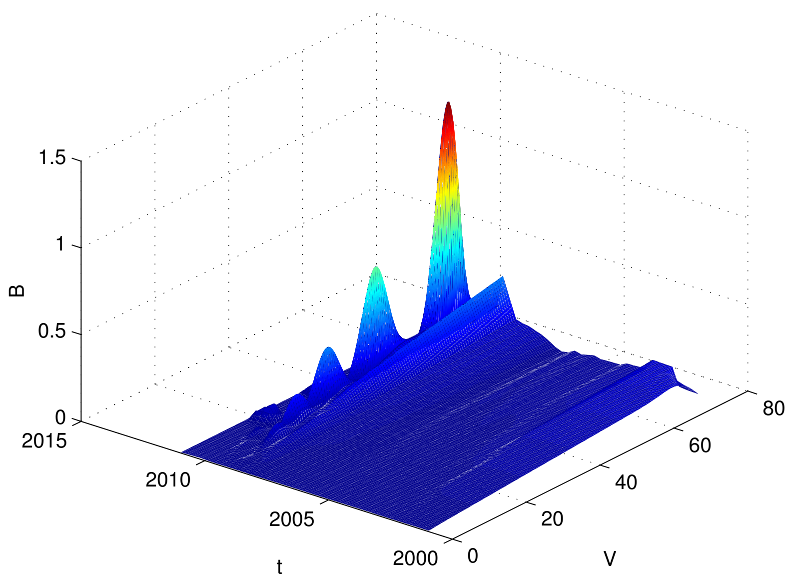


(b)

Figure 6.6: For company A, the surface graph at the top (a) corresponds to debt surface for the delay model while the graph at the bottom (b) corresponds to the absolute value of the difference between the debt with the delay model and the debt with Merton's model. We take $T = L = 10$ and the function g is the quadratic interpolation of the standard deviation of daily returns $Sdret$ in the memory part. For the Merton's model the volatility used is the mean of the memory part of $Sdret$.

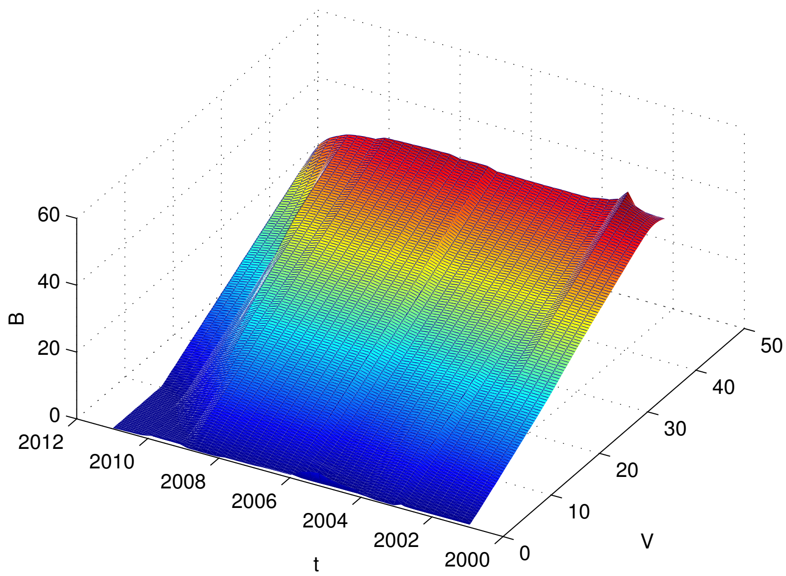


(a)

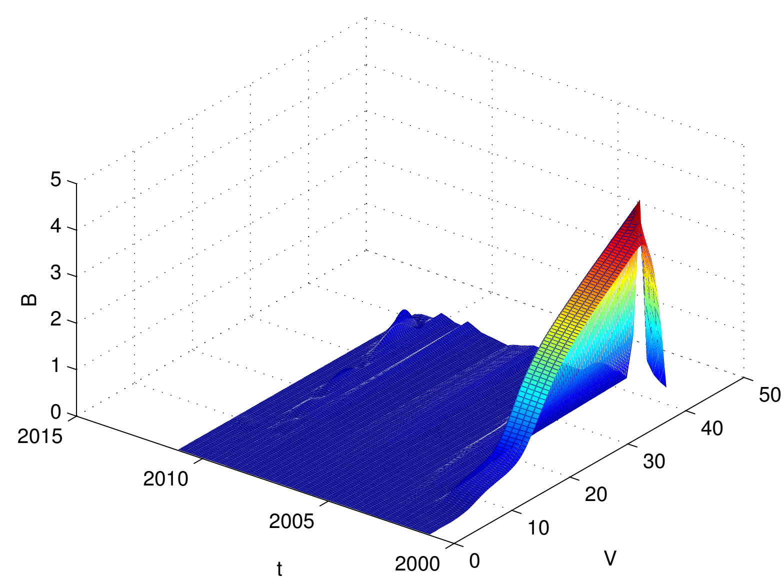


(b)

Figure 6.7: For company B, the surface graph at the top (a) corresponds to debt surface for the delay model while the graph at the bottom (b) corresponds to the absolute value of the difference between the debt with the delay model and the debt with Merton's model. We take $T = L = 10$ and the function g is the quadratic interpolation of the standard deviation of daily returns $Sdret$ in the memory part. For the Merton's model the volatility used is the mean of the memory part of $Sdret$.



(a)



(b)

Figure 6.8: For company C, the surface graph at the top (a) corresponds to debt surface for the delay model while the graph at the bottom (b) corresponds to the absolute value of the difference between the debt with the delay model and the debt with Merton's model. We take $T = L = 10$ and the function g is the quadratic interpolation of the standard deviation of daily returns $Sdret$ in the memory part. For the Merton's model the volatility used is the mean of the memory part of $Sdret$.

CONCLUSION

We suggest a delay model for pricing corporate liabilities. Using replication and self-financed strategy we have derived a RPDE which includes nonconstant volatility and time delay in the firm value. With this model which is an extension of the work of Merton [38], we demonstrated again that, any claim whose value depends on firm value and time is a solution to our RPDE under some suitable boundary and terminal conditions. We the RPDE and provide a formula for the price of equity value, debt value and loan guarantees considering a delay on the firm value. Our expectations on improvement on existing models were met.

Another aspect of our work was to analyze the risk structure of the firm. From the studies of the monotonicity we found a necessary condition for the yield spread to be a valid measure of risk. One can conclude that if we fix the maturity date, the risk premium is a valid measure of risk. Unfortunately, in the presence of different maturities, the risk premium cannot in general be considered as a valid measure of risk. Since our work only involves one single portfolio and a single maturity, both the standard deviation on the debt and the risk premium will be a valid measure of risk.

We apply our method to pricing risky debt and loan guarantees. We obtain numerical approximations for the company value using the implicit Euler-Maruyama scheme. We also use a combination of finite difference and finite volume method to approximate the RPDE. We use our approximation scheme to test our model against real market data. Finally, we compare the delay model on the company value with Merton's model. It turned out that our model always gives better prediction of the future price of the value of the company. We further looked at the difference between the two models for the debt value. This model can be used to obtain a formula for many other financial claims.

The immediate plan is to simulate the debt value for the same companies previously simulated for the company value. This will confirm the prediction indicated by the firm

value simulations. I also intend to investigate an application of my model to an actuarial cost of insurance.

In my future research, I will continue to develop better models for real world applications. I intend to investigate an application of optimal control on my model following Oksendal's approach in his paper titled "Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations" (joint work with Agnes Sulem and Tusheng Zhang). .

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Publications:

1. E. Kemajou et al., The Isbell-hull of a di-space, "Topology and its Applications", 10.1016/j.topol.2011.02.016, article in press, http://www.sciencedirect.com/science/journal/01668641, 2012.

In progress:

1. E. Kemajou et al, A Stochastic Delay Model for Corporate Claims: An Implicit Euler-Maruyama Scheme and data testing,
2. E. Kemajou et al, Pricing Debt and Loan Guarantees.