An Invariant of Basic Sets of Smale Flows

Michael C. Sullivan
Southern Illinois University Carbondale, msulliva@math.siu.edu

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AN IN Variant OF BASIC SETS OF SMALE FLOWs

MICHAEL C. SULLIVAN

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ABSTRACT. We consider one dimensional flows which arise as hyperbolic invariant sets of a smooth flow on a manifold. Included in our data is the twisting in the local stable and unstable manifolds. A topological invariant sensitive to this twisting is obtained.

1. Introduction

One dimensional flows have been classified by the work of R. Bowen & J. Franks [4], W. Parry & D. Sullivan [9], and Franks [7]. Here, we consider the problem when some additional information is included. For a one-dimensional flow which is a hyperbolic invariant set of a flow on a manifold, it is usual to study the first return map on a Markov partition of a cross section. It is possible to assign an orientation to the elements of the partition. Then the first return map will either reverse or preserve orientation on each of the partition elements. We define a twist matrix $A(t)$ by incorporating orientation information into the incidence matrix of a Markov partition. Then the computation $\det(I - A(t)) \mod (t^2 = 1)$ gives our invariant. This result was conjectured in [11]. The invariant of [9], now known as the Parry-Sullivan invariant, can be recovered by setting $t = 1$. We shall call the invariant of this paper the full Parry-Sullivan invariant.

This paper is organized as follows. Section 2 gives basic definitions and results of the theory of Smale flows and maps. Section 3 develops the notion of a subshift of finite type with parity. We prove a modified version of the theorem of Williams that classifies non-negative integral matrices according the topological conjugacy classes of their associated subshifts of finite type. This is used in Section 4 in the proof our main result, Theorem 4.1. In Section 5 we give an application for Smale flows in 3 dimensions. Here basic sets can be modeled with templates, which are branched 2-manifolds with semi-flows whose periodic orbits are ambient isotopic to those of the original flow.

2. Basic Definitions and Background

If the chain-recurrent set $\mathcal{R}$ of flow has a hyperbolic structure then a result of Smale’s shows that $\mathcal{R}$ is the union of a finite collection of disjoint invariant compact sets, called basic sets, each of which contains a dense orbit [10]. Bowen showed that one-dimensional basic sets are homeomorphic to suspensions of subshifts of finite type [3], defined below.

Let $\mathcal{S}$ be a finite set of symbols and $\rightarrow$ be a relation on $\mathcal{S}$. Let $\Sigma = \{ s \in \prod_{i=1}^{\infty} \mathcal{S} | s_i \rightarrow s_{i+1}, \forall i \}$. We often say $\Sigma$ is the set of allowed sequences. Let $\sigma: \Sigma \rightarrow \Sigma$ be the rightward
shift map, \( \sigma(s) = s' \), where \( s'_i = s_{i-1} \). Then \((\sigma, \Sigma)\) is a subshift of finite type. When \( \Sigma \) is given the topology induced by the product topology it is a Cantor set.

Given an \( n \times n \) matrix \( A \) of nonnegative integers one can construct a subshift of finite type through the edge shift formulation, as described in Section 3. If the matrix entries are all zeros and ones then one can also define a subshift of finite type through vertex shifts. Let

\[
\Sigma_A = \{ x \in \prod_{k=0}^{\infty} \{1, \ldots, n\} \mid A_k x_{k+1} = 1 \text{ for all } k \}
\]

and define the right shift map \( \sigma : \Sigma_A \to \Sigma_A \) by \( \sigma(x) = y \) where \( y_k = x_{k-1} \). Given a matrix of zeros and ones the edge and vertex subshifts associated to it are topologically conjugate.

To define the suspension flow of \( \sigma \) let \( X_A = \Sigma_A \times [0, 1]/\sim \), where \( \sim \) identifies \((x, 1)\) with \((\sigma(x), 0)\) and the flow \( \phi_t \) on \( X_A \) is given by

\[
\phi_t(x, s) = (x, s + t)
\]

for \( s + t \in [0, 1) \) and for other \( t \) by using the identification as needed.

Two matrices are said to be flow equivalent if their induced subshifts of finite type give rise to topologically equivalent suspension flows. A matrix \( A \) is irreducible if for each \((i, j)\) there is a power \( n \) such that the \((i, j)\) entry of \( A^n \) is nonzero. In terms of the corresponding subshift and suspension, irreducibility is equivalent to the existence of a dense orbit and so we are dealing with a single basic set. The suspension flow of a permutation matrix consists of a single closed orbit. Permutation matrices are thus said to form the trivial flow equivalence class.

**Theorem 2.1 (Franks).** Suppose that \( A \) and \( B \) are non-negative irreducible integer matrices neither of which is in the trivial flow equivalence class. The matrices \( A \) and \( B \) are flow equivalent if and only if

\[
\det(I_n - A) = \det(I_m - B)
\]

and

\[
\frac{Z^m}{(I_n - A)Z^n} \cong \frac{Z^m}{(I_m - B)Z^m},
\]

where \( n \) and \( m \) are the sizes of \( A \) and \( B \) respectively and \( I_n \) and \( I_m \) are identity matrices.

The main tool in the proof of Theorem 2.1 is Theorem 2.2 below, a generalization of which we shall need later. First we define some more terms.

Two nonnegative square integer matrices \( A \) and \( B \) are strong shift equivalent if there exist nonnegative square integer matrices \( A = A_1, A_2, \ldots, A_{k+1} = B \) and nonnegative integer matrices \( R_1, S_1, R_2, S_2, \ldots, R_k, S_k \) such that \( A_i = R_i S_i \) and \( A_{i+1} = S_i R_i \) for \( i = 1, \ldots, k \).

**Theorem 2.2 (Williams).** Suppose \( A \) and \( B \) are nonnegative square integer matrices and \( \sigma_A \) and \( \sigma_B \) are the corresponding subshifts of finite type. Then \( \sigma_A \) is topologically conjugate to \( \sigma_B \) if and only if \( A \) is strong shift equivalent to \( B \).

The gist of this result is as follows. The matrices \( A \) and \( B \) can be thought of as incidence matrices for maps on a finite open-closed partition of the Cantor sets \( \Sigma_A \) and \( \Sigma_B \). The equivalence relation is generated by these operations: relabeling the partition elements, refining a partition element into two open-closed sets, and combining two partition elements into one. These can each be realized by the matrix move that generates strong shift equivalence. See [12] or Appendix A of [5].
3. Subshifts with Parity

The purpose of this section is to prove the following analog of Theorem 2.2.

**Theorem 3.1.** Parity matrices are parity-wise topologically conjugate if and only if they are strong shift equivalent.

We now present the relevant definitions. A **subshift of finite type with parity**, or just a **shift with parity** for short, is a subshift of finite type together with a **parity map** on the symbol set, $p : S \to \{-1, 1\}$. We define a function $p : \Sigma \to \{-1, 1\}$ by $p(s) = p(s_0)$. Let $\mathcal{C} = \{s \in \Sigma | s \text{ is periodic} \} / \sigma$, be called the set of cycles. Define $p : \mathcal{C} \to \{-1, 1\}$ by $p(c) = p(s_0)p(s_1) \cdots p(s_{n-1})$, where $s = s_0s_1 \cdots s_{n-1}$ is a periodic element of $\Sigma$ of least period $n$ that represents the class $c$.

**Definition 3.2.** Let $(\sigma, p)$ and $(\sigma', p')$ be shifts with parity. Then they are parity-wise conjugate if there is a homeomorphism $h : \Sigma \to \Sigma'$ such that

$$h \circ \sigma = \sigma' \circ h$$

and

$$p = p' \circ h.$$  

It follows that parity is preserved under the induced homeomorphism on the cycle sets. If only the first equation holds, $\sigma$ and $\sigma'$ are topologically conjugate.

It is technically necessary to define what it means to iterate shifts with parity. If $(\sigma : \Sigma \to \Sigma, p : S \to \{-1, 1\})$ is a shift with parity then its $k$th iterate $(\sigma^{<k>} : \Sigma^{<k>} \to \Sigma^{<k>}, p^{<k>} : S^{<k>} \to \{-1, 1\})$, is defined as follows. Let $S^{<k>} = \{(s_0, \ldots, s_{k-1}) \in S^k | s_i \mapsto s_{i+1}, i = 0, \ldots, k - 2\}$. We also let $(s_0, \ldots, s_{k-1})^{<\leq 2>} = (s_0', \ldots, s_{k-1}')$ mean $s_{k-1} \mapsto s_0'$. These give a subshift of finite type:

$$\sigma^{<k>} : \Sigma^{<k>} \to \Sigma^{<k>}.$$  

Let $p^{<k>}((s_0, \ldots, s_{k-1})) = p(s_0) \cdots p(s_{k-1})$. We can extend $p^{<k>}$ to cycles in $\Sigma^{<k>}$. 

A subshift with parity can be specified by giving a labeled directed graph or a matrix whose elements take the form $a + bt$, where $a$ and $b$ are nonnegative integers.

Given a graph $G$ with edges labeled $\pm 1$ and vertices numbered 1 through $n$, we define an $n \times n$ matrix $M(t)$ by $M_{ij} = a_{ij} + b_{ij}t$ with $a_{ij}$ the number of edges from vertex $i$ to $j$ with label $+1$, and $b_{ij}$ the number of such edges labeled $-1$. Matrices of this form will be called **parity matrices**. Given a parity matrix we can construct a labeled graph.

We go from an $n \times n$ parity matrix to the shift with parity as follows. Let $S$ be a set of $\sum_{i,j} a_{ij} + b_{ij}$ symbols, think of them as edges on the graph. We partition $S$ into $2n^2$ disjoint, possibly empty, subsets. Call them $A_{ij}$ and $B_{ij}$ for $i$ and $j$ equal to 1 through $n$, with $|A_{ij}| = a_{ij}$ and $|B_{ij}| = b_{ij}$. Also, let $S_{ij} = A_{ij} \cup B_{ij}$. Now we define the relation, $\mapsto$ on $S$. Let $s \mapsto s'$, read $s$ can be followed by $s'$, mean $s \in S_{ik}$ and $s' \in S_{kj}$ for some $i, j$ and $k$. The set of allowed sequences and the shift map are given in the usual way. The parity map is defined by

$$p(s) = \begin{cases} 1 & \text{if } s \in A_{ij} \text{ for some } i \text{ and } j \\ -1 & \text{if } s \in B_{ij} \text{ for some } i \text{ and } j \end{cases}$$  

**Remark.** If a parity matrix's entries are all zeros, ones and $t$'s, and we then set $t$ equal to minus one, the parity matrix becomes the **structure matrix** first defined by Bowen and
Franks [4] and [5, page 79]. In their context parity corresponded to whether the shift map was orientation preserving or not. This is also how we shall use parity in the next section.

**Lemma 3.3.** Let \((\sigma, p)\) and \((\sigma', p')\) be the (edge) shifts with parity for the \(n \times n\) parity matrices \(M\) and \(M^2\) respectively. Then \((\sigma^{<2>}, p^{<2>})\) is parity-wise conjugate to \((\sigma', p')\).

**Proof.** Let \(a_{ij} + b_{ij} t = M_{ij}\) and \(a'_{ij} + b'_{ij} t = (M^2)_{ij}\). Let \((S, \Sigma), (S', \Sigma')\) and \((S^{<2>}, \Sigma^{<2>}, \Sigma^{<2>})\), have their usual meanings.

Partition \(S^{<2>}\) and \(S'\) as follows. Since \(\sigma\) and \(\sigma'\) are edge shifts we can define maps from and to, each on \(S\) and \(S'\) to \(\{1, \ldots, n\}\) so that for \(s_1\) and \(s_2\) in \(S\)

\[to(s_1) = from(s_2) \iff s_1 \leftrightarrow s_2,\]

and for \(s'_1\) and \(s'_2\) in \(S'\)

\[to(s'_1) = from(s'_2) \iff s'_1 \leftrightarrow s'_2.\]

The partition is given by

\[A_{ij}^{<2>} = \{(s_0, s_1) \in S^{<2>} | from(s_0) = i, to(s_1) = j, \text{ and } p^{<2>}((s_0, s_1)) = 1\},\]

\[B_{ij}^{<2>} = \{(s_0, s_1) \in S^{<2>} | from(s_0) = i, to(s_1) = j, \text{ and } p^{<2>}((s_0, s_1)) = -1\},\]

\[A'_{ij} = \{(s_0, s_1) \in S' | from(s_0) = i, to(s_1) = j, \text{ and } p'(s_0, s_1) = 1\},\]

\[B'_{ij} = \{(s_0, s_1) \in S' | from(s_0) = i, to(s_1) = j, \text{ and } p'(s_0, s_1) = -1\}.\]

From the definition of matrix multiplication and the fact the \(t^2 = 1\) we have,

\[|A_{ij}^{<2>}| = \sum_{k=1}^{n} a_{ik} a_{kj} + b_{ik} b_{kj} = a'_{ij} = |A'_{ij}|, \quad \text{and}\]

\[|B_{ij}^{<2>}| = \sum_{k=1}^{n} a_{ik} b_{kj} + b_{ik} a_{kj} = b'_{ij} = |B'_{ij}|.\]

Clearly, \(|S^{<2>}| = |S'|\). Hence we can find a bijection \(H : S^{<2>} \to S'\) such that \(H(A_{ij}^{<2>}) = A'_{ij}\) and \(H(B_{ij}^{<2>}) = B'_{ij}\). This bijection induces a homeomorphism \(h : \Sigma^{<2>} \to \Sigma'\) given by

\[h(\ldots(s_0, s_1)(s_2, s_3)\ldots) = (\ldots H(s_0, s_1) H(s_2, s_3)\ldots).\]

This gives us a parity-wise conjugacy. \(\square\)

**Definition 3.4.** Strong shift equivalence is defined just as in section 2 but now the \(R_i\) and \(S_i\) matrices can have entries of the form \(a + bt\) for nonnegative integers \(a\) and \(b\).

We break the proof of Theorem 3.1 up into a series of lemmas. The presentation we give follows the pattern of the proof of Theorem 2.2 given in [5, Theorem A.1], which the reader will want to have on hand.

**Lemma 3.5.** If \(A\) and \(B\) are parity matrices with \(A = RS\) and \(B = SR\), where \(R\) and \(S\) are as in Definition 3.4, then \((\sigma_A, \Sigma_A, p_A)\) is parity-wise topologically conjugate to \((\sigma_B, \Sigma_B, p_B)\).
Proof. See the second paragraph of the proof of Theorem A.1 in [5]. To check that the conjugacy given there is parity-wise use our Lemma 3.3. \qed

By induction Lemma 3.5 gives the “if” direction of Theorem 3.1.

**Lemma 3.6.** If $A$ is an $n \times n$ parity matrix then $A$ is strong shift equivalent to a matrix $A'$ whose entries are zeros, ones and $t$'s.

Proof. The proof so similar to the proof of Lemma A.2 in [5] that we merely note that one replaces $R_{ij}$ on the bottom of page 99 with

$$R_{ij} = \begin{cases} 
1 & \text{if } U_j \text{ is on the edge emanating from } V_i \text{ and the edge is labeled } +1, \\
 t & \text{if } U_j \text{ is on the edge emanating from } V_i \text{ and the edge is labeled } -1, \\
0 & \text{otherwise.}
\end{cases}$$

The reader should have no trouble reconstructing the details. \qed

**Lemma 3.7.** Suppose $A(t)$ is a $n \times n$ matrix of zeros, ones and $t$'s. Let $k = \sum_{ij} A_{ij}(1)$, the number of edges. Then $A(t)$ is shift equivalent to a $k \times k$ matrix $B(t)$ where each row of $B(t)$ has only zeros and ones or only zeros and $t$'s.

Proof. In the graph associated with $A(t)$ number the edges by $e_{(i-1)n+j}$, for the edge starting at vertex $i$ and ending on vertex $j$. Now number the edges $e_1, \ldots, e_k$, so as to preserve the previous order. Let $p$ give the label $\mp 1$ of the edges.

Let $R$ be the $n \times k$ matrix given by

$$R_{ij} = \begin{cases} 
1 & \text{if } e_j \text{ emanates from } v_i, \\
0 & \text{otherwise.}
\end{cases}$$

Let $S$ be the $k \times n$ matrix given by

$$S_{ij} = \begin{cases} 
1 & \text{if } e_i \text{ ends on } v_j \text{ and } p(e_i) = 1, \\
 t & \text{if } e_i \text{ ends on } v_j \text{ and } p(e_i) = -1, \\
0 & \text{otherwise.}
\end{cases}$$

It is easy to see that $A = RS$. The $i$-th row of $R$ is a list of the edges emanating from $v_i$, at most one of which may end at $v_j$. The edges which end at $v_j$ are listed according to their parity in the $j$-th column of $S$. Since $RS = \sum_k R_{ik} S_{kj}$, we have $(RS)_{ij} = 0$ if and only if there is an edge connecting $v_i$ and $v_j$. It should also be clear that the parity carries through correctly. Thus, $A = RS$.

Since each edge ends at only one vertex, each row of $S$ will be all zeros except for a single one or $t$. From this we can conclude that $B = SR$ has the desired form. \qed

**Example.** Let $A(t) = \begin{bmatrix} 1 & t \\ t & 0 \end{bmatrix}$. Then $R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $S = \begin{bmatrix} 1 & 0 \\ 0 & t \\ t & 0 \end{bmatrix}$. Thus, we get $B(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & t \\ t & t & 0 \end{bmatrix}$. 

In the next lemma and in Section 4 will use Markov partitions for homeomorphisms of Cantor sets. See [5] for the definition. Let \( A(t) \) be a matrix of the form of Lemma 3.7. A Markov partition of \( \Sigma_A \) can be constructed as follows. Let

\[
u_i = \{ u \in \Sigma_A | u_0 = i \}.
\]

Then \( U = \{ u_1, \ldots, u_k \} \) is the canonical Markov partition associated with \( A(t) \). We give \( U \) a parity convention \( p : U \to \{ \pm 1 \} \) induced by noting that \( p \) on \( \Sigma_A \) is constant on elements of \( U \).

Conversely, given a Markov partition and a parity convention an associated parity matrix is given by

\[
A_{ij}(U) = \begin{cases} 
1 & \text{if } u_i \cap \sigma(u_j) \neq \emptyset \text{ and } p(u_i) = 1, \\
t & \text{if } u_i \cap \sigma(u_j) \neq \emptyset \text{ and } p(u_i) = -1, \\
0 & \text{otherwise}.
\end{cases}
\]

One checks that given \( U, A(U) \) generates the same Markov partition it came from.

**Definition 3.8.** Let \( U \) and \( V \) be Markov partitions of the same shift space. We say that \( U \) refines \( V \) (denoted \( U > V \)) if each element of \( U \) is contained in an element of \( V \). By \( U \cap V \) we shall mean the Markov partition given by \( \{ R_i \cap R_j | R_i \in U, R_j \in V \} \). We also let \( U(m, n) \) denote the Markov partition given by \( \sigma^m(U) \cap \cdots \cap \sigma^n(U) \), for integers \( m \) and \( n \).

**Lemma 3.9.** Suppose \( U \) and \( V \) are Markov partitions for \( (\sigma, \Sigma, p) \) with associated matrices \( A \) and \( B \) respectively. If \( U > V \) and either \( V(0, 1) > U \) or \( V(-1, 0) > U \) then there are matrices \( R \) and \( S \) of zeros, ones and \( t \)'s, with \( A = SR \) and \( B = RS \).

**Proof.** The proof is similar to Lemma A.3 of [5]. One must change the matrix \( S_{kl} \) on page 101 to

\[
S_{kl} = \begin{cases} 
1 & \text{if } \phi \neq v_r \cap \sigma(v_l) \subset u_k \text{ and } p(u_k) = 1, \text{ where } u_k \subset v_r, \\
t & \text{if } \phi \neq v_r \cap \sigma(v_l) \subset u_k \text{ and } p(u_k) = -1, \text{ where } u_k \subset v_r, \\
0 & \text{otherwise}.
\end{cases}
\]

\( \square \)

The remainder of the proof of Theorem 3.1 follows from induction. See [5].

4. MAIN THEOREM

Consider a one-dimensional basic set along with the local stable manifolds of the orbits. Call such an object a ribbon set. For a flow on a 3-manifold a ribbon set would consist of annuli, Möbius bands, and infinite strips. Two ribbon sets are topologically equivalent if there is an orbit preserving homeomorphism between them that respects the flow direction.

A twist matrix \( A(t) \) of a basic set is a parity matrix for a first return map of a cross section, where the parity is determined by whether the map reverses or preserves orientation on a given partition element. The Markov partition of the Cantor set must fine enough for the parity to be uniform on each piece, but this can always be done. [5]

Two twist matrices are in the same twist-wise flow equivalence class if they can be realized as twist matrices of topologically equivalent ribbon sets.

**Theorem 4.1.** If two twist matrices \( A(t) \) and \( B(t) \) are in the same twist-wise flow equivalence class then \( \det(I - A(t)) = \det(I - B(t)) \) mod \( t^3 = 1 \).

We define three moves or relations on twist matrices.
• The shift move: $A(t) \sim B(t)$ if there exist matrices $R$ and $S$ as in section 3 such that $A(t) = RS$ and $B(t) = SR$.

• The expansion move: $A(t) \sim B(t)$ if

$$A(t) = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad B(t) = \begin{bmatrix} 0 & a_{11} & \cdots & a_{1n} \\
1 & 0 & \cdots & 0 \\
0 & a_{21} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_{n1} & \cdots & a_{nn} \end{bmatrix},$$

or vice versa. This notion was first introduced in [9].

• The twist move: $A(t) \prec B(t)$ if

$$B(t) = \begin{bmatrix} a_{11} & ta_{12} & \cdots & ta_{1n} \\
at_{a_{21}} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
t_{a_{n1}} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

where $A(t)$ is the same as above. All we have done is multiplied the first row and column of $A(t)$ by $t$ and used $t^2 = 1$.

The expansion and twist moves can be visualized on graphs as follows. In Figure 1a a new vertex and edge labeled +1 have been inserted. This is expansion. In Figure 1b the parity of the edges adjoining the vertex have all been flipped, except for the edge coming back to its starting place. This is the twist move. We can apply the expansion and twist moves to any vertex by first relabeling the vertices with shift moves.

We claim that the three relations, shift, expansion and twist, generate the twist-wise flow equivalence relation and that $\det(I - A(t)) \mod (t^2 = 1)$ is invariant under them.

**Lemma 4.2.** The shift, expansion and twist moves leave $\det(I - A(t)) \mod (t^2 = 1)$ invariant.

**Proof.** If $A(t) \sim B(t)$ then they have the same set of non-zero eigenvalues. Hence, $\det(I - A(t)) = \det(I - B(t))$. Let $A(t) \sim B(t)$. Computing $\det(I - B(t))$ along the first column gives $\det(I - B(t)) = (I - A(t))_{11} + \det(I - A(t)) - (I - A(t))_{11}$, where $(I - A(t))_{11}$ is the $(1, 1)$ cofactor of $I - A(t)$. If we apply the twist move to $I - A(t)$ directly the result is still $I - B(t)$. Thus, $\det(I - B(t)) = t^2 \det(I - A(t)) = \det(I - A(t))$, modulo $t^2 = 1$. \hfill $\Box$

**Lemma 4.3.** Two twist matrices are in the same twist-wise flow equivalence class if and only if they can be connected by a finite sequence of shift, expansion and twist moves.

**Proof.** Theorem 3.1 shows that applying the shift move does not alter the twist-wise flow equivalence class. In the expansion case it is clear from Figure 1a that the underlying one-dimensional flow (the inverse limit of the semi-flow on the graph) is unchanged. For the twist move think of the one-dimensional flow as a basic set embedded in a flow on a manifold. The local stable manifolds form a ribbon set. The twist move can be realized as an isotopy of the ribbon set, as Figure 2 tries to show, so twist-wise topological flow equivalence is preserved.

For the converse we consider the self-homeomorphism of a Cantor set cross section given by the first return map of a basic set. Let $C = C_1 \cup C_2$ be a open-closed decomposition of
a Cantor set $C$ and let $f : C \to C$ be a homeomorphism. The expansion along $C_1$ is defined as follows. Let $i : C_1 \to C_1'$ be an identification between $C_1$ and a copy $C_1'$ of $C_1$. Let $C' = C_1 \cup C_1' \cup C_2$ be a disjoint union. Define $f' : C' \to C'$ by $i$ on $C_1$, $f \circ i^{-1}$ on $C_1'$ and $f$ on $C_2$. See Figure 3. For a twist matrix $A(t)$ of zeros, ones and $t$'s, using the vertex shift, expansion along the block $\{\ldots 1 \ldots \}$ gives the subshift corresponding to the expansion of $A$. Using the edge shift for $A(t)$ with parity, expansion corresponds to expansion along $\bigcup \{ \ldots k \ldots \}$ over $k$ where edge $k$ emanates from vertex 1. The new first edge has parity $+1$.

Suppose the orientation of $f$ on $C_1$ is constant. Reversing the orientation assigned to $C_1$ will be called twisting $C_1$. If we twist the block $\{\ldots 1 \ldots \}$ of a subshift with parity given by $A(t)$ the resulting subshift with parity corresponds to the subshift with parity associated to the matrix resulting from applying the twist move to $A(t)$.

Suppose then that $A(t)$ and $B(t)$ are twist-wise equivalent. We can then regard them as arising from partitions of different cross sections of the same underlying flow, $\phi_t$.

Let $C_A$ and $C_B$ be Cantor set cross sections corresponding to $A$ and $B$ respectively. Since $C_A$ and $C_B$ are zero-dimensional we can take them to be disjoint.

Let $C_{A_0}$ be the subset of $C_A$ that meets $C_B$ under $\phi_0$ before coming back to $C_A$. Let $C_{A_i}$ be the subset of $C_A$ that passes through $C_A$ exactly once before meeting $C_B$. Continuing this until we exhaust $C_A$ allows us to write $C_A = C_{A_0} \cup \cdots \cup C_{A_n}$, a finite disjoint union, where $C_{A_i}$ is the subset of $C_A$ that passes through $C_A$ exactly $i$ times before meeting $C_B$.

Likewise partition $C_B = C_{B_0} \cup \cdots \cup C_{B_m}$, but where $C_{B_i}$ is the subset of $C_B$ exactly $i$ times before meeting $C_A$ under the reverse flow $\phi_{-t}$.

We can now make Markov partitions for $C_A$ and $C_B$ that are refinements of the partitions just described, such that the parity matrices $A'$ and $B'$ of these Markov partitions induced by the flow have only zeros, ones and $t$'s as entries. It is clear that $A'$ and $B'$ are parity-wise topologically conjugate to $A$ and $B$ respectively.

Now we use twist moves to “comb out” any twisting between the layers of $C_A$ and $C_B$.

If the map induced by the flow reverses orientation between a partition element $C_{A_i}$ in $C_A$ and one in $C_{A_0}$ then apply the twist move to the offending partition element of $C_{A_i}$. If needed we first refine $C_{A_i}$ so that only the twisted edge emanates from it. This pushes the twist to the left in Figure 4. With repeated applications all twisting between the layers of $C_A$ can be combed to the far left. Likewise we comb out all twisting between layers of $C_B$ to the far right.

We momentarily regard $C_A \cup C_B$ as a single cross section. If there is any twisting between $C_{A_0}$ and $C_{B_0}$ push it away, say to the far right.

Now apply (reverse) expansion moves to collapse $C_A$ to $C_{A_0}$ and $C_B$ to $C_{B_0}$. But $C_{B_0}$ is just a forward translate of $C_{A_0}$. Thus the $A'$ and $B'$ matrices, and hence $A$ and $B$, are in the same twist-wise flow equivalence class.

Lemmas 4.2 and 4.3 combine to prove Theorem 4.1.

5. Templates

A template is a smooth branched 2-manifold which supports a semi-flow. For a Smale flow on a 3-manifold nontrivial basic sets can be modeled by templates in the following sense. The inverse limit of the semi-flow of the template recovers the basic set. Further, any finite link of closed orbits in the basic set is ambient isotopic to a corresponding link on the
template. Also, the twisting in the local stable manifolds is preserved in the twisting of the bands of the template. The foundations of template theory are in [1, 2]. Also see [8].

The Figure 5a is the Lorenz template and Figure 5b is called the Horseshoe template. The classification theorem of [7] depends only on an incidence matrix which is \[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\] for both these templates. However, the twist matrix is \[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
t & t & t
\end{bmatrix}
\] for the Horseshoe template. Thus, \( \det(I - A(t)) = -1 \) for the Lorenz template and \(-t\) for the Horseshoe, and they are distinguished. More generally, consider a template with a single branch line with \( n \) bands emanating form it. Each bands loops back and stretches over the entire branch line. So, the dynamics of a return map are a full shift on \( n \) symbols. Suppose \( k \) of the bands have a even number of half twists while \( t = n - k \) have an odd number of have twists. Then the twist matrix is

\[
\begin{bmatrix}
1 & \ldots & 1 \\
\vdots & \ddots & \vdots \\
1 & \ldots & 1 \\
t & \ldots & t
\end{bmatrix}
\]

Thus, the full Parry-Sullivan invariant is \( 1 - k - lt \). However, the template in Figure 5c, though unorientable, has the same invariant as the Lorenz template, \(-1\).

**Note added.** The set \( \{1, t\} \) under multiplication modulo \( t^2 = 1 \), is of course a group isomorphic to \( \mathbb{Z}_2 \). The entries of the parity matrix are just elements of the group ring \( \mathbb{Z} \mathbb{Z}_2 \). The only properties of \( \mathbb{Z}_2 \) used in Section 3 were group properties. Hence the definitions are meaningful and the analogous results hold true for any group, \( G \). If \( G \) is Abelian then an analog of the Parry-Sullivan invariant should hold as well. Here however, one has to replace the twist move with a \( g \)-move for each nonidentity element \( g \in G \) consisting of multiplying the first row of the matrix by \( g \) and the first column by \( g^{-1} \).

**References**


Figure 1. Flow direction is down and counter clockwise.
Figure 2. Flow direction is down.
Figure 3. Expansion
FIGURE 4. Dark boxes represent $C_A$, gray boxes $C_B$
Figure 5. Templates