The Distribution of Certain Combinatorial Arrays

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THE DISTRIBUTION OF CERTAIN COMBINATORIAL ARRAYS

by

Yahya Dabab

B.S., King Khalid University, 2007

A Research Paper
Submitted in Partial Fulfillment of the Requirements for the
Master of Science Degree

Department of Mathematics
in the Graduate School
Southern Illinois University Carbondale
December 2010
RESEARCH PAPER APPROVAL

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Yahya Dabab

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Master of Science

in the field of Mathematics

Approved by:

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Graduate School
Southern Illinois University Carbondale
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AN ABSTRACT OF THE RESEARCH PAPER OF

YAHYA DABAB, for the Master of Science in Mathematics, presented on 2nd of December, at Southern Illinois University Carbondale.

TITLE: The Distribution of Certain Combinatorial Arrays

MAJOR PROFESSOR: Dr. Lane Clark

I determine the distribution function of the binomial coefficients and the Narayana numbers. I then present results which provide numerical evidence for our theorems.
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INTRODUCTION

The distribution of values of number-theoretic functions $\psi : \mathbb{P} \rightarrow \mathbb{P}$ is an important research area in analytic number theory. The seminal example is the distribution of prime numbers: Let $\psi : \mathbb{P} \rightarrow \mathbb{P}$ where $\psi(n)$ denotes the $n$th prime. The function $\pi(x) = |\{\psi(n) : n \in \mathbb{P}\} \cap [0, x]|$ is the number of primes at most real $x$. Chebyshev proved $\frac{c_1 x}{\log x} < \pi(x) < \frac{c_2 x}{\log x}$ for all sufficiently large $x$ where $c_1$ and $c_2$ are appropriate constants. This determined the order of magnitude of $\pi(x)$. Hadamard and Vallée Poussin refined the inequalities of Chebyshev proving the Prime Number Theorem: $\pi(x) \sim \frac{x}{\log x}$. This determined the distribution of primes (see Apostol [1]).

A combinatorial array can be viewed as a function $A : X^2 \rightarrow \mathbb{N}$ where $X = \mathbb{N}$ or $X = \mathbb{P}$. Its distribution function $|\{A(n, k) : (n, k) \in X^2\} \cap [0, x]|$ is the number of values of $A(n, k)$ at most real $x$. In this paper I determine the distribution function of the well-known arrays of the binomial coefficients and the Narayana numbers.

Chapter 1 gives the basic definitions and properties used in the paper.

Chapter 2 introduces the array of binomial coefficients. First I prove several properties of the binomial coefficients which I require. I then determine the distribution function of the binomial coefficients. Additionally, I then present results which provide numerical evidence for our theorem.

Chapter 3 introduces the array of Narayana numbers. First I prove several properties of the Narayana numbers which I require. I then determine the distribution function of the Narayana numbers. Additionally, I then present results which provide numerical evidence for our theorem.
CHAPTER 1
BACKGROUND

1.1 INTRODUCTION

An integer array is either a function (1) $A : \mathbb{N}^2 \to \mathbb{N}$ such that $A(n,0) = A(n,n) = 1$ and $A(n,k) = 0$ for $k > n$ or (2) $A : \mathbb{P}^2 \to \mathbb{N}$ such that $A(n,1) = A(n,n) = 1$ and $A(n,k) = 0$ for $k > n$. With a slight abuse of notation, we write $A$ for the set $\{A(n,k)\}$.

Suppose $A$ is an integer array. The distribution function of the array $A$ is the cardinality of the sets $A(x)$ defined below where $x \geq 0$: In case (1), $A(x) = \{A(n,k) : 2 \leq k \leq n - 2, n \geq 4\} \cap [0,x]$ and in case (2), $A(x) = \{A(n,k) : 2 \leq k \leq n - 1, n \geq 3\} \cap [0,x]$.

1.1.1 Definitions and Properties

Definition. Let $\mathbb{N}$ denote the non-negative integers; $\mathbb{P}$ denote the positive integers; $\mathbb{Z}$ denote the integers; and $\mathbb{R}$ denote the real numbers.

Definition. Suppose $f,g : E \to \mathbb{C}$ where $E = \mathbb{P}$ or $E = \mathbb{N}$. Then

$$f = o(g) \text{ or } f \ll g \text{ if and only if } \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$  

Example 1.1.1. We have $x^{-\alpha} = o(x^{-\beta})$ for constants $\alpha > \beta > 0$ because

$$\lim_{x \to \infty} \frac{x^{-\alpha}}{x^{-\beta}} = \lim_{x \to \infty} \frac{1}{x^{\alpha-\beta}} = 0.$$  

Example 1.1.2. We have $\ln x = o(x^\epsilon)$ for constant $\epsilon > 0$. L’Hospital Rule implies

$$\lim_{x \to \infty} \frac{\ln x}{x^\epsilon} = \lim_{x \to \infty} \frac{1/x}{\epsilon x^{\epsilon-1}} = \lim_{x \to \infty} \frac{1}{\epsilon x^\epsilon} = 0.$$  

Definition. Let $x \in \mathbb{R}$. The floor of $x$ is the largest integer not greater than $x$. We denote the floor of $x$ by $\lfloor x \rfloor$.

Example 1.1.3. $\lfloor 6.5 \rfloor = 6$, $\lfloor e \rfloor = 2$, and $\lfloor \log_3 30 \rfloor = 3$.  

2
**Definition.** Let \( x \in \mathbb{R} \). The ceiling of \( x \) is the smallest integer not less than \( x \). We denote the ceiling of \( x \) by \( \lceil x \rceil \).

**Example 1.1.4.** \( \lceil 6.5 \rceil = 7 \), \( \lceil e \rceil = 3 \), and \( \lceil \log_{30} 3 \rceil = 4 \).

**Lemma 1.1.1.** Suppose \( n \in \mathbb{Z} \). Then \( \lfloor n \rfloor = \lceil n \rceil = n \). Further for \( x \in \mathbb{R} \), we have \( \lfloor x + n \rfloor = \lfloor x \rfloor + n \) and \( \lceil x + n \rceil = \lceil x \rceil + n \).

**Lemma 1.1.2.** For \( x \in \mathbb{R} \), \( x - 1 \leq \lfloor x \rfloor \leq x \leq \lceil x \rceil \leq x + 1 \).

**Definition.** Let \( n \) and \( k \) be integers such that \( n \geq k \geq 0 \). Then

\[
(n)_k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1).
\]

We have \((n)_0 = 1\) for all \( n \geq 0 \).

**Lemma 1.1.3.** For \( n \geq k \geq 1 \), we have \((n-k)^k \leq (n)_k \leq n^k\).

**Proof.** Suppose \( n \geq k \geq 1 \). Then \((n)_k = n(n-1)(n-2)\cdots(n-k+1)\). Hence \((n-k)^k \leq n(n-1)(n-2)\cdots(n-k+1) \leq n^k\). \(\square\)

**Definition.** Suppose \( a < b \) are real numbers. A function \( f \) is a *decreasing function* on the interval \([a, b]\) if for any \( x_1 \) and \( x_2 \) in \([a, b]\), \( x_1 \leq x_2 \implies f(x_1) \geq f(x_2) \). A function \( f \) is an *increasing function* on the interval \([a, b]\) if for any \( x_1 \) and \( x_2 \) in \([a, b]\), \( x_1 \leq x_2 \implies f(x_1) \leq f(x_2) \).

**Lemma 1.1.4.** Let \( f \) be a differentiable function on the interval \((a, b)\). Then

1- If \( \frac{d}{dx}f(x) < 0 \), for \( x \in (a, b) \), then \( f \) is decreasing on \((a, b)\).

2- If \( \frac{d}{dx}f(x) > 0 \), for \( x \in (a, b) \), then \( f \) is increasing on \((a, b)\).

3- If \( \frac{d}{dx}f(x) = 0 \), for \( x \in (a, b) \), then \( f \) is constant on \((a, b)\).
1.1.2 Lattice Paths in the Plane

The pair \((k, l) \in \mathbb{Z}^2\) is called a step. The steps \((1, 0)\) and \((0, 1)\) are the directions horizontal \((H)\) and vertical \((V)\), respectively. For \(P = (a, b), Q = (c, d) \in \mathbb{Z}^2\), \(P \pm Q = (a \pm c, b \pm d) \in \mathbb{Z}^2\).

We call \((P_0, P_1, \ldots, P_m) \in (\mathbb{Z}^2)^{m+1}\) a lattice path in \(\mathbb{Z}^2\) or a lattice path. Here \((P_0, P_1, \ldots, P_m)\) has initial vertex \(P_0\) and length \(m\). The map \(\phi : (P_0, P_1, \ldots, P_m) \mapsto (s_1, s_2, \ldots, s_m)\) defined by \(s_k = P_k - P_{k-1}\ (k \in [m])\) is a bijection from lattice paths with initial vertex \(P_0\) to sequences of steps. Further \(P_k = P_0 + \sum_{j=1}^{k} s_j\ (k \in [m])\) (see Clark [2]).

**Example 1.1.5.** The lattice path \(P= ((0, 0), (1, 0), (2, 0), (2, 1), (2, 2), (2, 3), (3, 3), (3, 4), (3, 5), (4, 5))\) from \((0, 0)\) to \((4, 5)\) corresponds to the sequence of steps \((H, H, V, V, H, V, V, H)\).
CHAPTER 2
ARRAY OF BINOMIAL COEFFICIENTS

2.1 THE BINOMIAL COEFFICIENTS

The array of binomial coefficients \( B : \mathbb{N}^2 \to \mathbb{N} \) is defined by 
\[
B(n, k) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ for } 0 \leq k \leq n \text{ and } B(n, k) = 0 \text{ otherwise.}
\]
Hence \( B(n, 0) = B(n, n) = 1 \) for all \( n \in \mathbb{N} \). It is immediately seen that \( B \) is in fact an array. Table 2.1 gives the nonzero values of \( B(n, k) \) for \( 0 \leq n \leq 7 \).

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<thead>
<tr>
<th>( B(n, k) = \binom{n}{k} )</th>
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<td>35</td>
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<td>21</td>
<td>7</td>
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</tr>
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</table>

Table 2.1. Nonzero values of \( B(n, k) \) for \( 0 \leq n \leq 7 \)

The binomial coefficients have a nice geometric description. Suppose \((P_0, P_1, ..., P_{n_1+n_2})\) is a rectangular path from \((a, b)\) to \((a + n_1, b + n_2)\) \((a, b \in \mathbb{Z}, n_1, n_2 \in \mathbb{N})\). Then \((P_0, P_1, ..., P_{n_1+n_2}) \leftrightarrow \) to the \((n_1+n_2)\)-tuple \((s_1, s_2, ..., s_{n_1+n_2})\) of \(\{ H, V \} \) with \(n_1\) \(H\)’s and \(n_2\) \(V\)’s. Such an \((n_1+n_2)\)-tuple \((s_1, s_2, ..., s_{n_1+n_2}) \leftrightarrow \) to the \(n_1\)-subset \(\{ k \in [n_1+n_2] : s_k = H \} \) of the coordinates \([n_1+n_2]\) of the tuple. Hence the number of rectangular paths from \((a, b)\) to \((a + n_1, b + n_2)\) is \(\binom{n_1+n_2}{n_1} = \binom{n_1+n_2}{n_2}\) (see Clark [2]).
2.2 DISTRIBUTION FUNCTION OF BINOMIAL COEFFICIENTS

Let

\[ B = \left\{ \binom{n}{k} : 2 \leq k \leq n-2, n \geq 4 \right\} \]

\[ B_k = \left\{ \binom{n}{k} : n \geq 2k \right\} = \left\{ \binom{2k}{k}, \binom{2k+1}{k}, \ldots, \binom{n}{k}, \ldots \right\} . \]

For all \( x \in [0, \infty) \), let \( B(x) = B \cap [0, x] \) and \( B_k(x) = B_k \cap [0, x] \).

**Lemma 2.2.1.** We have,

\[ B = \bigcup_{k=2}^{\infty} B_k . \]

Hence, for all \( x \in [0, \infty) \),

\[ B(x) = \bigcup_{k=2}^{\infty} B_k(x) . \]

**Proof.** (\( \supseteq \)) First, we will show that \( \bigcup_{k=2}^{\infty} B_k \subseteq B \). Suppose \( \binom{n}{k} \in B_k \) where \( k \geq 2 \). Hence \( n \geq 2k \) implies \( n-1 \geq 2k-1 \geq k \) since \( k \geq 2 \) . Then \( \binom{n}{k} \in B \) for all \( k \geq 2 \). Hence \( \bigcup_{k=2}^{\infty} B_k \subseteq B \).

(\( \subseteq \)) Second, we will show that \( B \subseteq \bigcup_{k=2}^{\infty} B_k \). Suppose \( \binom{n}{k} \in B \). If \( n \geq 2k \), then \( \binom{n}{k} \in B_k \). If \( n \leq 2k-1 \), then \( n \geq 2n-2k+1 = 2(n-k)+1 > 2(n-k) \). Hence, \( \binom{n}{k} = \binom{n}{n-k} \in B_{n-k} \) and \( n-k \geq 2 \). So \( \binom{n}{k} \in B_{n-k} \). Hence \( B \subseteq \bigcup_{k=2}^{\infty} B_k \).

Consequently

\[ B = \bigcup_{k=2}^{\infty} B_k . \]

Hence, for all \( x \in [0, \infty) \),

\[ B(x) = B \cap [0, x] = \left( \bigcup_{k=2}^{\infty} B_k \right) \cap [0, x] = \bigcup_{k=2}^{\infty} \left( B_k \cap [0, x] \right) = \bigcup_{k=2}^{\infty} B_k(x) . \]

\[ \square \]

**Lemma 2.2.2.** For all \( k \geq 1 \), \( (a) \binom{2k}{k} \geq 3^{k-1} \) \( (b) \binom{2k-1}{k} \geq 3^{k-1} \).
Proof. (a) We show that \((\binom{2k}{k}) \geq 3^{k-1}\) by induction on \(k \geq 1\). The inequality is true when \(k = 1\) \([\binom{2}{1} > 3^0]\). Assume that the inequality holds at \(k\), then
\[
(\binom{2k+2}{k+1}) = \frac{(2k+2)(2k+1)(2k)!}{(k+1)^2 2^k t} = \frac{4k+2}{k+1} (\binom{2k}{k}) \geq \frac{4k+2}{k+1} 3^{k-1} \geq 3^k \iff \frac{4k+2}{k+1} \geq 3 \iff 4k + 2 \geq 3k + 3 \iff k \geq 1.
\]
Hence, \((\binom{2k+2}{k+1}) \geq 3^k\).

(b) We show that \((\binom{2k-1}{k}) \geq 3^{k-1}\) by induction on \(k \geq 1\). The inequality is true when \(k = 1\) \([\binom{1}{1} \geq 3^0]\). Assume that the inequality holds at \(k\), then
\[
(\binom{2k+1}{k+1}) = \frac{(2k+1)(2k-1)!}{k(k+1)(k-1)!} = \frac{4k+2}{k+1} (\binom{2k-1}{k}) \geq \frac{4k+2}{k+1} 3^{k-1} \geq 3^k \iff \frac{4k+2}{k+1} \geq 3 \iff 4k + 2 \geq 3k + 3 \iff k \geq 1.
\]
Hence, \((\binom{2k+1}{k+1}) \geq 3^k\).

**Theorem 2.2.3.** For any constant \(\epsilon\) such that \(0 < \epsilon \leq \frac{1}{6}\), we have
\[
|B(x)| = \sqrt{2x} + o\left(x^{\frac{1}{4} + \epsilon}\right).
\]

*Proof.* From Lemma 2.2.1, for all \(x \in [0, \infty)\)
\[
B_2(x) \subseteq B(x) = \bigcup_{k=2}^{\infty} B_k(x).
\]
Hence, for all \(x \in [0, \infty)\),
\[
|B_2(x)| \leq |B(x)| = \bigcup_{k=2}^{\infty} |B_k(x)| \leq |B_2(x)| + \sum_{k=3}^{\infty} |B_k(x)|. \tag{1}
\]
To avoid trivialities, we take fixed real \(x \geq 20\) in what follows. Hence, \(B_2(x), B_3(x) \neq \emptyset\).

We need to find \(|B_2(x)|\) where \(B_2(x) = \left\{\binom{1}{2}, \binom{5}{2}, \binom{6}{2}, \ldots\right\} \cap [0, x]\) is a finite set. Since \(\binom{1}{2} < \binom{5}{2} < \binom{6}{2} \leq \cdots\), there exists \(n \in \mathbb{P}\) such that \(\binom{n}{2} \leq x < \binom{n+1}{2}\) which we now find. We have \(\binom{n}{2} \leq x \iff \frac{n^2 - n}{2} \leq x \iff n^2 - n - 2x \leq 0\). Considering \(n\) to be real, \(f(n) = n^2 - n - 2x\) has zeros \(\frac{1 + \sqrt{8x+1}}{2}\) and \(\frac{1 - \sqrt{8x+1}}{2}\). Hence \(f(n) \leq 0\) on \(\left[\frac{1 - \sqrt{8x+1}}{2}, \frac{1 + \sqrt{8x+1}}{2}\right]\). Consequently, \(f\left(\frac{1 + \sqrt{8x+1}}{2}\right) \leq 0 < f\left(\frac{1 + \sqrt{8x+1}}{2} + 1\right)\). Hence the largest such integer \(n\) is \(\left\lfloor \frac{1 + \sqrt{8x+1}}{2} \right\rfloor\).

So, we get that
\[
|B_2(x)| = \left\lfloor \frac{1 + \sqrt{8x+1}}{2} \right\rfloor - 3.
\]
We will simplify \( \left\lfloor \frac{1 + \sqrt{8x + 1}}{2} \right\rfloor - 3 \). We write
\[
\left\lfloor \frac{1 + \sqrt{8x + 1}}{2} \right\rfloor - 3 = \frac{1 + \sqrt{8x + 1}}{2} - 3 - \delta(x), \\
\delta(x) \in [0, 1)
\]
\[
= \frac{1 + \sqrt{8x + 1}}{2} - (3 + \delta(x)) = \frac{1 + \sqrt{8x + 1}}{2} - \beta(x), \\
\beta(x) \in [3, 4)
\]
\[
= \frac{\sqrt{8x + 1}}{2} - (\beta(x) - 1/2) = \frac{\sqrt{8x + 1}}{2} - \alpha(x), \\
\alpha(x) \in [5/2, 7/2).
\]

Set \( r(x) = \sqrt{8x+1} - \sqrt{8x} > 0 \) for \( x \geq 1 \). For \( x \in [1, \infty) \), \( \frac{d}{dx} r(x) = \frac{4}{\sqrt{8x+1}} - \frac{4}{\sqrt{8x}} < 0 \), hence, \( r(x) \) is a decreasing function there. Therefore, \( r(x) \in (0, 1) \). So, we have
\[
\frac{\sqrt{8x + 1}}{2} - \alpha(x) = \sqrt{\frac{8x}{2}} + \frac{r(x)}{2} - \alpha(x) = \sqrt{2x} + \phi(x) \text{ where } |\phi(x)| \leq 4.
\]
Hence,
\[
|B_2(x)| = \sqrt{2x} + \phi(x) \tag{2}
\]
where \( |\phi(x)| \leq 4 \).

We next need to bound \( k \) from above so that \( B_k(x) = \left\{ \binom{2k}{k}, \binom{2k+1}{k}, \ldots, \binom{n}{k}, \ldots \right\} \cap [0, x] \neq \phi \) where \( B_k(x) = \phi \) if \( \binom{2k}{k} > x \). Lemma 2.2.2 (a) asserts \( \binom{2k}{k} \geq 3^{k-1} \) for all \( k \geq 1 \). Further, \( 3^{k-1} > x \Leftrightarrow k > \log_3 x + 1 \). Set \( k = \lfloor \log_3 x + 1 \rfloor \). Then \( k > \log_3 x + 1 \Rightarrow \binom{2k}{k} > x \Rightarrow B_k(x) = \phi \). Therefore, inequality (1) becomes
\[
|B_2(x)| \leq |B(x)| = | \bigcup_{k=2}^{\lfloor \log_3 x \rfloor + 1} B_k(x) | \leq |B_2(x)| + \sum_{k=3}^{\lfloor \log_3 x \rfloor + 1} |B_k(x)| . \tag{3}
\]

We need to find an upper bound for \( |B_k(x)| \) \((k \geq 3)\) where \( B_k(x) = \left\{ \binom{2k}{k}, \binom{2k+1}{k}, \ldots, \binom{n}{k} \right\} \cap [0, x] \) and \( \binom{n}{k} \leq x < \binom{n+1}{k} \). To do this, we find any \( n \geq 2k \geq 6 \) such that \( \binom{n}{k} > x \), hence, \( |B_k(x)| \leq n-1-2k+1 = n-2k \leq n-1 \). For \( n \geq 2k \geq 6 \), we have \( \binom{n}{k} = \binom{n}{k} \geq 2k \rightarrow \frac{k}{k+1} > \frac{n-k}{k} \geq \binom{n-k}{k} \geq x \Leftrightarrow n-k \geq kx^{1/k} \Leftrightarrow n \geq k + kx^{1/k} \).

Set \( f(k) = \left( \frac{k+1}{k} \right)^k \), hence, \( \lim_{k \to \infty} f(k) = e \). Then, \( \frac{d}{dk} f(k) = \left( \frac{k+1}{k} \right)^k \left\{ -\frac{1}{k+1} + \ln \left( \frac{k+1}{k} \right) \right\} \). Set \( g(k) = \frac{1}{k+1} + \ln \left( \frac{k+1}{k} \right) \), hence, \( \lim_{k \to \infty} g(k) = 0 \). Then, \( \frac{d}{dk} g(k) = \left\{ -\frac{1}{k+1} + \ln \left( \frac{k+1}{k} \right) \right\} - \frac{1}{(k+1)^2} + \frac{k+1}{k(k+1)} \right\} \).
\(-\frac{1}{k(k+1)^2}\) < 0 for all \(k \geq 3\). Therefore \(g(k) > 0\) for all \(k \geq 3\). Hence, \(\frac{d}{dk}f(k) > 0\) for all \(k \geq 3\). Consequently, \(\frac{64}{27} = f(3) \leq f(k) \leq e\) for all \(k \geq 3\). Set \(n = \lceil 2kx^\frac{1}{k} \rceil\).

Then \(n = \lceil 2kx^\frac{1}{k} \rceil \geq 2kx^\frac{1}{k} - 1 \geq k + kx^\frac{1}{k} \iff x \geq \left(\frac{k+1}{k}\right)^k\) which is true for all \(k \geq 3\) since \(x \geq 20 \geq \frac{64}{27}\). Hence \(n = \lceil 2kx^\frac{1}{k} \rceil \geq k + kx^\frac{1}{k} \geq 6\) for all \(k \geq 3\) since \(x \geq 20\).

Therefore \(\left(\frac{n-k}{k}\right)^k \geq x\) for all \(k \geq 3\). Then \(\binom{n}{k} > x \Rightarrow |B_k(x)| \leq n - 1 \leq \lceil 2kx^\frac{1}{k} \rceil\).

Consequently,

\[
0 \leq \sum_{k=3}^{\lceil \log_3 x \rceil + 1} |B_k(x)| \leq \sum_{k=3}^{\lceil \log_3 x \rceil + 1} \lceil 2kx^\frac{1}{k} \rceil \leq \sum_{k=3}^{\lceil \log_3 x \rceil + 1} 2kx^\frac{1}{k} \leq 2(\log_3 x + 2)^2 x^\frac{1}{x}. \quad (4)
\]

For all \(x \geq 20\), equality (2) and inequalities (3) and (4) imply

\[
\sqrt{2x + \phi(x)} \leq |B(x)| \leq \sqrt{2x + \phi(x)} + 2(\log_3 x + 2)^2 x^\frac{1}{x}
\]

\[
\iff \phi(x) \leq |B(x)| - \sqrt{2x} \leq \phi(x) + 2(\log_3 x + 2)^2 x^\frac{1}{x}. \quad (5)
\]

For any constant \(0 < \epsilon \leq \frac{1}{6}, \frac{2(\log_3 x + 2)^2 x^\frac{1}{x}}{x^{\frac{1}{x} + \epsilon}} = \frac{2(\log_3 x + 2)^2}{x^{\frac{1}{x}} + \epsilon} \longrightarrow 0\) as \(x \longrightarrow \infty\). Hence,

\[
\phi(x), 2(\log_3 x + 2)^2 x^\frac{1}{x} = o(x^{\frac{1}{x} + \epsilon}). \quad (6)
\]

Consequently, (5) and (6) imply

\[
|B(x)| = \sqrt{2x} + o(x^{\frac{1}{x} + \epsilon}). \quad \square
\]

### 2.3 NUMERICAL RESULTS

We tabulated \(|B(x)|\) for many \(x\) in the range \(100 \leq x \leq 1000000\). The crude upper bound in the proof of Theorem 2.2.3 shows that we need only consider the \(B_k(1000000)\) where \(k \leq \lceil \log_3 1000000 \rceil + 1 = 14\). We needed only to find the sets \(B_2(1000000), B_3(1000000), B_4(1000000), B_5(1000000), B_6(1000000), B_7(1000000), B_8(1000000), B_9(1000000), B_{10}(1000000)\) and \(B_{11}(1000000)\) since \(B_k(1000000) = \phi\) for all \(k \geq 12\). Further \(B_2(1000000) \cap B_3(1000000) = \{120, 1540, 7140\}\), \(B_2(1000000) \cap B_4(1000000) = \{210\}\), \(B_2(1000000) \cap B_5(1000000) = \{3003, 11628\}\),
\[ B_2(1000000) \cap B_6(1000000) = \{3003\}, \quad B_2(1000000) \cap B_8(1000000) = \{24310\}, \]
\[ B_5(1000000) \cap B_6(1000000) = \{3003\}, \quad B_2(1000000) \cap B_5(1000000) \cap B_6(1000000) = \{3003\} \text{ and } B_k(1000000) \cap B_l(1000000) = \phi \text{ for all other distinct } k, l \geq 2. \]

For example, let \( x = 10000 \). The crude upper bound for \( k \) is \( \lceil \log_3 10000 \rceil + 1 = 7 \). The Inclusion-Exclusion Principle and the above information imply
\[
|B(10000)| = \sum_{k=2}^{7} |B_k(10000)| - |B_2(10000) \cap B_3(10000)| - |B_2(10000) \cap B_4(10000)|
- |B_2(10000) \cap B_5(10000)| - |B_2(10000) \cap B_6(10000)|
- |B_5(10000) \cap B_6(10000)| + |B_2(10000) \cap B_5(10000) \cap B_6(10000)|. 
\]

We have
\[
B_2(10000) = \left\lfloor \frac{1 + \sqrt{8(10000) + 1}}{2} \right\rfloor - 3 = 138
\]
\[
B_3(10000) = \left\{ \binom{6}{3}, \binom{7}{3}, \ldots, \binom{40}{3} \right\} \Rightarrow |B_3(10000)| = 40 - 5 = 35
\]
\[
B_4(10000) = \left\{ \binom{8}{4}, \binom{9}{4}, \ldots, \binom{23}{4} \right\} \Rightarrow |B_4(10000)| = 23 - 7 = 16
\]
\[
B_5(10000) = \left\{ \binom{10}{5}, \binom{11}{5}, \ldots, \binom{18}{5} \right\} \Rightarrow |B_5(10000)| = 18 - 9 = 9
\]
\[
B_6(10000) = \left\{ \binom{12}{6}, \binom{13}{6}, \ldots, \binom{16}{6} \right\} \Rightarrow |B_6(10000)| = 16 - 11 = 5
\]
\[
B_7(10000) = \left\{ \binom{14}{7}, \binom{15}{7} \right\} \Rightarrow |B_7(10000)| = 15 - 13 = 2
\]
Then, \( |B(10000)| = 138 + 35 + 16 + 9 + 5 + 2 - 3 - 1 - 1 - 1 - 1 + 1 = 199 \), hence,
\[
\frac{|B(10000)| - \sqrt{20000}}{\sqrt{10000}} = 0.575.
\]

Table 2.2 with \( \epsilon = \frac{1}{6} \) below gives the distribution function of the array \( \{B(n, k)\} \) which provides numerical evidence for Theorem 2.2.3.
<table>
<thead>
<tr>
<th>(x)</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
<th>700</th>
<th>800</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>B(x)</td>
<td>)</td>
<td>16</td>
<td>21</td>
<td>32</td>
<td>37</td>
<td>44</td>
<td>48</td>
</tr>
<tr>
<td>(\sqrt{2x})</td>
<td>14.142</td>
<td>20.000</td>
<td>24.495</td>
<td>28.284</td>
<td>31.623</td>
<td>34.641</td>
<td>37.420</td>
<td>40.000</td>
</tr>
<tr>
<td>(</td>
<td>B(x)</td>
<td>-\sqrt{2x}) (\sqrt{x})</td>
<td>0.185</td>
<td>0.071</td>
<td>0.433</td>
<td>0.435</td>
<td>0.553</td>
<td>0.545</td>
</tr>
<tr>
<td>(x)</td>
<td>900</td>
<td>1000</td>
<td>2000</td>
<td>3000</td>
<td>4000</td>
<td>5000</td>
<td>6000</td>
<td>7000</td>
</tr>
<tr>
<td>(</td>
<td>B(x)</td>
<td>)</td>
<td>59</td>
<td>64</td>
<td>90</td>
<td>110</td>
<td>127</td>
<td>143</td>
</tr>
<tr>
<td>(\sqrt{2x})</td>
<td>42.426</td>
<td>44.721</td>
<td>63.246</td>
<td>77.460</td>
<td>89.443</td>
<td>100</td>
<td>109.545</td>
<td>118.322</td>
</tr>
<tr>
<td>(</td>
<td>B(x)</td>
<td>-\sqrt{2x}) (\sqrt{x})</td>
<td>0.552</td>
<td>0.609</td>
<td>0.598</td>
<td>0.594</td>
<td>0.593</td>
<td>0.608</td>
</tr>
<tr>
<td>(x)</td>
<td>8000</td>
<td>9000</td>
<td>10000</td>
<td>20000</td>
<td>30000</td>
<td>40000</td>
<td>50000</td>
<td>60000</td>
</tr>
<tr>
<td>(</td>
<td>B(x)</td>
<td>)</td>
<td>178</td>
<td>190</td>
<td>199</td>
<td>278</td>
<td>336</td>
<td>385</td>
</tr>
<tr>
<td>(\sqrt{2x})</td>
<td>126.491</td>
<td>134.164</td>
<td>141.421</td>
<td>200</td>
<td>244.949</td>
<td>282.843</td>
<td>316.228</td>
<td>346.410</td>
</tr>
<tr>
<td>(</td>
<td>B(x)</td>
<td>-\sqrt{2x}) (\sqrt{x})</td>
<td>0.575</td>
<td>0.588</td>
<td>0.575</td>
<td>0.551</td>
<td>0.525</td>
<td>0.510</td>
</tr>
<tr>
<td>(x)</td>
<td>70000</td>
<td>80000</td>
<td>90000</td>
<td>100000</td>
<td>200000</td>
<td>300000</td>
<td>400000</td>
<td>1000000</td>
</tr>
<tr>
<td>(</td>
<td>B(x)</td>
<td>)</td>
<td>496</td>
<td>534</td>
<td>563</td>
<td>592</td>
<td>818</td>
<td>989</td>
</tr>
<tr>
<td>(\sqrt{2x})</td>
<td>374.166</td>
<td>400</td>
<td>424.264</td>
<td>447.214</td>
<td>632.456</td>
<td>774.597</td>
<td>894.427</td>
<td>1414.214</td>
</tr>
<tr>
<td>(</td>
<td>B(x)</td>
<td>-\sqrt{2x}) (\sqrt{x})</td>
<td>0.460</td>
<td>0.473</td>
<td>0.462</td>
<td>0.457</td>
<td>0.414</td>
<td>0.391</td>
</tr>
</tbody>
</table>

Table 2.2. The distribution function of \(B(n, k)\)
CHAPTER 3
ARRAY OF NARAYANA NUMBERS

3.1 THE NARAYANA NUMBERS

The array of Naryana numbers \( N : \mathbb{N}^2 \rightarrow \mathbb{N} \) is defined by

\[
N(n,k) = \frac{n!}{k!(n-k)!} \cdot \frac{n!}{(k-1)!(n-k+1)!} = \binom{n}{k} \binom{n}{k-1}
\]

for \( 1 \leq k \leq n \) and \( N(n,k) = 0 \) otherwise. Hence \( N(n,1) = N(n,n) = 1 \) for all \( n \in \mathbb{P} \). It is immediately seen that \( N \) is in fact an array. The maximum of the Narayana numbers \( N(n,k) \) occurs at \( \left\lceil \frac{n}{2} \right\rceil \) when \( n \) is odd and at \( \left\lceil \frac{n}{2} \right\rceil \) and \( \left\lceil \frac{n}{2} \right\rceil + 1 \) when \( n \) is even. Moreover, \( N(n, \left\lceil \frac{n}{2} \right\rceil) \geq \frac{4n-1}{\left\lceil \frac{n}{2} \right\rceil^2} \) for \( n \in \mathbb{P} \).

Table 3.1 gives the nonzero values of \( N(n,k) \) for \( 1 \leq n \leq 8 \).

<table>
<thead>
<tr>
<th>( N(n,k) = )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{n!}{k!(n-k)!} \cdot \frac{n!}{(k-1)!(n-k+1)!} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( n )</td>
<td>1</td>
<td>6</td>
<td>6</td>
<td>1</td>
<td>10</td>
<td>20</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>( 1 )</td>
<td>28</td>
<td>196</td>
<td>490</td>
<td>196</td>
<td>28</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1. Nonzero values of \( N(n,k) \) for \( 1 \leq n \leq 8 \)

The Narayana numbers have a nice geometric description. A lattice path from \((0,0)\) to \((n,n)\) that does not go below the straight line \( y = x \) is called a Catalan path \( P \). Let \( \text{Cat}(n) \) denote the set of Catalan paths from \((0,0)\) to \((n,n)\). Then \( |\text{Cat}(n)| = \frac{\binom{2n}{n}}{n+1} \) is the Catalan number \( C_n \). Suppose a Catalan path \( P = (s_1, s_2, ..., s_{2n}) \) where each \( s_j \in \{V = (0,1), H = (1,0)\} \). We say \( P \) has a peak at \( r \) if \( (s_{r-1}, s_r) = (V,H) \). Let \( \text{Cat}(n,k) \) denote the set of Catalan paths from \((0,0)\) to \((n,n)\) with \( k \) peaks. Then the statistics \( |\text{Cat}(n,k)| = N(n,k) \) \((1 \leq k \leq n)\). There is an extensive literature about the Narayana numbers (see Clark [3]).
3.2 DISTRIBUTION FUNCTION OF NARAYANA NUMBERS

Let

\[ N = \{ N(n, k) : 2 \leq k \leq n - 1, n \geq 3 \} \]

\[ N_k = \{ N(n, k) : n \geq 2k - 1 \} = \{ N(2k - 1, k), N(2k, k), \ldots, N(n, k), \ldots \} . \]

For all \( x \in [0, \infty) \), let \( N(x) = N \cap [0, x] \) and \( N_k(x) = N_k \cap [0, x] \).

**Lemma 3.2.1.** \( N(n, k) = N(n, n - k + 1) \) for all \( 1 \leq k \leq n \)

**Proof.** We have \( \binom{n}{k} = \binom{n}{n-k} \) and \( \binom{n}{k} = \binom{n}{n-k+1} \). Then, \( N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1} = \frac{1}{n} \binom{n}{n-k+1} \binom{n}{n-k} = N(n, n - k + 1) \).

**Lemma 3.2.2.** We have,

\[ N = \bigcup_{k=2}^{\infty} N_k . \]

Hence, for all \( x \in [0, \infty) \),

\[ N(x) = \bigcup_{k=2}^{\infty} N_k(x) . \]

**Proof.** (\( \supseteq \)) First, we will show that \( \bigcup_{k=2}^{\infty} N_k \subseteq N \). Suppose \( N(n, k) \in N_k \) where \( k \geq 2 \). Hence \( n \geq 2k - 1 \) implies \( n - 1 \geq 2k - 2 \geq k \) since \( k \geq 2 \). Then \( N(n, k) \in N \) for all \( k \geq 2 \). Hence \( \bigcup_{k=2}^{\infty} N_k \subseteq N \).

(\( \subseteq \)) Second, we will show that \( N \subseteq \bigcup_{k=2}^{\infty} N_k \). Suppose \( N(n, k) \in N \). If \( n \geq 2k - 1 \), then \( N(n, k) \in N_k \). If \( n \leq 2k - 2 \), then \( n \geq 2n - 2k + 2 = 2(n - k + 1) > 2(n - k + 1) - 1 \). Hence, \( N(n, k) = N(n, n - k + 1) \in N_{n-k+1} \) and \( n - k + 1 \geq 2 \). So \( N(n, k) \in N_{n-k+1} \). Hence \( N \subseteq \bigcup_{k=2}^{\infty} N_k \).

Consequently

\[ N = \bigcup_{k=2}^{\infty} N_k . \]

Hence, for all \( x \in [0, \infty) \),

\[ N(x) = N \cap [0, x] = \left( \bigcup_{k=2}^{\infty} N_k \right) \cap [0, x] = \bigcup_{k=2}^{\infty} (N_k \cap [0, x]) = \bigcup_{k=2}^{\infty} N_k(x) . \]

\[ \square \]
Lemma 3.2.3. For all \( k \geq 2 \),

\[
N(k, k) < N(k + 1, k) < N(k + 2, k) < \cdots.
\]

Proof. For \( n \geq k \), we have

\[
N(n, k) < N(n + 1, k)
\]

\[
\Leftrightarrow \frac{1}{n} \binom{n}{k} \frac{n}{k-1} < \frac{1}{n+1} \binom{n+1}{k} \frac{n+1}{k-1}
\]

\[
\Leftrightarrow \frac{1}{n} \binom{n}{k} \frac{n}{k-1} < \frac{1}{n+1} \binom{n}{k} \frac{n}{k-1} \frac{n+1}{n+1-k} \frac{n+1}{n+2-k}
\]

\[
\Leftrightarrow (n + 1 - k)(n + 2 - k) < n(n + 1)
\]

\[
\Leftrightarrow n^2 + 3n + 2 - k(2n + 3) + k^2 < n^2 + n
\]

\[
\Leftrightarrow k^2 - (2n + 3)k + 2n + 2 < 0
\]

\[
\Leftrightarrow k^2 - 2nk - 3k + 2n + 2 < 0
\]

\[
\Leftrightarrow 2n(k - 1) > k^2 - 3k + 2
\]

\[
\Leftrightarrow 2n(k - 1) > (k - 2)(k - 1)
\]

\[
\Leftrightarrow n > \frac{k - 2}{2}
\]

which is true. This implies our result. \( \square \)

To our knowledge the following result is new.

Theorem 3.2.4. For any constant \( \epsilon \) such that \( 0 < \epsilon \leq \frac{1}{4} \), we have

\[
|N(x)| = \sqrt{2x} + o(x^{\frac{1}{4} + \epsilon}).
\]

Proof. From Lemma 3.2.2, for all \( x \in (0, \infty) \)

\[
N_2(x) \subseteq N(x) = \bigcup_{k=2}^{\infty} N_k(x).
\]
Hence, for all $x \in (0, \infty)$,

$$|N_2(x)| \leq |N(x)| = \bigcup_{k=2}^{\infty} N_k(x) \leq |N_2(x)| + \sum_{k=3}^{\infty} |N_k(x)|. \quad (7)$$

To avoid trivialities, we take fixed real $x \geq 20$ in what follows. Hence, $N_2(x), N_3(x) \neq \phi$.

We need to find $|N_2(x)|$ where $N_2(x) = \{N(3, 2), N(4, 2), N(5, 2), \ldots \} \cap [0, x]$ is finite set. From Lemma 3.2.3, we have $N(3, 2) < N(4, 2) < N(5, 2) < \cdots$, hence, there exists $n \in \mathbb{P}$ such that $N(n, 2) \leq x < N(n + 1, 2) \iff \binom{n}{2} \leq x < \binom{n+1}{2}$ which we now find. We have $\binom{n}{2} \leq x \iff \frac{n^2 - n}{2} \leq x \iff n^2 - n - 2x \leq 0$. From the proof of Theorem 2.2.3, we have $n = \left\lfloor \frac{1 + \sqrt{8x + 1}}{2} \right\rfloor$. So, we get that

$$|N_2(x)| = \left\lfloor \frac{1 + \sqrt{8x + 1}}{2} \right\rfloor - 2.$$  

Similarly as in the proof of Theorem 2.2.3, we conclude that $\left\lfloor \frac{1 + \sqrt{8x + 1}}{2} \right\rfloor - 2 = \sqrt{2x} + \phi(x)$ where $|\phi(x)| \leq 3$. Hence,

$$|N_2(x)| = \sqrt{2x} + \phi(x) \quad (8)$$

where $|\phi(x)| \leq 3$.

We next need to bound $k$ from above so that $N_k(x) = \{N(2k-1, k), N(2k, k), \ldots, N(n, k), \ldots \} \cap [0, x] \neq \phi$ where $N_k(x) = \phi$ if $N(2k-1, k) > x$. Lemma 2.2.2 (a) - (b) asserts $\binom{2k}{k}, \binom{2k-1}{k} \geq 3^{k-1}$ for all $k \geq 1$. Further, $\frac{1}{2k-1} \binom{2k-1}{k-1} \geq \frac{1}{2k-1} \binom{2k}{k} \geq \frac{2^{2k-3}}{2k-1} \geq 3^{k-2} \implies 3^{k-1} \geq 2k-1$ which is true for all $k \geq 3$. Therefore $N(2k-1, k) \geq 3^{k-2} > x \iff k > \log_3 x + 2$. Set $k = \lceil \log_3 x \rceil + 3$. Then $k > \log_3 x + 2 \implies N(2k-1, k) > x \implies N_k(x) = \phi$. Therefore, inequality (7) becomes

$$|N_2(x)| \leq |N(x)| = \bigcup_{k=2}^{\lceil \log_3 x \rceil + 2} N_k(x) \leq |N_2(x)| + \sum_{k=3}^{\lceil \log_3 x \rceil + 2} |N_k(x)|. \quad (9)$$

We need to find an upper bound for $|N_k(x)|$ $(k \geq 3)$ where $N_k(x) = \{N(2k-1, k), N(2k, k), \ldots, N(n, k) \} \cap [0, x]$ and $N(n, k) \leq x < N(n+1, k)$. To do this, we
find any \( n \geq 2k - 1 \geq 5 \) such that \( N(n, k) > x \), hence \( |N_k(x)| \leq n - 2k + 2 \leq n - 1 \). For \( n \geq 2k - 1 \geq 5 \), we have

\[
N(n, k) = \frac{1}{n} \binom{n}{k} \left( \frac{n}{k-1} \right) = \frac{k}{n(n-k+1)} \left( \frac{n}{k} \right)^2
\]

\[
= \frac{k}{n(n-k+1)} \cdot \frac{(n)_k^2}{k!^2} \geq \frac{k}{n(n-k+1)} \cdot \frac{(n-k)^{2k}}{k^{2k}}
\]

\[
= \frac{k}{n(n-k+1)} \left( \frac{n-k}{k} \right)^{2k} \geq \frac{1}{n^2} \left( \frac{n-k}{k} \right)^{2k}.
\]

Set \( f(k) = \left( \frac{k+1}{k} \right)^{2k-2} \). Then \( \frac{d}{dk} f(k) = \left( \frac{k+1}{k} \right)^{2k-2} \{ 2 \ln \left( \frac{k+1}{k} \right) - \frac{(2k-2)(k+2)}{k(k+1)} \} < 0 \) for all \( k \geq 3 \). Hence \( f(k) = \left( \frac{k+1}{k} \right)^{2k-2} \leq f(3) = \frac{256}{6561} \) for all \( k \geq 3 \). Set \( n = [2k^2x^{\frac{1}{k-1}}] \).

Then \( n = [2k^2x^{\frac{1}{k-1}}] \geq 2k^2x^{\frac{1}{2k-2}} - 1 \geq k + k^2x^{\frac{1}{2k-2}} \iff x \geq \left( \frac{k+1}{k} \right)^{2k-2} \) which is true since \( x \geq 20 > \frac{256}{6561} \geq \left( \frac{k+1}{k} \right)^{2k-2} \) for all \( k \geq 3 \). Hence \( n = [2k^2x^{\frac{1}{k-1}}] \geq k + k^2x^{\frac{1}{2k-2}} \geq 12 \) for all \( k \geq 3 \). Therefore \( \left( \frac{n-k}{k} \right)^{2k} \geq k^2x^{\frac{k}{k-1}} \Rightarrow \left( \frac{n-k}{k} \right)^{2k} \geq \frac{k^2x^{\frac{k}{k-1}}}{\left( \frac{k^2x^{\frac{1}{k-1}}}{4} \right)} \geq 4 \) which is true since \( 3^2 = 9 \).

Then \( N(n, k) > x \Rightarrow |N_k(x)| \leq n - 1 \leq [2k^2x^{\frac{1}{k-1}}] \). Consequently,

\[
0 \leq \sum_{k=3}^{[\log x]+2} |N_k(x)| \leq \sum_{k=3}^{[\log x]+2} [2k^2x^{\frac{1}{k-1}}] \leq \sum_{k=3}^{[\log x]+2} [2k^2x^{\frac{1}{k-1}}] \leq 2(\log x + 3)^3x^{\frac{1}{4}}. \tag{10}
\]

For all \( x \geq 20 \), equality (8) and inequalities (9) and (10) imply

\[
\sqrt{2x} + \phi(x) \leq |N(x)| \leq \sqrt{2x} + \phi(x) + 2(\log x + 3)^3x^{\frac{1}{4}}
\]

\[
\iff \phi(x) \leq |N(x)| - \sqrt{2x} \leq \phi(x) + 2(\log x + 3)^3x^{\frac{1}{4}}. \tag{11}
\]

For any constant \( 0 < \epsilon \leq \frac{1}{4} \), \( \frac{2(\log x + 3)^3x^{\frac{1}{4}}}{x^{\frac{1}{4} + \epsilon}} = \frac{2(\log x + 3)^3}{x^\epsilon} \to 0 \) as \( x \to \infty \).

Hence,

\[
\phi(x), 2(\log x + 3)^3x^{\frac{1}{4}} = o(x^{\frac{1}{4} + \epsilon}) \tag{12}
\]

Consequently, (11) and (12) imply

\[
|N(x)| = \sqrt{2x} + o(x^{\frac{1}{4} + \epsilon}). \quad \square
\]
3.3 NUMERICAL RESULTS

We tabulated $|N(x)|$ for many $x$ in the range $500 \leq x \leq 1000000$. The crude upper bound in the proof of Theorem 3.2.4 shows that we need only consider the $N_k(1000000)$ where $k \leq \lceil \log_3 1000000 \rceil + 2 = 15$. We needed only to find the sets $N_2(1000000)$, $N_3(1000000)$, $N_4(1000000)$, $N_5(1000000)$, $N_6(1000000)$ and $N_7(1000000)$ since $N_k(1000000) = \phi$ for all $k \geq 8$. Further $N_2(1000000) \cap N_3(1000000) = \{105\}$, $N_2(1000000) \cap N_4(1000000) = \{1176, 4950\}$ and $N_k(1000000) \cap N_l(1000000) = \phi$ for all other distinct $k, l \geq 2$.

For example, let $x = 10000$. The crude upper bound for $k$ is $\lceil \log_3 10000 \rceil + 2 = 8$. The Inclusion-Exclusion Principle and the above information imply

$$|N(10000)| = \sum_{k=2}^{8} |N_k(10000)| - |N_2(10000) \cap N_3(10000)| - |N_2(10000) \cap N_4(10000)|$$

We have

- $N_2(10000) = \left\lceil \frac{1+\sqrt{8(10000)+1}}{2} \right\rceil - 2 = 139$
- $N_3(10000) = \{N(5,3), N(6,3), \ldots, N(19,3)\} \Rightarrow |N_3(10000)| = 19 - 4 = 15$
- $N_4(10000) = \{N(7,4), N(8,4), \ldots, N(12,4)\} \Rightarrow |N_4(10000)| = 12 - 6 = 6$
- $N_5(10000) = \{N(9,5), N(10,5)\} \Rightarrow |N_5(10000)| = 10 - 8 = 2$
- $N_6(10000) = N_7(10000) = N_8(10000) = \phi \Rightarrow |N_6(10000)| = |N_7(10000)| = |N_8(10000)| = 0$

Then, $|N(10000)| = 139 + 15 + 6 + 2 + 0 + 0 + 0 - 1 - 2 = 159$, hence, $\frac{|N(10000)| - \sqrt{20000}}{\sqrt{10000}} = 0.175$.

Table 3.2 with $\epsilon = \frac{1}{4}$ below gives the distribution function of the array $\{N(n,k)\}$ which provides numerical evidence for Theorem 3.2.4.
<table>
<thead>
<tr>
<th>$x$</th>
<th>500</th>
<th>600</th>
<th>700</th>
<th>800</th>
<th>900</th>
<th>1000</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>N(x)</td>
<td>$</td>
<td>36</td>
<td>39</td>
<td>42</td>
<td>45</td>
<td>48</td>
</tr>
<tr>
<td>$\sqrt{2x}$</td>
<td>31.623</td>
<td>34.641</td>
<td>37.417</td>
<td>40</td>
<td>42.426</td>
<td>44.721</td>
<td>63.246</td>
</tr>
<tr>
<td>$\frac{</td>
<td>N(x)</td>
<td>-\sqrt{2x}}{\sqrt{x}}$</td>
<td>0.195</td>
<td>0.177</td>
<td>0.173</td>
<td>0.176</td>
<td>0.185</td>
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<table>
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<tr>
<th>$x$</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
<th>6000</th>
<th>7000</th>
<th>8000</th>
<th>9000</th>
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<td>$</td>
<td>N(x)</td>
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<td>88</td>
<td>101</td>
<td>113</td>
<td>125</td>
<td>134</td>
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<tr>
<td>$\sqrt{2x}$</td>
<td>77.460</td>
<td>89.443</td>
<td>100</td>
<td>109.544</td>
<td>118.322</td>
<td>126.491</td>
<td>134.164</td>
</tr>
<tr>
<td>$\frac{</td>
<td>N(x)</td>
<td>-\sqrt{2x}}{\sqrt{x}}$</td>
<td>0.192</td>
<td>0.182</td>
<td>0.183</td>
<td>0.199</td>
<td>0.187</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>$x$</th>
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<th>20000</th>
<th>30000</th>
<th>40000</th>
<th>50000</th>
<th>60000</th>
<th>70000</th>
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</thead>
<tbody>
<tr>
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<td>225</td>
<td>273</td>
<td>314</td>
<td>349</td>
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<tr>
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<td>200</td>
<td>244.949</td>
<td>282.843</td>
<td>316.228</td>
<td>346.410</td>
<td>374.166</td>
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<tr>
<td>$\frac{</td>
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<td>0.176</td>
<td>0.161</td>
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<table>
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<th>100000</th>
<th>200000</th>
<th>300000</th>
<th>400000</th>
<th>1000000</th>
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</thead>
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<td>465</td>
<td>490</td>
<td>685</td>
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<td>447.213</td>
<td>632.456</td>
<td>774.597</td>
<td>894.427</td>
<td>1414.213</td>
</tr>
<tr>
<td>$\frac{</td>
<td>N(x)</td>
<td>-\sqrt{2x}}{\sqrt{x}}$</td>
<td>0.141</td>
<td>0.135</td>
<td>0.135</td>
<td>0.117</td>
<td>0.110</td>
</tr>
</tbody>
</table>

Table 3.2. The distribution function of $N(n,k)$
REFERENCES


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Major Professor: Dr. Lane Clark