On the Dynamics of Stochastic Differential Equations (Ellis B. Stouffer Colloquium, University of Kansas)

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ON THE DYNAMICS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Deterministic ODE’s: Stable Manifolds

Consider the ODE

\[ dx(t) = h(x(t)) \, dt \]  \hspace{1cm} (ODE)

on \( \mathbb{R}^d \) driven by a vector field \( h : \mathbb{R}^d \to \mathbb{R}^d \) of class \( C^k_b \); viz. \( h \) has all derivatives \( D^j h, 1 \leq j \leq k \), continuous and globally bounded.

Assume hyperbolic equilibrium at 0: \( h(0) = 0; \) \( Dh(0) \in L(\mathbb{R}^d) \) has all eigenvalues off imaginary axis.

Then (ODE) has a \( C^k_b \) flow \( \phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \) satisfying the following properties

(i) \( \phi(\cdot, x) \) is the unique solution of (ODE) through \( x \in \mathbb{R}^d \).
(ii) \( \phi(t, 0) = 0, t \in \mathbb{R} \).
(iii) Group property:

\[ \phi(t + s, \cdot) = \phi(t, \cdot) \circ \phi(s, \cdot), \]

for all \( s, t \in \mathbb{R} \).
(iv) Local flow-invariant stable/unstable \( C^k \) manifolds in a neighborhood of 0 (figure).

Above behavior is “generic” among all vector fields.
The Flow

\[ \phi(t_1, \cdot) \]

\[ \phi(t_2, \cdot) \]

\[ \phi(t_1, x) \]

\[ \phi(t_1 + t_2, x) \]
Local Stable/Unstable Manifolds

\[ \tilde{S}, \tilde{U} \quad \phi(t, \cdot) \quad \mathbb{R}^d \]

\[ \mathbb{R} \]

\[ 0 \quad t \]

\[ \mathbb{R} \]
What happens if the vector field is noisy??
SDE’s: Stable Manifolds

- Formulate a *Local Stable Manifold Theorem* for stochastic differential equations (SDE’s) driven by “white noise” (Brownian motion) (or general noise with stationary ergodic increments-Stratonovich or Itô type.)
- Start with the existence of a stochastic flow for SDE.
- Concept of a hyperbolic stationary trajectory. The stationary trajectory is a solution of the forward /backward anticipating SDE for all time (Stratonovich case).
- Existence of a stationary random family of asymptotically invariant stable and unstable manifolds within a stationary neighborhood of the hyperbolic stationary solution.
- The stable and unstable manifolds are dynamically characterized using forward and backward solutions of anticipating versions of the (Stratonovich) SDE.
- Proof based on Ruelle-Oseledec (non-linear) multiplicative ergodic theory and anticipating stochastic calculus.
Formulation of the Theorem

Stratonovich SDE

\[ dx(t) = h(x(t)) \, dt + \sum_{i=1}^{m} g_i(x(t)) \circ dW_i(t), \quad (I) \]

on \( \mathbb{R}^d \) driven by \( m \)-dimensional Brownian motion \( W := (W_1, \cdots, W_m) \).

\((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P) := \) canonical filtered Wiener space.

\( \Omega := \) space of all continuous paths \( \omega : \mathbb{R} \to \mathbb{R}^m, \omega(0) = 0 \), in Euclidean space \( \mathbb{R}^m \), with compact open topology;

\( \mathcal{F} := \) Borel \( \sigma \)-field of \( \Omega \);

\( \mathcal{F}_t := \) sub-\( \sigma \)-field of \( \mathcal{F} \) generated by the evaluations \( \omega \to \omega(u), \quad u \leq t, \quad t \in \mathbb{R} \).

\( P := \) Wiener measure on \( \Omega \).

\( h, g_i : \mathbb{R}^d \to \mathbb{R}^d, 1 \leq i \leq m, \) vector fields on \( \mathbb{R}^d \). For some \( k \geq 1, \delta \in (0, 1), \) \( h \) is \( C_b^{k,\delta} \), viz. \( h \) has all derivatives \( D^j h, 1 \leq j \leq k \), continuous and globally bounded, \( D^k h \) Hölder continuous with exponent \( \delta \).

\( g_i, 1 \leq i \leq m, \) globally bounded and in \( C_b^{k+1,\delta} \).

\( \theta : \mathbb{R} \times \Omega \to \Omega \) is the (ergodic) Brownian shift

\[ \theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \omega \in \Omega. \]
Let $\phi : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ be the stochastic flow generated by (I) $\left( \phi(t, \cdot, \omega) = \left[ \phi(-t, \cdot, \theta(t, \omega)) \right]^{-1}, t < 0 \right)$. Then $\phi$ is a perfect cocycle:

$$
\phi(t + s, \cdot, \omega) = \phi(t, \cdot, \theta(s, \omega)) \circ \phi(s, \cdot, \omega),
$$

for all $s, t \in \mathbb{R}$ and all $\omega \in \Omega$ ([I-W], [A-S], [A]).

Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of $\mathbb{R}^d$. $(\phi, \theta)$ is a “random vector-bundle morphism” over the “base” probability space $\Omega$. 
The Cocycle

$\Omega$ $\omega$ $\theta(t_1, \cdot, \omega)$ $\theta(t_2, \cdot, \theta(t_1, \omega))$

$\mathbb{R}^d$ $x$

$\phi(t_1, \cdot, \omega)$ $\phi(t_2, \cdot, \theta(t_1, \omega))$

$\Omega$ $\omega$ $\theta(t_1, \omega)$ $\theta(t_1 + t_2, \omega)$

$t = 0$ $t = t_1$ $t = t_1 + t_2$

$\phi(t_1, x, \omega)$ $\phi(t_1 + t_2, x, \omega)$
Definition

The SDE (I) has a stationary trajectory if there exists an $\mathcal{F}$-measurable random variable $Y : \Omega \to \mathbb{R}^d$ such that

$$\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \quad (1)$$

for all $t \in \mathbb{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $\phi(t, Y) = Y((\theta(t)))$.

If (1) holds on a sure event $\Omega_t$ that may depend on $t$, then there are “perfect” versions of the stationary random variable $Y$ and of the flow $\phi$ such that (1) and the cocycle property hold for all $\omega \in \Omega$ ([Sc]).

Let $\phi(t, Y)$ be a stationary solution of (I). Cocycle property of $\phi$ implies that the linearization

$$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$$

along the stationary solution is also a $d \times d$-matrix-valued cocycle. Using Kolmogorov’s theorem, the random variables

$$\sup_{x \in \mathbb{R}^d} \frac{|D_2\phi(t, x)|}{(1 + |x|^\gamma)}, \gamma > 0,$$

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have moments of all orders. If \( E \log^+ |Y| < \infty \), then
\( E \log^+ |D_2 \phi(1, Y)| < \infty \). Apply Oseledec’s Theorem to get a 
non-random finite Lyapunov spectrum:

\[
\lim_{n \to \infty} \frac{1}{n} \log |D_2 \phi(n, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbb{R}^d).
\]

Spectrum takes finitely many values \( \{\lambda_i\}_{i=1}^p \) with non-random multiplicities \( q_i \), \( 1 \leq i \leq p \), and \( \sum_{i=1}^p q_i = d \) ([Ru.1], Theorem I.6).

**Definition**

Stationary trajectory \( \phi(t, Y) \) of (I) is **hyperbolic** if
\( E \log^+ |Y(\cdot)| < \infty \), and if the linearized cocycle
\( (D_2 \phi(n, Y(\omega), \omega), \theta(n, \omega)) \) has a non-vanishing Lyapunov spec-
trum

\[
\{\lambda_p < \cdots < \lambda_{i_0+1} < \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_2 < \lambda_1\}
\]
i.e. \( \lambda_i \neq 0 \) for all \( 1 \leq i \leq p \).

Define \( \lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\} \) if at least one \( \lambda_i < 0 \). If all \( \lambda_i > 0 \), set \( \lambda_{i_0} = -\infty \). (This implies that \( \lambda_{i_0-1} \) is
the smallest positive Lyapunov exponent of the linearized flow, if at least one \( \lambda_i > 0 \); in case all \( \lambda_i \) are negative, set
\( \lambda_{i_0-1} = \infty \).)
Let $\rho \in \mathbb{R}^+$, $x \in \mathbb{R}^d$.

$B(x, \rho) :=$ open ball in $\mathbb{R}^d$, center $x$ and radius $\rho$;

$\bar{B}(x, \rho) :=$ corresponding closed ball;

$\mathcal{C}(\mathbb{R}^d) :=$ the class of all non-empty compact subsets of $\mathbb{R}^d$ with Hausdorff metric $d^*$:

\[
d^*(A_1, A_2) := \sup\{d(x, A_1) : x \in A_2\} \lor \sup\{d(y, A_2) : y \in A_1\}
\]

where $A_1, A_2 \in \mathcal{C}(\mathbb{R}^d)$;

\[
d(x, A_i) := \inf\{|x - y| : y \in A_i\}, x \in \mathbb{R}^d, i = 1, 2;
\]

$\mathcal{B}(\mathcal{C}(\mathbb{R}^d)) :=$ Borel $\sigma$-algebra on $\mathcal{C}(\mathbb{R}^d)$ with respect to the metric $d^*$.
Theorem 1 (The Stable Manifold Theorem) (M. + Scheutzow, 1997)

Assume that the coefficients of SDE (I) satisfy the given hypotheses. Suppose $\phi(t, Y)$ is a hyperbolic stationary trajectory of (I) with $E \log^+ |Y| < \infty$.

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0 - 1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$,

(ii) $\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds $\tilde{S}(\omega), \tilde{U}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{S}(\omega)$ is the set of all $x \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$|\phi(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}$$

for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \to \infty} \frac{1}{t} \log |\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0}$$

(2)

for all $x \in \tilde{S}(\omega)$. Each stable subspace $S(\omega)$ of the linearized flow $D_2\phi$ is tangent at $Y(\omega)$ to the submanifold $\tilde{S}(\omega)$, viz. $T_{Y(\omega)}\tilde{S}(\omega) = S(\omega)$. In particular, $\dim \tilde{S}(\omega) = \dim S(\omega)$ and is non-random.
(b) \( \limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup_{x_1 \neq x_2} \sup_{x_1, x_2 \in \tilde{S}(\omega)} \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq \lambda_{i_0} \).

(c) (Cocycle-invariance of the stable manifolds):

There exists \( \tau_1(\omega) \geq 0 \) such that

\[
\phi(t, \cdot, \omega)(\tilde{S}(\omega)) \subseteq \tilde{S}(\theta(t, \omega)), \quad t \geq \tau_1(\omega). \tag{3}
\]

Also

\[
D_2\phi(t, Y(\omega), \omega)(S(\omega)) = S(\theta(t, \omega)), \quad t \geq 0. \tag{4}
\]

(d) \( \tilde{U}(\omega) \) is the set of all \( x \in \bar{B}(Y(\omega), \rho_2(\omega)) \) with the property that

\[
|\phi(-n, x, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{(-\lambda_{i_0-1} + \epsilon_2)n} \tag{5}
\]

for all integers \( n \geq 0 \). Also

\[
\limsup_{t \to \infty} \frac{1}{t} \log |\phi(-t, x, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}. \tag{6}
\]

for all \( x \in \tilde{U}(\omega) \). Furthermore, the unstable subspace \( U(\omega) \) of \( D_2\phi \) is the tangent space to \( \tilde{U}(\omega) \) at \( Y(\omega) \), viz. \( T_{Y(\omega)}\tilde{U}(\omega) = U(\omega) \). In particular, \( \dim \tilde{U}(\omega) = \dim U(\omega) \) and is non-random.

(e) \( \limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup_{x_1 \neq x_2} \sup_{x_1, x_2 \in \tilde{U}(\omega)} \left\{ \frac{|\phi(-t, x_1, \omega) - \phi(-t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq -\lambda_{i_0-1}. \)
(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ such that

$$\phi(-t, \cdot, \omega)(\tilde{U}(\omega)) \subseteq \tilde{U}(\theta(-t, \omega)), \quad t \geq \tau_2(\omega).$$

(7)

Also

$$D_2\phi(-t, Y(\omega), \omega)(U(\omega)) = U(\theta(-t, \omega)), \quad t \geq 0.$$ (8)

(g) The submanifolds $\tilde{U}(\omega)$ and $\tilde{S}(\omega)$ are transversal, viz.

$$\mathbb{R}^d = T_{Y(\omega)}\tilde{U}(\omega) \oplus T_{Y(\omega)}\tilde{S}(\omega).$$

(9)

(h) The mappings

$$\Omega \to C(\mathbb{R}^d), \quad \Omega \to C(\mathbb{R}^d),$$

$$\omega \mapsto \tilde{S}(\omega) \quad \omega \mapsto \tilde{U}(\omega)$$

are $(\mathcal{F}, \mathcal{B}(C(\mathbb{R}^d)))$-measurable.

Assume, further, that $h, g_i, 1 \leq i \leq m$, are $C_b^\infty$ Then the local stable and unstable manifolds $\tilde{S}(\omega), \tilde{U}(\omega)$ are $C^\infty$. 

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\[ \tilde{S}(\theta(t, \omega)) \]

\[ \tilde{U}(\theta(t, \omega)) \]

\[ \Omega \]

\[ \omega \]

\[ \theta(t, \omega) \]

\[ t > \tau_1(\omega) \]

A picture is worth a 1000 words!
\[ \tilde{S}(\theta(-t, \omega)) \quad \tilde{U}(\theta(-t, \omega)) \]

\[ Y(\theta(-t, \omega)) \]

\[ \phi(-t, \cdot, \omega) \]

\[ \theta(-t, \cdot) \]

\[ t > \tau_2(\omega) \]
Sketch of Proof

Linearization and Substitution

Assume regularity conditions on the coefficients $h, g_i$. By the Substitution Rule, $\phi(t,Y(\omega),\omega)$ is a stationary solution of the anticipating Stratonovich SDE

$$
\begin{aligned}
\dot{\phi}(t,Y) &= h(\phi(t,Y)) \, dt + \sum_{i=1}^{m} g_i(\phi(t,Y)) \circ dW_i(t), \quad t > 0 \\
\phi(0,Y) &= Y.
\end{aligned}
$$

([N-P]).

Linearize the SDE (I) along the stationary trajectory. By substitution, match the solution of the linearized equation with the linearized cocycle $D_2\phi(t,Y(\omega),\omega)$. Hence $D_2\phi(t,Y(\omega),\omega), \, t \geq 0$, solves the SDE:

$$
\begin{aligned}
\dot{D}_2\phi(t,Y) &= Dh(\phi(t,Y)) D_2\phi(t,Y) \, dt \\
&\quad + \sum_{i=1}^{m} Dg_i(\phi(t,Y)) D_2\phi(t,Y) \circ dW_i(t), \quad t > 0 \\
D_2\phi(0,Y) &= I.
\end{aligned}
$$

([III])

$D_2, D$ denotes spatial (Fréchet) derivatives.

Similarly, the backward trajectories

$$
\phi(t,Y), \, D_2\phi(t,Y), \, t < 0,
$$
solve the corresponding backward Stratonovich SDE’s:

\[
\begin{aligned}
\frac{d\phi(t, Y)}{dt} &= -h(\phi(t, Y)) dt - \sum_{i=1}^{m} g_i(\phi(t, Y)) \circ dW_i(t), \quad t < 0 \\
\phi(0, Y) &= Y. \\
\frac{dD\phi(t, Y)}{dt} &= -Dh(\phi(t, Y))D\phi(t, Y) dt \\
&\quad - \sum_{i=1}^{m} Dg_i(\phi(t, Y)) D\phi(t, Y) \circ dW_i(t), \quad t < 0 \\
D\phi(0, Y) &= I.
\end{aligned}
\]

Above SDE’s (II)-(III) give dynamic characterizations of the stable and unstable manifolds.

The following lemma is used to construct the shift-invariant sure event appearing in the statement of the local stable manifold theorem. Gives “perfect versions” of the ergodic theorem and Kingman’s subadditive ergodic theorem.

**Lemma 1**

(i) Let \( h : \Omega \to \mathbb{R}^+ \) be \( \mathcal{F} \)-measurable and such that

\[
\int_{\Omega} \sup_{0 \leq u \leq 1} h(\theta(u, \omega)) dP(\omega) < \infty.
\]
Then there is a sure event $\Omega_1 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_1) = \Omega_1$ for all $t \in \mathbb{R}$, and

$$\lim_{t \to \infty} \frac{1}{t} h(\theta(t, \omega)) = 0$$

for all $\omega \in \Omega_1$.

(ii) Suppose $f : \mathbb{R}^+ \times \Omega \to \mathbb{R} \cup \{-\infty\}$ is a measurable process on $(\Omega, \mathcal{F}, P)$ satisfying the following conditions

(a) $E \sup_{0 \leq u \leq 1} f^+(u) < \infty$, $E \sup_{0 \leq u \leq 1} f^+(1 - u, \theta(u)) < \infty$

(b) $f(t_1 + t_2, \omega) \leq f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ for all $t_1, t_2 \geq 0$ and all $\omega \in \Omega$.

Then there is sure event $\Omega_2 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_2) = \Omega_2$ for all $t \in \mathbb{R}$, and a fixed number $f^* \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{t \to \infty} \frac{1}{t} f(t, \omega) = f^*$$

for all $\omega \in \Omega_2$.

Proof

[Mo.1], Lemma 7. $\square$
**Theorem 2** ([O], 1968)

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(\theta : \mathbb{R}^+ \times \Omega \to \Omega\) a measurable family of ergodic \(P\)-preserving transformations. Let \(T : \mathbb{R}^+ \times \Omega \to L(\mathbb{R}^d)\) be measurable, such that \((T, \theta)\) is an \(L(\mathbb{R}^d)\)-valued cocycle. Suppose that

\[
E \sup_{0 \leq t \leq 1} \log^+ \| T(t, \cdot) \| < \infty, \quad E \sup_{0 \leq t \leq 1} \log^+ \| T(1 - t, \theta(t, \cdot)) \| < \infty.
\]

Then there is a set \(\Omega_0 \in \mathcal{F}\) of full \(P\)-measure such that \(\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0\) for all \(t \in \mathbb{R}^+\), and for each \(\omega \in \Omega_0\), the limit

\[
\lim_{n \to \infty} \left[ T(t, \omega)^* \circ T(t, \omega) \right]^{1/(2t)} = \Lambda(\omega)
\]

exists in the uniform operator norm. Each \(\Lambda(\omega)\) has a discrete non-random spectrum

\[
e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_p}
\]

where the \(\lambda_i\)'s are distinct. Each \(e^{\lambda_i}\) has an eigen-space \(F_i(\omega)\) and a fixed non-random multiplicity \(m_i := \dim F_i(\omega)\). Define

\[
E_1(\omega) := \mathbb{R}^d, \quad E_i(\omega) := \left[ \bigoplus_{j=1}^{i-1} F_j(\omega) \right]^\perp, \quad 1 < i \leq p.
\]

Then

\[
E_p(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = \mathbb{R}^d
\]
\[ \lim_{t \to \infty} \frac{1}{t} \log \| T(t, \omega)x \| = \lambda_i(\omega), \quad \text{if} \quad x \in E_i(\omega) \setminus E_{i+1}(\omega), \]

and

\[ T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega)) \]

for all \( t \geq 0, \ 1 \leq i \leq p. \)

**Proof.**

Based on the discrete version of Oseledec’s multiplicative ergodic theorem and Lemma 1. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]), “perfect” infinite-dimensional version and application to SFDE’s. \( \square \)
Spectral Theorem

\[ T(t, \omega) \]

\[ E_1 = \mathbb{R}^d \]

\[ E_2(\omega) \]

\[ E_3(\omega) \]

\[ \Omega \]

\[ \omega \]

\[ \theta(t, \omega) \]

\[ \theta(t, \cdot) \]

\[ \mathbb{R}^d \]

\[ E_2(\theta(t, \omega)) \]

\[ E_3(\theta(t, \omega)) \]
Apply Theorem 2 with $T(t, \omega) := D_2\phi(t, Y(\omega), \omega)$. Then linearized cocycle has random invariant stable and unstable subspaces $\{S(\omega), U(\omega) : \omega \in \Omega\}$:

$$D_2\phi(t, Y(\omega), \omega)(S(\omega)) = S(\theta(t, \omega)),$$

$$D_2\phi(-t, Y(\omega), \omega)(U(\omega)) = U(\theta(-t, \omega)), \quad t \geq 0.$$

[Mo.1].
Estimates on the non-linear cocycle

**Theorem 3**  (M. + Schutzow [M-S.2])

There exists a jointly measurable modification of the trajectory random field of (I), denoted by \( \{ \phi_{s,t}(x) : -\infty < s, t < \infty, \ x \in \mathbb{R}^d \} \), with the following properties:

Define \( \phi : \mathbb{R} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d \) by

\[
\phi(t, x, \omega) := \phi_{0,t}(x, \omega), \quad x \in \mathbb{R}^d, \omega \in \Omega, t \in \mathbb{R}.
\]

Then for all \( \omega \in \Omega, \epsilon \in (0, \delta), \gamma, \rho, T > 0, 1 \leq |\alpha| \leq k, \phi(t, \cdot, \omega) \) is \( C^{k,\epsilon}, 0 < \epsilon < \delta \), and the quantities

\[
\sup_{0 \leq s, t \leq T, x \in \mathbb{R}^d} \frac{|\phi_{s,t}(x, \omega)|}{1 + |x| (\log^+ |x|)^\gamma}, \quad \sup_{0 \leq s, t \leq T, x \in \mathbb{R}^d} \frac{|D_x^\alpha \phi_{s,t}(x, \omega)|}{(1 + |x|)^\gamma},
\]

\[
\sup_{x \in \mathbb{R}^d} \sup_{0 \leq s, t \leq T, 0 < |x' - x| \leq \rho} \frac{|D_x^\alpha \phi_{s,t}(x, \omega) - D_x^\alpha \phi_{s,t}(x', \omega)|}{|x - x'|^\epsilon (1 + |x|)^\gamma},
\]

are finite. The random variables defined by the above expressions have \( p \)-th moments for all \( p \geq 1 \).
∥ · ∥_{k,ε} := C^{k,ε}-norm on $C^{k,ε}$ mappings $\bar{B}(0, \rho) \rightarrow \mathbb{R}^d$.

**Lemma 2**

Assume that $\log^+ |Y(\cdot)|$ is integrable. Then the cocycle $\phi$ satisfies

$$\int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \| \phi(t_2, Y(\theta(t_1, \omega)) + (\cdot), \theta(t_1, \omega)) \|_{k,\epsilon} dP(\omega) < \infty$$

(10)

for any fixed $0 < T, \rho < \infty$ and any $\epsilon \in (0, \delta)$. Furthermore, the linearized flow $(D_2 \phi(t, Y(\omega), \omega), \theta(t, \omega)), t \geq 0$, is an $L(\mathbb{R}^d)$-valued perfect cocycle and

$$\int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \| D_2 \phi(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega)) \|_{L(\mathbb{R}^d)} dP(\omega) < \infty$$

(11)

for any fixed $0 < T < \infty$. The forward cocycle

$(D_2 \phi(t, Y(\omega), \omega), \theta(t, \omega), t > 0)$ has a non-random finite Lyapunov spectrum $\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}$. Each Lyapunov exponent $\lambda_i$ has a non-random multiplicity $q_i, 1 \leq i \leq m,$ and $\sum_{i=1}^{m} q_i = d$. The backward linearized cocycle $(D_2 \phi(t, Y(\omega), \omega), \theta(t, \omega), t < 0)$ admits a “backward” non-random finite Lyapunov spectrum:

$$\lim_{t \to -\infty} \frac{1}{t} \log |D_2 \phi(t, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbb{R}^d),$$

taking values in $\{-\lambda_i\}_{i=1}^{m}$ with non-random multiplicities $q_i, 1 \leq i \leq m,$ and $\sum_{i=1}^{m} q_i = d.$

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The Auxiliary Cocycle

To apply Ruelle’s discrete non-linear ergodic theorem ([Ru.1], Theorem 5.1, p. 292), introduce the following auxiliary cocycle $Z : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$. This a “centering” of the flow $\phi$ about the stationary solution:

$$Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \quad (16)$$

for $t \in \mathbb{R}, x \in \mathbb{R}^d, \omega \in \Omega$.

**Lemma 3**

$(Z, \theta)$ is a perfect cocycle on $\mathbb{R}^d$ and $Z(t, 0, \omega) = 0$ for all $t \in \mathbb{R}$, and all $\omega \in \Omega$.

The proof of the local stable-manifold theorem (Theorem 1) uses a discretization argument that requires the following lemma.

**Lemma 4**

Suppose that $\log^+ |Y(\cdot)|$ is integrable. Then there is a sure event $\Omega_3 \in \mathcal{F}$ with the following properties:

(i) $\theta(t, \cdot)(\Omega_3) = \Omega_3$ for all $t \in \mathbb{R}$,

(ii) For every $\omega \in \Omega_3$ and any $x \in \mathbb{R}^d$, the statement

$$\limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)| < 0 \quad (17)$$
implies

\[
\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| = \limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)|. \quad (18)
\]
Ruelle’s Non-linear Ergodic Theorem

**Theorem 4 ([Ru.1], 1979)**

Let \( \Omega \ni \mapsto F_\omega \in C^{k,\epsilon}(\mathbb{R}^d,0;\mathbb{R}^d,0) \) be measurable such that \( E \log^+ \|F|\bar{B}(0,1)\|_{k,\epsilon} < \infty \). Set \( F^n(\omega) := F_{\theta(n-1,\omega)} \circ \cdots \circ F_{\theta(1,\omega)} \circ F_\omega \).

Suppose \( \lambda < 0 \) is not in the spectrum of the cocycle \((DF^n_\omega(0),\theta(n,\omega))\). Then there is a sure event \( \Omega_0 \in \mathcal{F} \) such that \( \theta(1,\cdot)(\Omega_0) \subseteq \Omega_0 \), and measurable functions \( 0 < \alpha(\omega) < \beta(\omega) < 1, \gamma(\omega) > 1 \) with the following properties:

(a) If \( \omega \in \Omega_0 \), the set

\[
V^\lambda_\omega := \{ x \in \bar{B}(0,\alpha(\omega)) : |F^n_\omega(x)| \leq \beta(\omega)e^{n\lambda} \text{ for all } n \geq 0 \}
\]

is a \( C^{k,\epsilon} \) submanifold of \( \bar{B}(0,\alpha(\omega)) \).

(b) If \( x_1, x_2 \in V^\lambda_\omega \), then

\[
|F^n_\omega(x_1) - F^n_\omega(x_2)| \leq \gamma(\omega)|x_1 - x_2|e^{n\lambda}
\]

for all integers \( n \geq 0 \). If \( \lambda' < \lambda \) and \([\lambda', \lambda]\) is disjoint from the spectrum of \((DF^n_\omega(0),\theta(n,\omega))\), then there exists a measurable \( \gamma'(\omega) > 1 \) such that

\[
|F^n_\omega(x_1) - F^n_\omega(x_2)| \leq \gamma'(\omega)|x_1 - x_2|e^{n\lambda'}
\]

for all \( x_1, x_2 \in V^\lambda_\omega \) and all integers \( n \geq 0 \).

**Proof**

[Ru.1], Theorem 5.1, p. 292.
Construction of the Stable/Unstable Manifolds

- Use auxiliary cocycle \((Z, \theta)\). Set \(\tau := \theta(1, \cdot) : \Omega \to \Omega\).

Define maps \(F_\omega, F^m_\omega : \mathbb{R}^d \to \mathbb{R}^d:\)

\[
F_\omega(x) := Z(1, x, \omega) \quad x \in \mathbb{R}^d
\]

\[
F^m_\omega := F_{\tau^{n-1}(\omega)} \circ \cdots \circ F_{\tau(\omega)} \circ F_\omega
\]

for all \(\omega \in \Omega\). Then cocycle property for \(Z\) gives \(F^m_\omega = Z(n, \cdot, \omega)\) for each \(n \geq 1\). \(F_\omega\) is \(C^{k, \epsilon}(\epsilon \in (0, \delta))\) and \(((DF_\omega)(0) = D_2\phi(1, Y(\omega), \omega))\).

- Integrability of the map \(\omega \mapsto \log^+ \|D_2\phi(1, Y(\omega), \omega)\|_{L(\mathbb{R}^d)}\) (Lemma 2) implies discrete cocycle \(((DF^m_\omega)(0), \theta(n, \omega), n \geq 0)\) has same non-random Lyapunov spectrum as that of linearized continuous cocycle \((D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)\), viz. \(\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}\), where each \(\lambda_i\) has fixed multiplicity \(q_i, 1 \leq i \leq m\) (Lemma 2).

- If \(\lambda_i > 0\) for all \(1 \leq i \leq m\), then take \(\tilde{S}(\omega) := \{Y(\omega)\}\) for all \(\omega \in \Omega\). Theorem is trivial in this case. Hence assume there is at least one \(\lambda_i < 0\).

- Use discrete non-linear ergodic theorem of Ruelle (Theorem 4) and its proof to obtain a sure event \(\Omega_i^* \in \mathcal{F}\) such that \(\theta(t, \cdot)(\Omega_i^*) = \Omega_i^*\) for all \(t \in \mathbb{R}\), \(\mathcal{F}\)-measurable positive random variables \(\rho_1, \beta_1 : \Omega_i^* \to (0, \infty), \rho_1 < \beta_1, \)
and a random family of $C^k, \epsilon \ (\epsilon \in (0, \delta))$ submanifolds of $\tilde{B}(0, \rho_1(\omega))$ denoted by $\tilde{S}_d(\omega), \omega \in \Omega_1^*$, and satisfying the following properties for each $\omega \in \Omega_1^*$: $\tilde{S}_d(\omega)$ is the set of all $x \in \tilde{B}(0, \rho_1(\omega))$ such that

$$|Z(n, x, \omega)| \leq \beta_1(\omega)e^{(\lambda_{i_0}+\epsilon_1)n}, \quad n \in \mathbb{Z}^+ \quad (21)$$

$\tilde{S}_d(\omega)$ is tangent at 0 to the stable subspace $S(\omega)$ of the linearized flow $D_2\phi$, viz. $T_0\tilde{S}_d(\omega) = S(\omega)$. Therefore $\dim \tilde{S}_d(\omega)$ is non-random by ergodicity of $\theta$. Also

$$\limsup_{n \to \infty} \frac{1}{n} \log \left[ \sup_{x_1 \neq x_2, x_1, x_2 \in \tilde{S}_d(\omega)} \frac{|Z(n, x_1, \omega) - Z(n, x_2, \omega)|}{|x_1 - x_2|} \right] \leq \lambda_{i_0}. \quad (22)$$

The $\theta(t, \cdot)$-invariant sure event $\Omega_1^* \in \mathcal{F}$ is constructed using the ideas in Ruelle’s proof (of Theorem 5.1 in [Ru.1], p. 293), combined with the estimate (10) of Lemma 2 and the subadditive ergodic theorem (Lemma 1 (ii)).

- For each $\omega \in \Omega_1^*$, let $\tilde{S}(\omega)$ be as defined in part (a) of the theorem. Then by definition of $\tilde{S}_d(\omega)$ and $Z$:

$$\tilde{S}(\omega) = \tilde{S}_d(\omega) + Y(\omega). \quad (23)$$

Since $\tilde{S}_d(\omega)$ is a $C^k, \epsilon \ (\epsilon \in (0, \delta))$ submanifold of $\tilde{B}(0, \rho_1(\omega))$, then $\tilde{S}(\omega)$ is a $C^k, \epsilon \ (\epsilon \in (0, \delta))$ submanifold of $\tilde{B}(Y(\omega), \rho_1(\omega))$. 

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Furthermore, $T_Y(\omega)\tilde{S}(\omega) = T_0\tilde{S}_d(\omega) = S(\omega)$. Hence $\dim \tilde{S}(\omega) = \dim S(\omega) = \sum_{i=i_0}^m q_i$, and is non-random.

- (22) implies that

$$
\limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)| \leq \lambda_{i_0} \quad (24)
$$

for all $\omega$ in $\Omega_1^*$ and all $x \in \tilde{S}_d(\omega)$. Lemma 4 implies there is a sure event $\Omega_2^* \subseteq \Omega_1^*$ such that $\theta(t, \cdot)(\Omega_2^*) = \Omega_2^*$ for all $t \in \mathbb{R}$, and

$$
\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| \leq \lambda_{i_0} \quad (25)
$$

for all $\omega \in \Omega_2^*$ and all $x \in \tilde{S}_d(\omega)$. Therefore (2) holds.

- To prove (b), let $\omega \in \Omega_1^*$. By (22), there is a positive integer $N_0 := N_0(\omega)$ (independent of $x \in \tilde{S}_d(\omega)$) such that $Z(n, x, \omega) \in B(0, 1)$ for all $n \geq N_0$. Let $\Omega_4^* := \Omega_2^* \cap \Omega_3$, where $\Omega_3$ is the shift-invariant sure event defined in the proof of Lemma 4. Then $\Omega_4^*$ is a sure event and $\theta(t, \cdot)(\Omega_4^*) = \Omega_4^*$ for all $t \in \mathbb{R}$. By cocycle property, Mean-Value theorem and the ergodic theorem (Lemma 1(i)), we get (b).

- To prove the invariance property (4), apply the Oseledec theorem to $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$. Get a sure
\(\theta(t, \cdot)\)-invariant event, also denoted by \(\Omega^*_1\), such that
\[ D_2 \phi(t, Y(\omega), \omega)(S(\omega)) \subseteq S(\theta(t, \omega)) \]
for all \(t \geq 0\) and all \(\omega \in \Omega^*_1\). Equality holds because \(D_2 \phi(t, Y(\omega), \omega)\) is injective and \(\dim S(\omega) = \dim S(\theta(t, \omega))\) for all \(t \geq 0\) and all \(\omega \in \Omega^*_1\).

- To prove the asymptotic invariance property (3), use ideas from Ruelle’s Theorems 5.1 and 4.1 in [Ru.1], to pick random variables \(\rho_1, \beta_1\) and a sure event (also denoted by) \(\Omega^*_1\) such that \(\theta(t, \cdot)(\Omega^*_1) = \Omega^*_1\) for all \(t \in \mathbb{R}\), and for any \(\epsilon \in (0, \epsilon_1)\) and every \(\omega \in \Omega^*_1\), there exists a positive \(K_1^\epsilon(\omega)\) for which the inequalities

\[
\begin{align*}
\rho_1(\theta(t, \omega)) &\geq K_1^\epsilon(\omega) \rho_1(\omega) e^{(\lambda_{i_0} + \epsilon)t}, \\
\beta_1(\theta(t, \omega)) &\geq K_1^\epsilon(\omega) \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon)t}
\end{align*}
\]

(26)

hold for all \(t \geq 0\). Use (b) to obtain a sure event \(\Omega^*_5 \subseteq \Omega^*_4\) such that \(\theta(t, \cdot)(\Omega^*_5) = \Omega^*_5\) for all \(t \in \mathbb{R}\), and for any \(0 < \epsilon < \epsilon_1\) and \(\omega \in \Omega^*_4\), there exists \(\beta_5^\epsilon(\omega) > 0\) (independent of \(x\)) with

\[
|\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \beta_5^\epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t}
\]

(27)

for all \(x \in \tilde{S}(\omega), t \geq 0\). Fix \(t \geq 0, \omega \in \Omega^*_5\) and \(x \in \tilde{S}(\omega)\). Let \(n\) be a non-negative integer. Then the cocycle
property and (27) imply that

\[ |\phi(n, \phi(t, x, \omega), \theta(t, \omega)) - Y(\theta(n, \theta(t, \omega)))| \]
\[ = |\phi(n + t, x, \omega) - Y(\theta(n + t, \omega))| \]
\[ \leq \beta \epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)(n + t)} \]
\[ \leq \beta \epsilon(\omega) e^{(\lambda_{i_0} + \epsilon)t} e^{(\lambda_{i_0} + \epsilon_1)n}. \] (28)

If \( \omega \in \Omega_*^5 \), then it follows from (26), (27), (28) and the definition of \( \tilde{S}(\theta(t, \omega)) \) that there exists \( \tau_1(\omega) > 0 \) such that \( \phi(t, x, \omega) \in \tilde{S}(\theta(t, \omega)) \) for all \( t \geq \tau_1(\omega) \). This proves asymptotic invariance.

- Prove (d), the existence of the local unstable manifolds \( \tilde{U}(\omega) \), by running both the flow \( \phi \) and the shift \( \theta \) backward in time getting the cocycle \( (\tilde{Z}(t, \cdot, \omega), \tilde{\theta}(t, \omega), t \geq 0) \):

\[ \tilde{\phi}(t, x, \omega) := \phi(-t, x, \omega), \quad \tilde{Z}(t, x, \omega) := Z(-t, x, \omega), \quad \tilde{\theta}(t, \omega) := \theta(-t, \omega) \]

for all \( t \geq 0, \omega \in \Omega \). The linearized flow
\( (D_2 \tilde{\phi}(t, Y(\omega), \omega), \tilde{\theta}(t, \omega), t \geq 0) \) is an \( L(\mathbb{R}^d) \)-valued perfect cocycle with a non-random finite Lyapunov spectrum
\( \{-\lambda_1 < -\lambda_2 < \cdots < -\lambda_i < -\lambda_{i+1} < \cdots < -\lambda_m\} \) where
\( \{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\} \) is the Lyapunov spectrum of the forward linearized flow.
\( (D_2 \phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0) \). Apply first part of the proof to get stable manifolds for the backward flow \( \tilde{\phi} \) satisfying assertions (a), (b), (c). This gives unstable manifolds for the original flow \( \phi \), and (d), (e), (f) automatically hold.

- Measurability of the stable manifolds follows from the representations:

\[
\tilde{S}(\omega) = Y(\omega) + \tilde{S}_d(\omega)
\]

\[
\tilde{S}_d(\omega) = \lim_{n \to \infty} \bar{B}(0, \rho_1(\omega)) \cap \bigcap_{i=1}^{n} f_i(\cdot, \omega)^{-1}(\bar{B}(0, 1))
\]

\[
f_i(x, \omega) := \beta_1(\omega)^{-1} e^{-(\lambda_{i0} + \epsilon_1)i} Z(i, x, \omega), \quad x \in \mathbb{R}^d, \omega \in \Omega^*_1,
\]

for all integers \( i \geq 0 \). (Above limit is taken in the metric \( d^* \) on \( \mathcal{C}(\mathbb{R}^d) \).) Use joint continuity of translation and measurability of \( Y, f_i, \rho_1 \), finite intersections and the continuity of the maps

\[
\mathbb{R}^+ \ni r \mapsto \bar{B}(0, r) \in \mathcal{C}(\mathbb{R}^d).
\]

\[
\text{Hom}(\mathbb{R}^d) \ni f \mapsto f^{-1}(\bar{B}(0, 1)) \in \mathcal{C}(\mathbb{R}^d).
\]

- For \( h, g_i \) in \( C_b^\infty \), can adapt above argument to give a sure event in \( \mathcal{F} \), also denoted by \( \Omega^* \) such that \( \tilde{S}(\omega), \tilde{U}(\omega) \) are \( C^\infty \) for all \( \omega \in \Omega^* \). \( \square \)
Examples of Stationary Solutions

1. Fixed points:

\[ d\phi(t) = h(\phi(t)) \, dt + \sum_{i=1}^{m} g_i(\phi(t)) \circ dW_i(t) \]

\[ h(x_0) = g_i(x_0) = 0, \quad 1 \leq i \leq m \]

Take \( Y(\omega) = x_0 \) for all \( \omega \in \Omega \).

2. Linear affine case \( d = 1 \):

\[ d\phi(t) = \lambda \phi(t) \, dt + dW(t) \]

\( \lambda > 0 \) fixed, \( W(t) \in \mathbb{R} \). Take

\[ Y(\omega) := -\int_{0}^{\infty} e^{-\lambda u} \, dW(u), \]

\[ \theta(t, \omega)(s) = \omega(t+s) - \omega(t). \]

Check that \( \phi(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \), using integration by parts and variation of parameters.

3. Affine linear SDE in \( d = 2 \):

\[ d\phi(t) = A\phi(t) \, dt + GdW(t) \]

with \( A \) a fixed hyperbolic \( 2 \times 2 \)-diagonal matrix; \( G \) a constant matrix.

4. Non-linear transforms of (3) under a global diffeomorphism.
References


