2006

Bass Series for Small Witt Rings

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Published in *Communications in Algebra*, 34, 1753-1762.

BASS SERIES FOR SMALL WITT RINGS

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Throughout R is a finitely generated (abstract) Witt ring . We will usually assume $I^3R = 0$.

Our interest in Ext arises from a desire to examine combinatorial techniques coming from ring theory. The two principal objects of study for a local ring (A, m, k), are $\text{Ext}_A(k, k)$ and $\text{Ext}_A(k, A)$. The dimension of $\text{Ext}^n_A(k, k)$ is the rank of the n^{th} free module in a minimal free resolution of k. If A is also Artinian then every finitely generated injective A-module I is, by [8], the direct sum of $\mu(I)$ many copies of $E(k)$, the injective hull of k . The dimension of $\text{Ext}^n_A(k, A)$ is μ of the n^{th} injective module in a minimal injective resolution of A.

We often work with the generating functions for these dimensions. Specifically, set:

$$P_A(t) = \sum_{i \geq 0} (\dim \text{Ext}^i_A(k, k)) t^i$$

$$I_A(t) = \sum_{i \geq 0} (\dim \text{Ext}^i_A(k, A)) t^i$$

$$H_A(t) = \sum_{i \geq 0} \dim(m^i / m^{i+1}) t^i.$$

Here $P_A(t), I_A(t)$ and $H_A(t)$ are respectively the Poincaré series, the Bass series and the Hilbert series of A. We note that for Artinian A, the Hilbert series is in fact a polynomial. But it is not the Hilbert polynomial. Also, to avoid confusion with the Bass series we will write the fundamental ideal of a Witt ring R as IR instead of the usual I_R.

1. Elementary type case.

Lemma 1.1.

(1) *Suppose $I^3S = 0, I^3T = 0$ and $R = S \sqcap T$. Then:*

$$I_R(t) = \frac{I_S(t)H_S(-t) + I_T(t)H_T(-t) + t}{H_R(-t)}$$

(2) *Let S be any local Artinian Witt ring and let $R = S[E_1]$. Then $I_R(t) = I_S(t)$.*

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Proof. (1) Lescot [7] gives:

$$\frac{I_R(t)}{P_R(t)} = \frac{I_S(t)}{P_S(t)} + \frac{I_T(t)}{P_T(t)} + t$$

and [4] gives $P_W(t)^{-1} = H_W(-t)$ for any Witt ring with $I^3W = 0$.

(2) We apply Foxby-Thorup [5], since S is a free (hence flat) R-module. In their notation, $Q = IS$, $C = S/(IS \cap R)S = S/IR \cdot S \cong k[E_1]$ and $Q' = QC = IS/IR \cdot S \cong \{0, 1+e\}$, where $E_1 = \{1, e\}$. We obtain:

$$\mu_S^n(IS, S) = \sum_{i+j=n} \mu_C^i(E_1, C)\mu_R^j(IR, R)$$
$$I_S(t) = I_C(t)I_R(t).$$

Now C is Gorenstein (by Bass' criterion [2]) so $I_C(t) = 1$ and $I_S(t) = I_R(t)$. □

We note that (1.1)(1) also holds for S and T local Witt rings of elementary type, since again $P(t)^{-1} = H(-t)$, by [4]. In fact, (1.1)(1) should be stated for any Fröberg Witt ring and (1.1)(2) for any Witt ring. Here I assume always that R is non-real and finitely-generated (equivalently, local Artinian).

Lemma 1.2.

(1) *If R is of local type then $I_R(t) = 1$.*
(2) *If R is of quasi-local type, say $R = D_n[E_1]$, then:*

$$I_R(t) = \frac{n-t}{1-nt}.$$

Proof. (1) R is Gorenstein by [3], hence self-injective. Thus $\dim \mathrm{Ext}^0(k, R) = 1$ and $\dim \mathrm{Ext}^n(k, R) = 0$, for all $n > 0$.

(2) Let $S = \mathbb{Z}_4$ or $\mathbb{Z}_2[E_1]$. Then S is Gorenstein and again $I_S(t) = 1$. Now D_n is a product of n copies of S, so an easy induction argument using (1.1)(1) shows $I_{D_n}(t) = (n-t)/(1-nt)$. The result then follows from (1.1)(2). □

We introduce the following notation:

$$g = \dim IR/I^2R$$
$$h = \dim I^2R.$$

For an infinite series $p = \sum a_i t^i, q = \sum b_i t^i$ we write $p \geq q$ if $a_i \geq b_i$ for all i.

Set $e_n = \dim \mathrm{Ext}_R^n(k, k)$. When $I^3R = 0$ the Fröberg relation yields:

$$e_n = \sum_{i=0}^{[n/2]} (-1)^i \binom{n-i}{i} g^{n-2i} h^i.$$

For convenience, set $e_n = 0$ if $n < 0$. Also note that $e_0 = 1$ and $e_1 = g$.

Theorem 1.3. *Suppose $I^3R = 0$ and R is non-degenerate of elementary type. Let k denote $\mathbb{Z}/2\mathbb{Z}$.*

(1) $\dim \operatorname{Ext}^0(k, R) = h$.
(2) $\dim \operatorname{Ext}^n(k, R) \geq (h-1)(ge_{n-1} - (h+1)e_{n-2})$, *for all* $n \geq 1$.
(3)
$$I_R(t) \geq \frac{h - gt + t^2}{1 - gt + ht^2}.$$
(4) *The following are equivalent:*

(a) R is indecomposable.
(b) Equality holds in (3).
(c) Equality holds in (2) for all $n \geq 1$.
(d) Equality holds in (2) for some $n \geq 1$.

Proof. (1) $\operatorname{Ext}^0(k, R) = \operatorname{Hom}(k, R) \cong \operatorname{ann} IR = I^2R$, since R is non-degenerate. (2) and (3) are equivalent, as a simple computation shows. We prove (3). If R is of local type then:
$$I_R(t) = 1 = \frac{h - gt + t^2}{1 - gt + ht^2},$$
since $h = 1$. If R is quasi-local, say $R = D_n[E_1]$, then:
$$\begin{aligned} I_R(t) &= \frac{n-t}{1-nt} \\ &= \frac{n-t}{1-nt} \cdot \frac{1-t}{1-t} \\ &= \frac{n-(n+1)t+t^2}{1-(n+1)+nt^2} = \frac{h-gt+t^2}{1-gt+ht^2}, \end{aligned}$$
as desired. We note that this computation also proves (a)→(b) in (4) also. Finally, if $R = S \sqcap T$ then by induction:
$$\begin{aligned} I_R(t) &= \frac{I_S(t)H_S(-t) + I_T(t)H_T(-t) + t}{H_R(-t)} \\ &\geq \frac{(h_S - g_S t + t^2) + (h_T - g_T t + t^2) + t}{H_R(-t)} \\ &= \frac{h - (g-1)t + 2t^2}{H_R(-t)} \\ &\geq \frac{h - gt + t^2}{1 - gt + ht^2} \end{aligned}$$

For (4), we have already shown (a)→(b) and that (b)↔(c). Clearly (c)→(d). Thus it suffices to prove (d) implies (a). We will show the contrapositive. Suppose $R = S \sqcap T$. Then as above:
$$I_R(t) \geq \frac{h - (g-1)t + 2t^2}{H_R(-t)}.$$

Once again using the Fröberg relation $1/H_R(-t) = P_R(t)$ we get for $n > 0$:

$$\dim \operatorname{Ext}^n(k, R) \geq he_n - (g-1)e_{n-1} + 2e_{n-2}.$$

Now comparing coefficients in $(1 - gt + ht^2)P_R(t) = 1$ yields:

$$e_n - ge_{n-1} + he_{n-2} = 0 \qquad \text{for} \quad n > 0$$
$$e_n = ge_{n-1} - he_{n-2}.$$

Thus:

$$\begin{aligned}\dim \operatorname{Ext}^n(k, R) &\geq hge_{n-1} - (g-1)e_{n-1} - h^2 e_{n-2} + 2e_{n-2} \\ &= (h-1)ge_{n-1} - (h^2-1)e_{n-2} + ge_{n-1} + e_{n-2} \\ &= (h-1)(ge_{n-1} - (h+1)e_{n-2}) + ge_{n-1} + e_{n-2}\end{aligned}$$

Thus we are done if we show $ge_{n-1} + e_{n-2} > 0$ for every $n > 0$. If not, then for some $n > 0$ we have $e_{n-1} = e_{n-2} = 0$. Since $e_m = ge_{m-1} + he_{m-2}$ for all $m > 0$ we see that $e_m = 0$ for all $m \geq n-2$. But then $P_R(t)$ is a polynomial which is impossible as $P_R(t)H_R(-t) = 1$. □

We note that (1.3)(1) holds for any non-degenerate Witt ring with $I^3R = 0$ since the proof used only these two facts. The next section will show (1.3)(2) also holds for such R as does part of (4).

We also note that since (1.1)(1) holds for all elementary type Witt rings (even if $I^3R \neq 0$), it is possible to compute $I_R(t)$ for any R of elementary type. In place of (1.3)(4) one can show that if $I^{n+1}R = 0$:

$$\sum_{i=0}^{[n-1/2]} \sum_{j+k=2i+1} (-1)^k (\dim I^k R/I^{k+1}R)(\dim \operatorname{Ext}^j(k, R)) \geq 0$$

with equality iff R is indecomposable. I suggest replacing (1.3) with this result for Fröberg Witt rings.

2. General case.

Suppose

$$\xrightarrow{d_n} R^{e_n} \longrightarrow \cdots \xrightarrow{d_1} R^{e_1} \xrightarrow{d_0} R \xrightarrow{\varepsilon} k \longrightarrow 0$$

is a minimal free resolution (here again $e_n = \dim \operatorname{Ext}^n_R(k, k)$). For $n \geq 1$ set $F_n = R^{e_n}$ and set $F_0 = R$ and $F_{-1} = k$. Also for $n \geq 1$ set $Z_n = \ker d_{n-1} \subset F_n$ with $Z_0 = IR$ and $Z_{-1} = k$. We note that, by the construction of a minimal resolution, that e_{n+1} is the size of a minimal generating set for Z_n. The following is standard.

Lemma 2.1. $\operatorname{Ext}^{n+2}(k, R) = \operatorname{Ext}^1(Z_n, R)$, *for all* $n \geq 0$.

Proof. We show by induction that $\operatorname{Ext}^1(Z_n, R) = \operatorname{Ext}^{j+1}(Z_{n-j}, R)$ for $0 \leq j \leq n+1$. This is clear for $j = 0$. For $j > 0$ we have the exact sequence:

$$0 \longrightarrow Z_{n-j} \longrightarrow R^{e_{n-j}} \overset{d_{n-j-1}}{\longrightarrow} Z_{n-j-1} \longrightarrow 0.$$

R^{e_n} is free so $\operatorname{Ext}^m(R^{e_n}, R) = 0$ for all $m > 0$. The induced long exact sequence is:

$$0 \longrightarrow \operatorname{Ext}^{j+1}(Z_{n-j}, R) \xrightarrow{\delta} \operatorname{Ext}^{j+2}(Z_{n-(j+1)}, R) \longrightarrow 0.$$

□

Proposition 2.2. *Suppose that $I^3R = 0$ and R is non-degenerate. Then:*

(1) $\dim \mathrm{Ext}^n(k, R) \geq (h-1)(ge_{n-1} - (h+1)e_{n-2})$ *for all* $n \geq 1$.

(2)
$$I_R(t) \geq \frac{h - gt + t^2}{1 - gt + ht^2}.$$

Proof. A simple computation shows that (1) and (2) are equivalent. We again have the short exact sequence:

$$0 \longrightarrow Z_{n-1} \longrightarrow R^{e_{n-1}} \longrightarrow Z_{n-2} \longrightarrow 0$$

which yields:

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}(Z_{n-2}, R) \longrightarrow \mathrm{Hom}(R^{e_{n-1}}, R) \longrightarrow \mathrm{Hom}(Z_{n-1}, R)\\ \longrightarrow \mathrm{Ext}^1(Z_{n-2}, R) \longrightarrow \mathrm{Ext}^1(R^{e_{n-1}}, R) \equiv 0\end{aligned}$$

We get:

$$\dim \mathrm{Ext}^1(Z_{n-2}, R) = \ell(\mathrm{Hom}(Z_{n-2}, R)) - e_{n-1}(1+g+h) + \ell(\mathrm{Hom}(Z_{n-1}, R)) \tag{2.3}$$

.

Claim. $\ell(\mathrm{Hom}(Z_m, R)) \geq he_{m+1} + e_m$.

When $m = -1$, $Z_m = k$ so that $\ell(\mathrm{Hom}(Z_m, R)) = h$. Since $e_{-1} = 0$ and $e_0 = 1$, the **Claim** is proven in this case. Suppose $m \geq 0$. Now Z_m has e_{m+1} elements in a minimal generating set. So $\dim(\mathrm{Hom}(Z_m, I^2R)) = he_{m+1}$. Further, let $p(Z_m, R)$ denote the subgroup generated by the e_m many projections $R^{e_m} \to R$ restricted to Z_m. We assert that $\mathrm{Hom}(Z_m, I^2R) \cap p(Z_m, R) = 0$. Suppose instead that p is a non-zero member of the intersection. Write $p = \sum \pi_i|_{Z_m}$, where π_i denotes the projection of R^{e_m} onto its ith coordinate. We may assume π_1 is one of the summands of p. Now $I^2R \cdot F_m \subset IR \cdot Z_m$ by [4, 3.9,2.5]. Let σ be a non-zero element of I^2R, and set $x = (\sigma, 0, \dots, 0) \in I^2R \cdot F_m$. Then $p(x) = \sigma \neq 0$. But since $p \in \mathrm{Hom}(Z_m, I^2R)$, we have $p(x) \in p(I^2R \cdot F_m) \subset p(IR \cdot Z_m) \subset I^3R = 0$, a contradiction. This proves the assertion.

We thus have:

$$\mathrm{Hom}(Z_m, I^2R) \oplus p(Z_m, R) \subset \mathrm{Hom}(Z_m, R).$$

So $\ell(\mathrm{Hom}(Z_m, R) \geq he_{m+1} + e_m$,proving the **Claim**. Plugging into (2.3) yields:

$$\begin{aligned}\dim \mathrm{Ext}^n(k, R) &= \dim \mathrm{Ext}^1(Z_{n-2}, R)\\ &\geq he_{n-1} + e_{n-2} - e_{n-1}(1+g+h) + he_n + e_{n-1}\\ &= e_{n-2} - ge_{n-1} + he_n.\end{aligned}$$

Now from the Fröberg relation [4,3.9] $P_R(t)H_R(-t) = 1$ we have $e_n = ge_{n-1} - he_{n-2}$. So:

$$\begin{aligned}\dim \mathrm{Ext}^n(k, R) &\geq e_{n-2} - ge_{n-1} + ghe_{n-1} - h^2e_{n-2}\\ &= (h-1)(ge_{n-1} - (h+1)e_{n-2}).\end{aligned}$$

□

We remark that when $n = 1$ (2.2) says:

$$\dim \operatorname{Ext}^1(k, R) \geq (h-1)g,$$

since $e_0 = 1$ and $e_{-1} = 0$. We are unable to show equality holds in (2.2) iff R is indecomposable, except in this case of $n = 1$.

Probably should state that, by the proof, equality holds in (2.2)(1) iff

$$\text{(Em)} \qquad \operatorname{Hom}(Z_m, R) = \operatorname{Hom}(Z_m, I^2R) \oplus p(Z_m, R)$$

for $m = n-1, n-2$.

Corollary 2.4. *Suppose that $I^3R = 0$. Let K be the Koszul complex on a minimal generating set for IR and let $c_i = \dim_k \operatorname{Hom}_i(K)$. Then:*

$$\frac{h - gt + t^2}{1 - gt + ht^2} \leq \frac{\sum_{i=1}^{g-1} c_{g-i} t^i - t^{g+1}}{1 - \sum\_i = 1^g c_i t^{i+1}}.$$

Moreover, equality never occurs.

Proof. This inequality is (2.2) combined with the inequality of [1]. If equality holds then $I_R(t)$ is the common value, hence R is a Golod ring. Golod's result [6] says:

$$P_R(t) = \frac{(1+t)^g}{1 - \sum_{i=1}^{g} c_i t^{i+1}}.$$

But by [4, 3.9] R is a Fröberg ring, that is,

$$P_R(t) = \frac{1}{1 - gt + ht^2}.$$

Thus:

$$1 - c_1t^2 - \cdots - c_g t^{g+1} = (1+t)^g(1 - gt + ht^2),$$

which is impossible by a comparison of degrees. □

Remark. This is optional. We can give the first four terms of the inequality in (2.4) (assuming $g \geq 4$).

$$\begin{aligned} h &\leq c_g \\ g(h-1) &\leq c_{g-1} \\ (h-1)(g^2 - h - 1) &\leq c_g + c_{g-2} \\ g(h-1)(g^2 - 2h - 1) &\leq c_g c_2 + c_1 c_{g-1} + c_{g-3}. \end{aligned}$$

The first is always equality but already the second is not (ever?).

Lemma 2.5. *Suppose $D\langle 1,-a\rangle \subset D\langle 1,-c\rangle$, $D\langle 1,-b\rangle \subset D\langle 1,-d\rangle$ and $D\langle 1,-ab\rangle \subset D\langle 1,-cd\rangle$. Then $c \in D\langle 1,-bd\rangle$ and $d \in D\langle 1,-ac\rangle$.*

Proof. Note that $ac \in D\langle 1,-c\rangle$,$bd \in D\langle 1,-d\rangle$ and $abcd \in D\langle 1,-cd\rangle$. So:

$$\langle\langle -c,-bd\rangle\rangle \simeq \langle\langle -c,-abcd\rangle\rangle \simeq \langle\langle -d,-abcd\rangle\rangle \simeq \langle\langle -d,-ac\rangle\rangle.$$

Thus $c \in D\langle 1,-bd\rangle$ iff $d \in D\langle 1,-ac\rangle$.

Now since $-a \in D\langle 1,-c\rangle$,$-ab \in D\langle 1,-cd\rangle$ and $-bD\langle 1,-d\rangle$ we have:

$$\langle\langle -c,-b\rangle\rangle \simeq \langle\langle -c,ab\rangle\rangle \simeq \langle\langle -d,ab\rangle\rangle \simeq \langle\langle -d,-a\rangle\rangle.$$

By linkage, there exists a t such that:

$$\langle\langle -b,-c\rangle\rangle \simeq \langle\langle -b,-t\rangle\rangle \simeq \langle\langle -d,-t\rangle\rangle \simeq \langle\langle -d,-a\rangle\rangle.$$

In particular, $ct \in D\langle 1,-b\rangle$ and $t \in D\langle 1,-bd\rangle$. Now $D\langle 1,-b\rangle \subset D\langle 1,-b\rangle \cap D\langle 1,-d\rangle \subset D\langle 1,-bd\rangle$. Thus $c \in D\langle 1,-bd\rangle$. □

Theorem 2.6. *Suppose $I^3R = 0$ and R is non-degenerate. Then R is indecomposable iff $\dim \operatorname{Ext}^1(k,R) = (h-1)g$.*

Proof. We need only Em for $m = -1, 0$ by ? and the case $m = -1$ is clear as $Z_{-1} = k$ implies $\operatorname{Hom}(Z_{-1},R) = \operatorname{Hom}(Z_{-1},I^2R)$. Moreover, $Z_0 = IR$ so $p(Z_0,R) = id_{IR}$. So $\dim \operatorname{Ext}^1(k,R) = (h-1)g$ iff $\operatorname{Hom}(IR,R) = \{0, id_{IR}\} + \operatorname{Hom}(IR,I^2R)$.

Suppose first that $R = S \sqcap T$. Then $IR = IS \oplus IT$ so that the projection $IR \to IS$ is in $\operatorname{Hom}(IR,R)$ but not in $\{0, id_{IR}\} + \operatorname{Hom}(IR,I^2R)$. Thus $\dim \operatorname{Ext}^1(k,R) > (h-1)g$.

Now suppose that $\dim \operatorname{Ext}^1(k,R) > (h-1)g$. Then there exists an $\alpha \in \operatorname{Hom}(IR,R)$ such that $\alpha \notin \{0, id_{IR}\} + \operatorname{Hom}(IR,I^2R)$. Fix a basis $a_1,\ldots,a_g$ of G. If $\sigma \in I^2R$ then $0 = \alpha(\sigma\langle 1,-a_i\rangle) = \sigma \cdot \alpha(\langle 1,-a_i\rangle)$. Thus $\alpha(\langle 1,-a_i\rangle) \in IR$. Write:

$$\alpha(\langle 1,-a_i\rangle) = \langle 1,-b_i\rangle + \sigma_i,$$

for some $b_i \in G$ and $\sigma_i \in I^2R$. Now $\beta : IR \to I^2R$ by $\beta(\langle 1,-a_i\rangle) = \sigma_i$ is a well-defined homomorphism and $\alpha - \beta$ is still not in $\{0, id_{IR}\} + \operatorname{Hom}(IR,I^2R)$. We may thus assume that $\alpha(\langle 1,-a_i\rangle) = \langle 1,-b_i\rangle$, for $1 \le i \le g$. Note that some $b_i \ne 1$ as $\alpha \ne 0$. Let H be the subgroup of G generated by the b_i; then $H \ne 1$. Let K be the subgroup of G generated by the a_ib_i. Note that as $\alpha \ne id_{IR}$, for some i we have $a_i \ne b_i$ and so $K \ne 1$. Also, clearly each $a_i \in HK$ so that $G = HK$.

Claim. If $x \in H$ and $y \in K$ then $x \in D\langle 1,-y\rangle$.

For $1 \le i \le g$, if $c \in D\langle 1,-a_i\rangle$ then $0 = \alpha(\langle\langle -a_i,-c\rangle\rangle) = \langle 1,-c\rangle\alpha(\langle 1,-a_i\rangle) = \langle\langle -c,-b_i\rangle\rangle$. Thus $D\langle 1,-a_i\rangle \subset D\langle 1,-b_i\rangle$. Let $j \ne i$ and $1 \le j \le g$. Then $D\langle 1,-a_j\rangle \subset D\langle 1,-b_j\rangle$ also.

If $c \in D\langle 1,-a_ia_j\rangle$ then $\langle\langle -c,-a_i\rangle\rangle \simeq \langle\langle -c,-a_j\rangle\rangle$. Applying α yields $\langle\langle -c,-b_i\rangle\rangle \simeq \langle\langle -c,-b_j\rangle\rangle$. Thus $c \in D\langle 1,-b_ib_j\rangle$ and we have $D\langle 1,-a_ia_j\rangle \subset D\langle 1,-b_ib_j\rangle$. Then (2.5) gives $b_i \in D\langle 1,-a_jb_j\rangle$ for all $j \ne i$. Also $D\langle 1,-a_i\rangle \subset D\langle 1,-b_i\rangle$ implies that $b_i \in D\langle 1,a_i\rangle\cap$

$D\langle 1, b_i\rangle \subset D\langle 1, -a_i b_i\rangle$. Hence every b_i is in $D\langle 1, -y\rangle$ for every $y \in K$. So if $x \in H$ and $y \in K$ then $x \in D\langle 1, -y\rangle$, proving **Claim**.

To show we have a decomposition of R, we need to show $D\langle 1, -xy\rangle = D\langle 1, -x\rangle \cap D\langle 1, -y\rangle$ for all $x \in H$ and $y \in K$. Let $t = \prod a_{i_j} \in D\langle 1, -xy\rangle$. Set $t' = \prod b_{i_j}$, over the same indices as t. Then $0 = \alpha(\langle\langle -t, -xy\rangle\rangle) = \langle\langle -t', -xy\rangle\rangle$. Now $t' \in H$ so $y \in D\langle 1, -t'\rangle$. Thus $x = xy \cdot y \in D\langle 1, -t'\rangle$. Further, $tt' \in K$ and so $x \in D\langle 1, -tt'\rangle$. Thus $x \in D\langle 1, -t\rangle$, $t \in D\langle 1, -x\rangle$ and, as $t \in D\langle 1, -xy\rangle$, $t \in D\langle 1, -y\rangle$ also. Hence R decomposes. □

Remark. Here is a proof that if R decomposes then

$$\dim \operatorname{Ext}^n(k, R) > e_{n-2} - g e_{n-1} + h e_n,$$

for all $n \geq 0$. Namely, let π be the projection of Z_{n-1} onto the first coordinate. If $R = S \sqcap T$ then $IR = IS \oplus IT$. Let σ be the projection of IR onto IS. Set $\varphi = \sigma\pi$. Then $\varphi \in \operatorname{Hom}(Z_{n-1}, R)$. We check that $\varphi \notin \operatorname{Hom}(Z_{n-1}, I^2R) \oplus p(Z_{n-1}, R)$. Pick non-zero $\alpha \in I^2S$ and $\beta \in I^2T$ and let $\gamma = (\alpha, \beta) \in I^2R$. As in (2.2) $\gamma v_1 \in Z_{n-1}$, where $v_1 = (1, 0, \dots, 0)$. Now $\varphi(\gamma v_1) = \alpha$. But if $\psi \in \operatorname{Hom}(Z_{n-1}, I^2R)$ then $\psi(\gamma v_1) \in IR\psi(Z_{n-1}) = 0$. And if $\psi \in p(Z_{n-1}, R)$ is a combination of projections then $\psi(\gamma v_1) = \gamma$ or 0. Hence $\varphi \notin \operatorname{Hom}(Z_{n-1}, I^2R) \oplus p(Z_{n-1}, R)$. The result follows from the, as yet unwritten, corollary to (2.2).

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