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Research Article

An *L*-Moment-Based Analog for the Schmeiser-Deutsch Class of Distributions

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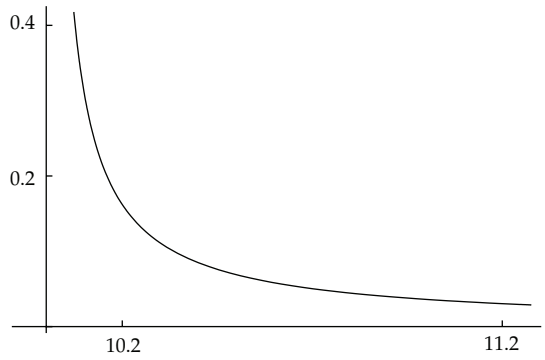
This paper characterizes the conventional moment-based Schmeiser-Deutsch (S-D) class of distributions through the method of *L*-moments. The system can be used in a variety of settings such as simulation or modeling various processes. A procedure is also described for simulating S-D distributions with specified *L*-moments and *L*-correlations. The Monte Carlo results presented in this study indicate that the estimates of *L*-skew, *L*-kurtosis, and *L*-correlation associated with the S-D class of distributions are substantially superior to their corresponding conventional product-moment estimators in terms of relative bias—most notably when sample sizes are small.

1. Introduction

The conventional moment-based Schmeiser-Deutsch (S-D) [1] class of distributions has demonstrated to be useful for modeling or simulating phenomena in the contexts of operations research and industrial engineering. Some examples include modeling stochastic inventory processes and lead time distributions [2–7], unpaced line efficiency [8, 9], two-stage production systems [10], stochastic activity networks [11, 12], and the newsboy problem [13]. Further, it is also common practice for methodologists to investigate the Type I error and power properties associated with inferential statistics (e.g., [14]). In many cases, these investigations may only require an elementary transformation to produce distributions with specified values of conventional skew, kurtosis, and Pearson correlations (or Gaussian copulas). The S-D class of distributions is particularly well suited for this task as it is computationally efficient because it only requires the knowledge of four parameters and an algorithm that generates zero-one uniform pseudorandom deviates.



$\gamma_1 = 10, \gamma_2 = 10000, \gamma_3 = 15, \gamma_4 = 0.45$

Skew: $\alpha_3 = 4.762$			Kurtosis: $\alpha_4 = 24.11$		
$\widehat{\alpha}_3$	95% C.I.	SE	$\widehat{\alpha}_4$	95% C.I.	SE
3.452	3.433, 3.470	0.00955	14.75	14.66, 14.83	0.04277
L-skew: $\tau_3 = 0.8048$			L-kurtosis: $\tau_4 = 0.7156$		
$\widehat{\tau}_3$	95% C.I.	SE	$\widehat{\tau}_4$	95% C.I.	SE
0.7308	0.7275, 0.7342	0.00172	0.7565	0.7565, 0.7600	0.00089

Figure 1: A graph of an S-D distribution based on the pdf in (2.4). The values of skew and kurtosis were determined based on (A.2) for α_3 and α_4 in the Appendix. The values of τ_3 and τ_4 were determined based on (3.4) and (3.5). The estimates $(\widehat{\alpha}_{3,4}; \widehat{\tau}_{3,4})$ and bootstrap confidence intervals (C.I.s) were based on resampling 25,000 statistics. Each sample statistic was based on a sample size of $n = 25$.

Specifically, the quantile function for generating S-D distributions can be succinctly described as in [1]:

$$X = q(U) = \begin{cases} \gamma_1 - \gamma_2(\gamma_4 - U)^{\gamma_3}, & \text{for } U \leq \gamma_4, \\ \gamma_1 + \gamma_2(U - \gamma_4)^{\gamma_3}, & \text{for } U > \gamma_4, \end{cases} \tag{1.1}$$

where U is zero-one uniformly distributed. The values of $\gamma_1 \in (-\infty, +\infty)$ and $\gamma_2 \geq 0$ are the location and scale parameters, while $\gamma_3 \geq 0$ and $\gamma_4 \in [0, 1]$ are the shape parameters that determine the skew and kurtosis. The system of equations for determining the parameters in (1.1) for a S-D distribution with prespecified values of mean, variance, skew, and kurtosis is given in the Appendix. Figure 1 gives an example of an S-D distribution.

There are problems associated with conventional moment-based estimators (e.g., skew and kurtosis $\widehat{\alpha}_{3,4}$ in Figure 1) insofar as they can be substantially biased, have high variance, or can be influenced by outliers. For example, inspection of Figure 1 indicates, on average, that the estimates of $\widehat{\alpha}_{3,4}$ attenuate 27.51% and 38.82% below their associated population parameters. Note that each estimate of $\widehat{\alpha}_{3,4}$ in Figure 1 was calculated based on samples of size $n = 25$ and the formulae currently used by most commercial software packages such as SAS, SPSS, and Minitab for computing skew and kurtosis.

However, L-moment-based estimators, such as L-skew ($\widehat{\tau}_3$), L-kurtosis ($\widehat{\tau}_4$), and the L-correlation, have been introduced to address the limitations associated with conventional moment-based estimators [15–18]. Specifically, some of the advantages that L-moments have over conventional moments are that they (i) exist whenever the mean of the distribution exists, (ii) are nearly unbiased for all sample sizes and distributions, and (iii) are more robust in the presence of outliers. For example, the estimates of $\widehat{\tau}_{3,4}$ in Figure 1 are relatively much closer to their respective parameters $\tau_{3,4}$ with smaller relative standard errors than their corresponding conventional moment-based analogs $\widehat{\alpha}_{3,4}$. More specifically, the estimates of $\widehat{\tau}_{3,4}$ that were simulated are, on average, 9.19% below and 5.7% above their respective parameters.

Thus, the present aim here is to characterize the S-D class of distributions through the method of L-moments. The characterization will enable researchers to model or simulate nonnormal distributions with specified values of L-skew, L-kurtosis, and L-correlation. The rest of the paper is outlined as follows. In Section 2, a summary of univariate L-moment theory is provided as well as additional properties associated with S-D distributions. In Section 3, the derivation of the system of equations for specifying values of L-skew and L-kurtosis for the S-D class of distributions is subsequently provided. In Section 4, the coefficient of L-correlation is introduced and the equations are developed for determining intermediate correlations for specified L-correlations associated with the S-D class of distributions. In Section 5, the steps for implementing a simulation procedure are described. Numerical examples and the results of a simulation are also provided to confirm the derivations and compare the new methodology with its conventional moment-based counterparts. In Section 6, the results of the simulation are discussed and concluding comments are made.

2. Preliminaries on Univariate L-Moments and the Schmeiser-Deutsch Class of Distributions

2.1. Univariate L-Moments

Let $Y_1, \ldots, Y_j, \ldots, Y_n$ be identically and independently distributed random variables each with continuous pdf $f(y)$, cdf $F(y)$, order statistics denoted as $Y_{1:n} \leq \cdots \leq Y_{j:n} \leq \cdots \leq Y_{n:n}$, and L-moments defined in terms of either linear combinations of (i) expectations of order statistics or (ii) probability weighted moments (β_i). For the purposes considered herein, the first four L-moments associated with $Y_{j:n}$ are expressed as [16, pages 20–22]

$$\begin{aligned}
\lambda_1 &= E[Y_{1:1}] = \beta_0, \\
\lambda_2 &= \frac{1}{2}E[Y_{2:2} - Y_{1:2}] = 2\beta_1 - \beta_0, \\
\lambda_3 &= \frac{1}{3}E[Y_{3:3} - 2Y_{2:3} + Y_{1:3}] = 6\beta_2 - 6\beta_1 + \beta_0, \\
\lambda_4 &= \frac{1}{4}E[Y_{4:4} - 3Y_{3:4} + 3Y_{2:4} - Y_{1:4}] = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0,
\end{aligned} \tag{2.1}$$

where the β_i are determined from

$$\beta_i = \int y\{F(y)\}^i f(y)dy, \tag{2.2}$$

where $i = 0,\ldots,3$. The coefficients associated with β_i in (2.2) are obtained from shifted orthogonal Legendre polynomials and are computed as shown in [16, page 20].

The L-moments λ_1 and λ_2 in (2.1) are measures of location and scale and are the arithmetic mean and one-half the coefficient of mean difference, respectively. Higher order L-moments are transformed to dimensionless quantities referred to as L-moment ratios defined as $\tau_r = \lambda_r/\lambda_2$ for $r \geq 3$, and where τ_3 and τ_4 are the analogs to the conventional measures of skew and kurtosis. In general, L-moment ratios are bounded in the interval $-1 < \tau_r < 1$ as is the index of L-skew (τ_3) where a symmetric distribution implies that all L-moment ratios with odd subscripts are zero. Other smaller boundaries can be found for more specific cases. For example, the index of L-kurtosis (τ_4) has the boundary condition for continuous distributions of [19] $(5\tau_3^2 - 1)/4 < \tau_4 < 1$.

2.2. The Schmeiser-Deutsch (S-D) Class of Distributions

The cdf and pdf associated with the S-D quantile function in (1.1) are expressed as in [1]:

$$F_X(x) = \begin{cases} \gamma_4 - \left(\dfrac{\gamma_1 - x}{\gamma_2}\right)^{1/\gamma_3}, & \text{if } \gamma_1 - \gamma_2\gamma_4^{\gamma_3} \leq x \leq \gamma_1 \\ \gamma_4 + \left(\dfrac{x - \gamma_1}{\gamma_2}\right)^{1/\gamma_3}, & \text{if } \gamma_1 \leq x \leq \gamma_1 + \gamma_2(1-\gamma_4)^{\gamma_3}, \end{cases} \tag{2.3}$$

$$f_X(x) = \frac{1}{\gamma_2\gamma_3}\left|\frac{\gamma_1 - x}{\gamma_2}\right|^{(1-\gamma_3)/\gamma_3}, \quad \text{for } \gamma_1 - \gamma_2\gamma_4^{\gamma_3} \leq x \leq \gamma_1 + \gamma_2(1-\gamma_4)^{\gamma_3}. \tag{2.4}$$

Setting $\gamma_4 = 0.50$ in (2.4) produces symmetric S-D densities with a lower bound of kurtosis of $\alpha_4 = -2$ as $\gamma_3 \to 0$. Positive (negative) skew is produced for cases where $0 \leq \gamma_4 < 0.50$ ($0.50 < \gamma_4 \leq 1$) and $\gamma_3 > 1$, where $\gamma_3 < 1$ reverses the direction of skew. For $\gamma_3 > 1$ ($0 < \gamma_3 < 1$), the unique mode (antimode) is located at γ_1 and where $\gamma_3 = 1$ produces uniform distributions. For example, the distribution in Figure 1 is bounded in the range of $9.9372 \leq x \leq 11.2748$ with mode, mean, and variance of 10, 10.042, and 0.0271, respectively. See the Appendix for the formulae for computing the moments associated with S-D distributions. In the next section, the system of L-moments for the class of S-D distributions is derived.

3. L-Moments for the Schmeiser-Deutsch (S-D) Class of Distributions

The derivation of the first four L-moments associated with the S-D class of distributions begins by defining the probability weighted moments based on (2.2) in terms of (1.1) as

$$\beta_i = \int_0^{\gamma_4} q(u,\gamma_1,\gamma_2,\gamma_3,\gamma_4)\{F(u)\}^i f(u)du + \int_{\gamma_4}^1 q(u,\gamma_1,\gamma_2,\gamma_3,\gamma_4)\{F(u)\}^i f(u)du, \tag{3.1}$$

where $F(u) = u$ and $f(u) = 1$ are the zero-one uniform cdf and pdf. As such, integrating (3.1) for $i = 0, 1, 2, 3$ and simplifying using (2.1) give λ_1, λ_2, τ_3, and τ_4 as

$$\lambda_1 = \gamma_1 + \frac{\gamma_2(1-\gamma_4)^{1+\gamma_3}}{(1+\gamma_3)} - \gamma_1\gamma_4 + \gamma_4\left(\gamma_1 - \frac{\gamma_2\gamma_4{}^{\gamma_3}}{(1+\gamma_3)}\right), \tag{3.2}$$

$$\lambda_2 = \frac{\left(-\gamma_2\gamma_4^{1+\gamma_3}(2\gamma_4 - \gamma_3 - 2) + \gamma_2(1-\gamma_4)^{1+\gamma_3}(\gamma_3 + 2\gamma_4)\right)}{((1+\gamma_3)(2+\gamma_3))}, \tag{3.3}$$

$$\begin{aligned}\tau_3 = & -\Big\{\Big((1-\gamma_4)^{1+\gamma_3}\left(\gamma_3(1-6\gamma_4) - \gamma_3^2 + 6(1-2\gamma_4)\gamma_4\right) \\ & +\gamma_4^{1+\gamma_3}\left(6 + 5\gamma_3 + \gamma_3^2 - 6(3+\gamma_3)\gamma_4 + 12\gamma_4^2\right)\Big) \\ & /\Big((3+\gamma_3)\left((2+\gamma_3-2\gamma_4)\gamma_4{}^{1+\gamma_3} + (1-\gamma_4)^{1+\gamma_3}(\gamma_3+2\gamma_4)\right)\Big)\Big\},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\tau_4 = & \Big\{-(1-\gamma_4)^{1+\gamma_3}\left((\gamma_3-2)(\gamma_3-1)\gamma_3 + 12(\gamma_3-2)(\gamma_3-1)\gamma_4 + 60(\gamma_3-2)\gamma_4^2 + 120\gamma_4^3\right) \\ & +\gamma_4^{1+\gamma_3}\left(-24 - 26\gamma_3 - 9\gamma_3^2 - \gamma_3^3 + 12(3+\gamma_3)(4+\gamma_3)\gamma_4 - 60(4+\gamma_3)\gamma_4^2 + 120\gamma_4^3\right)\Big\} \\ & /\Big\{(3+\gamma_3)(4+\gamma_3)\left(\gamma_4^{1+\gamma_3}(2\gamma_4 - \gamma_3 - 2) - (1-\gamma_4)^{1+\gamma_3}(\gamma_3+2\gamma_4)\right)\Big\}.\end{aligned} \tag{3.5}$$

Thus, given user-specified values of λ_1, λ_2, τ_3, and τ_4, (3.2)–(3.5) can be numerically solved to obtain the parameters for γ_1, γ_2, γ_3, and γ_4. Inspection of (3.4) and (3.5) indicates that the solutions to τ_3 and τ_4 are independent of the location and scale parameters (λ_1 and λ_2). As with the conventional S-D class of symmetric distributions ($\gamma_4 = 0.5$) the lower-bound of L-kurtosis $\tau_4 = -0.25$ is obtained as $\gamma_3 \to 0$.

Table 1 gives four examples of S-D distributions. These four distributions are used in the simulation portion of this study in Section 5. Distribution 1 was used by Lau [13] for stochastic modeling related to the newsboy problem and Distribution 2 was used by Aardal et al. [2] in the context of modeling inventory-control systems. The values of conventional skew and kurtosis for the four distributions were determined based on (A.2) in the Appendix. In the next section, we introduce the topic of the L-correlation and subsequently develop the methodology for simulating S-D distributions with specified L-correlations.

4. *L*-Correlations for the Schmeiser-Deutsch (S-D) Class of Distributions

The L-correlation [17, 18] is introduced by considering two random variables Y_j and Y_k with distribution functions $F(Y_j)$ and $F(Y_k)$, respectively. The second L-moments of Y_j and Y_k can alternatively be expressed as

$$\lambda_2(Y_j) = 2\mathrm{Cov}(Y_j, F(Y_j)), \tag{4.1}$$

$$\lambda_2(Y_k) = 2\mathrm{Cov}(Y_k, F(Y_k)). \tag{4.2}$$

Table 1: Four S-D distributions and their associated L-moments, conventional moments (C-moments), and S-D parameters used in the simulation. The values of L-skew (τ_3) and L-kurtosis (τ_4) are based on (3.4) and (3.5). The values of skew (α_3) and kurtosis (α_4) are based on the expressions in (A.2) of the Appendix.

Distribution	L-moments	C-moments	S-D parameters	
			L-moment	C-moment
(1) Moderate positive skew and platykurtic	$\lambda_1 = 0$	$\alpha_1 = 0$	$\gamma_1 = -0.8429$	$\gamma_1 = -0.8427$
	$\lambda_2 = 1/\sqrt{\pi}$	$\alpha_2 = 1$	$\gamma_2 = 4.6222$	$\gamma_2 = 4.6213$
	$\tau_3 = 0.1647$	$\alpha_3 = 0.5524$	$\gamma_3 = 1.5$	$\gamma_3 = 1.5$
	$\tau_4 = 0.01606$	$\alpha_4 = -0.9261$	$\gamma_4 = 0.25$	$\gamma_4 = 0.25$
(2) Symmetric and mesokurtic	$\lambda_1 = 0$	$\alpha_1 = 0$	$\gamma_1 = 0$	$\gamma_1 = 0$
	$\lambda_2 = 1/\sqrt{\pi}$	$\alpha_2 = 1$	$\gamma_2 = 22.568$	$\gamma_2 = 21.166$
	$\tau_3 = 0$	$\alpha_3 = 0$	$\gamma_3 = 3$	$\gamma_3 = 3$
	$\tau_4 = 0.2857$	$\alpha_4 = 0.7692$	$\gamma_4 = 0.50$	$\gamma_4 = 0.50$
(3) Positive skew and leptokurtic	$\lambda_1 = 0$	$\alpha_1 = 0$	$\gamma_1 = -0.2344$	$\gamma_1 = -0.1373$
	$\lambda_2 = 1/\sqrt{\pi}$	$\alpha_2 = 1$	$\gamma_2 = 59196.04$	$\gamma_2 = 34569.55$
	$\tau_3 = 0.3314$	$\alpha_3 = 3$	$\gamma_3 = 12.89$	$\gamma_3 = 12.89$
	$\tau_4 = 0.7011$	$\alpha_4 = 15$	$\gamma_4 = 0.4853$	$\gamma_4 = 0.4853$
(4) Negative skew and leptokurtic	$\lambda_1 = 0$	$\alpha_1 = 0$	$\gamma_1 = 0.6148$	$\gamma_1 = 0.3043$
	$\lambda_2 = 1/\sqrt{\pi}$	$\alpha_2 = 1$	$\gamma_2 = 2960.06$	$\gamma_2 = 1465.19$
	$\tau_3 = -0.8414$	$\alpha_3 = -4.181$	$\gamma_3 = 15$	$\gamma_3 = 15$
	$\tau_4 = 0.6483$	$\alpha_4 = 18.21$	$\gamma_4 = 0.70$	$\gamma_4 = 0.70$

The second L-comoments of Y_j toward Y_k and Y_k toward Y_j are

$$\lambda_2(Y_j, Y_k) = 2\mathrm{Cov}(Y_j, F(Y_k)), \tag{4.3}$$

$$\lambda_2(Y_k, Y_j) = 2\mathrm{Cov}(Y_k, F(Y_j)). \tag{4.4}$$

As such, the L-correlations of Y_j toward Y_k and Y_k toward Y_j are expressed as

$$\eta_{jk} = \frac{\lambda_2(Y_j, Y_k)}{\lambda_2(Y_j)}, \tag{4.5}$$

$$\eta_{kj} = \frac{\lambda_2(Y_k, Y_j)}{\lambda_2(Y_k)}. \tag{4.6}$$

The L-correlation in (4.5) (or (4.6)) is bounded such that $-1 \leq \eta_{jk} \leq 1$ where a value of $\eta_{jk} = 1$ ($\eta_{jk} = -1$) indicates a strictly increasing (decreasing) *monotone* relationship between the two variables. In general, we would also note that $\eta_{jk} \neq \eta_{kj}$.

In the context of the L-moment-based S-D class of distributions, suppose it is desired to simulate a T-variate distribution based on quantile functions of the forms in (1.1) with a specified L-correlation matrix and where each distribution has its own specified values of τ_3 and τ_4. Let $Z_1, \ldots, Z_T$ denote standard normal variables where the distribution functions and bivariate density function associated with Z_j and Z_k are expressed as

$$\Phi(z_j) = \Pr\{Z_j \le z_j\} = \int_{-\infty}^{z_j} (2\pi)^{-1/2} \exp\left\{\frac{-w_j^2}{2}\right\} dw_j, \tag{4.7}$$

$$\Phi(z_k) = \Pr\{Z_k \le z_k\} = \int_{-\infty}^{z_k} (2\pi)^{-1/2} \exp\left\{\frac{-w_k^2}{2}\right\} dw_k, \tag{4.8}$$

$$f_{jk} = \left(2\pi\left(1-\rho_{jk}^2\right)^{1/2}\right)^{-1} \exp\left\{-\left(2\left(1-\rho_{jk}^2\right)\right)^{-1}\left(z_j^2 + z_k^2 - 2\rho_{jk} z_j z_k\right)\right\}. \tag{4.9}$$

Using (4.7), it follows that the jth S-D distribution associated with (1.1) can be expressed as $q_j(\Phi(Z_j))$ since $\Phi(Z_j)$ is zero-one uniformly distributed. As such, using (4.5), the L-correlation of $q_j(\Phi(Z_j))$ toward $q_k(\Phi(Z_k))$ can be evaluated using the solved values of the parameters for $q_j(\Phi(Z_j))$, a specified *intermediate correlation* (IC) ρ_{jk} in (4.9), and the following integral generally expressed as

$$\eta_{jk} = 2\sqrt{\pi}\int\int_{-\infty}^{+\infty} q_j\left(\Phi(z_j), \gamma_{1_j}, \gamma_{2_j}, \gamma_{3_j}, \gamma_{4_j}\right)\Phi(z_k) f_{jk}\, dz_j\, dz_k, \tag{4.10}$$

where it is required that the location and scale parameters $(\gamma_{1_j}, \gamma_{2_j})$ in (4.10) are to be solved such that $q_j(\Phi(Z_j))$ will have the values of $\lambda_1 = 0$ and $\lambda_2 = 1/\sqrt{\pi}$ in (3.2) and (3.3), that is, set to the values of λ_1 and λ_2 associated with the unit normal distribution. Note that this requirement is not a limitation as the L-correlation is invariant to linear transformations [17]. Further, we would point out that the purpose of the IC (ρ_{jk}) in (4.9) and (4.10) is to adjust for the effect of the transformation $q_j(\Phi(Z_j))$, which is induced by the parameters, such that $q_j(\Phi(Z_j))$ has its specified L-correlation (η_{jk}) toward $q_k(\Phi(Z_k))$. Analogously, the L-correlation of $q_k(\Phi(Z_k))$ toward $q_j(\Phi(Z_j))$ is expressed as

$$\eta_{kj} = 2\sqrt{\pi}\int\int_{-\infty}^{+\infty} q_k\left(\Phi(z_k), \gamma_{1_k}, \gamma_{2_k}, \gamma_{3_k}, \gamma_{4_k}\right)\Phi(z_j) f_{jk}\, dz_k\, dz_j. \tag{4.11}$$

Note for the special case that if $q_j(\Phi(Z_j))$ in (4.10) and $q_k(\Phi(Z_k))$ in (4.11) have the same parameters, that is, $\gamma_{1_j} = \gamma_{1_k}; \gamma_{2_j} = \gamma_{2_k}; \gamma_{3_j} = \gamma_{3_k}; \gamma_{4_j} = \gamma_{4_k}$, then $\eta_{jk} = \eta_{kj}$. Provided in Algorithm 1 is source code written in Mathematica [20] that implements the computation

```
(* Intermediate Correlation for Distributions 1 and 2. See Table 4. *)
ρjk = 0.714645;

Needs["MultivariateStatistics"]
fjk = PDF[MultinormalDistribution[{0,0}, {{1,ρjk}, {ρjk,1}}], {Zj, Zk}];
Fj = CDF[NormalDistribution[0, 1], Zj];
Fk = CDF[NormalDistribution[0, 1], Zk];

(* S-D L-moment Parameters for Distribution j = 1 in Table 1 *)
γ1 = −0.8429;
γ2 = 4.6222;
γ3 = 1.5;
γ4 = 0.25;
(* Quantile function for (1.1) *)
q1j = γ1j − γ2j ∗ (γ4j − Fj)^γ3j;
q2j = γ1j + γ2j ∗ (Fj − γ4j)^γ3j;
Xj =Piecewise[{{q1j, Fj ≤ γ4j}, {q2j, Fj > γ4j}}]

(* Compute the specified correlation in Table 2 for Distributions 1 and 2 *)
ηjk = 2√π ∗ NIntegrate[Xj ∗ Fk ∗ fjk, {Zj, −10, 10}, {Zk, −10, 10}, Method → MultiDimensional]

0.70
```

Algorithm 1: Mathematica source code for computing intermediate correlations for specified L-correlations. The example is for distributions $j = 1$ and $k = 2$ (η_{12}) in Table 1. See Tables 2 and 4.

of an IC (ρ_{jk}) based on (4.10). The details for simulating S-D distributions with specified values of L-skew, L-kurtosis, and L-correlations are described in the next section.

5. The Procedure for Simulation and Monte Carlo Study

To implement a method for simulating S-D distributions with specified L-moments and L-correlations we suggest the following six steps.

(1) Specify the L-moments for T transformations of the forms in (1.1), that is, $q_1(\Phi(Z_1)), \ldots, q_T(\Phi(Z_T))$ and obtain the solutions for the parameters of γ_{1_j}, γ_{2_j}, γ_{3_j}, and γ_{4_j} by solving (3.2)–(3.5) with $\lambda_1 = 0$ and $\lambda_2 = 1/\sqrt{\pi}$ using the specified values of L-skew (τ_3) and L-kurtosis (τ_4) for each distribution. Specify a $T \times T$ matrix of L-correlations (η_{jk}) for $q_j(\Phi(Z_j))$ toward $q_k(\Phi(Z_k))$, where $j < k \in \{1, 2, \ldots, T\}$.

(2) Compute the (Pearson) intermediate correlations (ICs) ρ_{jk} by substituting the solutions of the parameters from Step (1) into (4.10) and then numerically integrate to solve for ρ_{jk} (see Algorithm 1 for an example). Repeat this step separately for all $T(T-1)/2$ pairwise combinations of correlations.

(3) Assemble the ICs into a $T \times T$ matrix and decompose this matrix using a Cholesky factorization. Note that this step requires the IC matrix to be positive definite.

(4) Use the results of the Cholesky factorization from Step (3) to generate T standard normal variables $(Z_1, \ldots, Z_T)$ correlated at the intermediate levels as follows:

$$\begin{aligned}
Z_1 &= a_{11}V_1, \\
Z_2 &= a_{12}V_1 + a_{22}V_2, \\
&\vdots \\
Z_j &= a_{1j}V_1 + a_{2j}V_2 + \cdots + a_{ij}V_i + \cdots + a_{jj}V_j, \\
&\vdots \\
Z_T &= a_{1T}V_1 + a_{2T}V_2 + \cdots + a_{iT}V_i + \cdots + a_{jT}V_j + \cdots + a_{TT}V_T,
\end{aligned} \tag{5.1}$$

where $V_1, \ldots, V_T$ are independent standard normal random variables and where a_{ij} represents the element in the ith row and the jth column of the matrix associated with the Cholesky factorization performed in Step (3).

(5) Substitute $Z_1, \ldots, Z_T$ from Step (4) into the following Taylor series-based expansion for the standard normal cdf [21]:

$$\Phi(Z_j) = \left(\frac{1}{2}\right) + \phi(Z_j)\left\{Z_j + \frac{Z_j^3}{3} + \frac{Z_j^5}{(3\cdot 5)} + \frac{Z_j^7}{(3\cdot 5\cdot 7)} + \cdots\right\}, \tag{5.2}$$

where $\phi(Z_j)$ denotes the standard normal pdf and where the absolute error associated with (5.2) is less than 8×10^{-16}.

(6) Substitute the zero-one uniform deviates, $\Phi(Z_j)$, generated from Step (5) into the T equations of the form of $q_j(\Phi(Z_j))$ to generate the S-D distributions with the specified L-moments and L-correlations.

To demonstrate the steps above, and evaluate the proposed method, a comparison between the proposed L-moment and conventional product-moment-based procedures is subsequently described. Specifically, the parameters for the distributions in Table 1 are used as a basis for a comparison using the specified correlation matrix in Table 2. Tables 3 and 4 give the solved IC matrices for the conventional moment and L-moment-based methods, respectively. See Algorithm 2 for an example of computing ICs for the conventional method. Tables 5 and 6 give the results of the Cholesky decompositions on the IC matrices, which are then used to create $Z_1, \ldots, Z_4$ with the specified ICs by making use of the formulae given in (5.1) of Step (4) with $T = 4$. The values of $Z_1, \ldots, Z_4$ are subsequently transformed to $\Phi(Z_1), \ldots, \Phi(Z_4)$ using (5.2) and then substituted into equations of the forms in (1.1) to produce $q_1(\Phi(Z_1)), \ldots, q_4(\Phi(Z_4))$ for both methods.

In terms of the simulation, a Fortran algorithm was written for both methods to generate 25,000 independent sample estimates for the specified parameters of (i) conventional skew (α_3), kurtosis (α_4), and Pearson correlation (ρ^*_{jk}); (ii) L-skew (τ_3), L-kurtosis (τ_4), and L-correlation (η_{jk}). All estimates were based on sample sizes of $n = 25$ and $n = 1000$. The formulae used for computing estimates of $\alpha_{3,4}$ were based on Fisher's k-statistics, that is, the formulae currently used by most commercial software packages such

Table 2: Specified correlation matrix for the distributions in Table 1.

	1	2	3	4
1	1			
2	0.70	1		
3	0.60	0.60	1	
4	0.35	0.60	0.40	1

Table 3: Intermediate correlations for the conventional moment procedure.

	1	2	3	4
1	1			
2	0.761371	1		
3	0.945345	0.813672	1	
4	0.810720	0.956258	0.910196	1

Table 4: Intermediate correlations for the *L*-moment procedure.

	1	2	3	4
1	1			
2	0.714645	1		
3	0.615758	0.584908	1	
4	0.362634	0.584908	0.332108	1

as SAS, SPSS, and Minitab, for computing indices of skew and kurtosis (where $\alpha_{3,4} = 0$ for the standard normal distribution). The formulae used for computing estimates of $\tau_{3,4}$ were Headrick's equations (2.4) and (2.6) [22]. The estimate for ρ_{jk}^{*} was based on the usual formula for the Pearson product-moment of correlation statistic and the estimate for η_{jk} was computed based on (4.5) using the empirical forms of the cdfs in (4.1) and (4.3). The estimates for ρ_{jk}^{*} and η_{jk} were both transformed using Fisher's z' transformation. Bias-corrected-accelerated-bootstrapped average (mean) estimates, confidence intervals (C.I.s), and standard errors were subsequently obtained for the estimates associated with the parameters $(\alpha_{3,4}, \tau_{3,4}, z'_{\rho_{jk}^{*}}, z'_{\eta_{jk}})$ using 10,000 resamples via the commercial software package Spotfire S+ [23]. The bootstrap results for the estimates of the means and C.I.s associated with $z'_{\rho_{jk}^{*}}$ and $z'_{\eta_{jk}}$ were transformed back to their original metrics (i.e., estimates for ρ_{jk}^{*} and η_{jk}). Further, if a parameter (P) was outside its associated bootstrap C.I., then an index of relative bias (RB) was computed for the estimate (E) as: RB $= ((E - P)/P) \times 100$. Note that the small amount of bias associated with any bootstrap C.I. containing a parameter was considered negligible and thus not reported. The results of the simulation are reported in Tables 7–12 and are discussed in the next section.

6. Discussion and Conclusion

One of the primary advantages that *L*-moments have over conventional moment-based estimators is that they can be far less biased when sampling is from distributions with more severe departures from normality (e.g. [16, 21]). Inspection of the simulation results in Tables 7 and 8 of this study clearly indicates that this is also the case for the S-D class of distributions.

```
(* Intermediate Correlation for Distributions 1 and 2. See Table 3. *)
ρjk = 0.761371;

Needs["MultivariateStatistics"]
fjk = PDF[MultinormalDistribution[{0,0},{{1,ρjk},{ρjk,1}}],{Zj,Zk}];
Fj = CDF[NormalDistribution[0,1],Zj];
Fk = CDF[NormalDistribution[0,1],Zk];

(* S-D C-Moment Parameters Distributions j = 1 and k = 2 in Table 1 *)
γ1j = −0.8427;
γ2j = 4.6213;
γ3j = 1.5;
γ4j = 0.25;
γ1k = 0.0;
γ2k = 21.166;
γ3k = 3.0;
γ4k = 0.5;

(* Quantile functions for (1.1) *)
q1j = γ1j − γ2j ∗ (γ4j − Fj)^γ3j;
q2j = γ1j + γ2j ∗ (Fj − γ4j)^γ3j;
Xj = Piecewise[{{q1j, Fj ≤ γ4j}, {q2j, Fj > γ4j}}]

q1k = γ1k − γ2k ∗ (γ4k − Fk)^γ3k;
q2k = γ1k + γ2k ∗ (Fk − γ4k)^γ3k;
Xk = Piecewise[{{q1k, Fk ≤ γ4k}, {q2k, Fk > γ4k}}]

(* Compute the specified correlation in Table 2 for Distributions 1 and 2. *)
ρ*jk =NIntegrate[Xj ∗ Xk ∗ fjk, {Zj, −10, 10}, {Zk, −10, 10}, Method → MultiDimensional]

0.70
```

Algorithm 2: Mathematica source code for computing intermediate correlations for specified conventional Pearson correlations. The example is for distributions $j = 1$ and $k = 2$ (ρ^*_{12}) in Table 1. See also Tables 2 and 3.

Table 5: Cholesky decompositions for the conventional moment procedure.

$a_{11} = 1$	$a_{12} = 0.761371$	$a_{13} = 0.945345$	$a_{14} = 0.810720$
0	$a_{22} = 0.648316$	$a_{23} = 0.144858$	$a_{24} = 0.522892$
0	0	$a_{33} = 0.292128$	$a_{34} = 0.232914$
0	0	0	$a_{44} = 0.122753$

Table 6: Cholesky decompositions for the L-moment procedure.

$a_{11} = 1$	$a_{12} = 0.714645$	$a_{13} = 0.615758$	$a_{14} = 0.362634$
0	$a_{22} = 0.699487$	$a_{23} = 0.207094$	$a_{24} = 0.465703$
0	0	$a_{33} = 0.760233$	$a_{34} = 0.016269$
0	0	0	$a_{44} = 0.807064$

Table 7: Skew (α_3) and kurtosis (α_4) results for distributions 2 and 4 in Table 1.

Dist	Parameter	Estimate	95% bootstrap C.I.	Standard error	Relative bias %
			$n = 25$		
2	$\alpha_3 = 0.0$	0.0019	−0.0055, 0.0094	0.00381	—
	$\alpha_4 = 0.7692$	1.166	1.1496, 1.1821	0.00825	51.59
4	$\alpha_3 = -4.181$	−3.581	−3.5917, −3.5702	0.00549	−14.35
	$\alpha_4 = 18.21$	13.61	13.5278, 13.6973	0.04337	−25.26
			$n = 1000$		
2	$\alpha_3 = 0.0$	−0.0005	−0.0013, 0.0004	0.00042	—
	$\alpha_4 = 0.7692$	0.7798	0.7778, 0.7817	0.00100	1.38
4	$\alpha_3 = -4.181$	−4.190	−4.1936, −4.1866	0.00177	0.22
	$\alpha_4 = 18.21$	18.44	18.4046, 18.4738	0.01775	1.26

Table 8: L-skew (τ_3) and L-kurtosis (τ_4) results for distributions 2 and 4 in Table 1.

Dist	Parameter	Estimate	95% bootstrap C.I.	Standard error	Relative bias %
			$n = 25$		
2	$\tau_3 = 0.0$	0.0002	−0.0014, 0.0018	0.00079	—
	$\tau_4 = 0.2857$	0.2981	0.2968, 0.2993	0.00063	4.34
4	$\tau_3 = -0.8414$	−0.8638	−0.8647, −0.8628	0.00049	2.66
	$\tau_4 = 0.6483$	0.6956	0.6935, 0.6975	0.00102	7.30
			$n = 1000$		
2	$\tau_3 = 0.0$	−0.0001	−0.0003, 0.0001	0.00010	—
	$\tau_4 = 0.2857$	0.2860	0.2859, 0.2862	0.00008	0.11
4	$\tau_3 = -0.8414$	−0.8422	−0.8423, −0.8420	0.00007	0.095
	$\tau_4 = 0.6483$	0.6498	0.6495, 0.6501	0.00015	0.23

Table 9: Correlation results for the conventional moment procedure $n = 25$.

Parameter	Estimate	95% bootstrap C.I.	Standard error	Relative bias %
$\rho^*_{12} = 0.70$	0.7083	0.7071, 0.7095	0.00121	1.19
$\rho^*_{13} = 0.60$	0.6396	0.6380, 0.6414	0.00146	6.60
$\rho^*_{14} = 0.35$	0.3837	0.3826, 0.3848	0.00065	9.63
$\rho^*_{23} = 0.60$	0.6342	0.6327, 0.6356	0.00122	5.70
$\rho^*_{24} = 0.60$	0.6342	0.6330, 0.6353	0.00097	5.70
$\rho^*_{34} = 0.40$	0.4958	0.4919, 0.5000	0.00272	23.95

Table 10: Correlation results for the L-moment procedure $n = 25$.

Parameter	Estimate	95% bootstrap C.I.	Standard error	Relative bias %
$\eta_{12} = 0.70$	0.7155	0.7140, 0.7170	0.00152	2.21
$\eta_{13} = 0.60$	0.6155	0.6137, 0.6173	0.00148	2.58
$\eta_{14} = 0.35$	0.3588	0.3562, 0.3611	0.00143	2.51
$\eta_{23} = 0.60$	0.6116	0.6099, 0.6136	0.00149	1.93
$\eta_{24} = 0.60$	0.6057	0.6038, 0.6075	0.00151	0.95
$\eta_{34} = 0.40$	0.4217	0.4177, 0.4259	0.00257	5.43

Table 11: Correlation results for the conventional moment procedure $n = 1000$.

Parameter	Estimate	95% bootstrap C.I.	Standard error	Relative bias %
$\rho^*_{12} = 0.70$	0.7001	0.6999, 0.7003	0.00018	—
$\rho^*_{13} = 0.60$	0.6004	0.6001, 0.6006	0.00021	0.07
$\rho^*_{14} = 0.35$	0.3502	0.3500, 0.3504	0.00011	—
$\rho^*_{23} = 0.60$	0.6002	0.5999, 0.6004	0.00019	—
$\rho^*_{24} = 0.60$	0.6004	0.6002, 0.6006	0.00015	0.07
$\rho^*_{34} = 0.40$	0.4009	0.4004, 0.4013	0.00027	0.23

Table 12: Correlation results for the L-moment procedure $n = 1000$.

Parameter	Estimate	95% bootstrap C.I.	Standard error	Relative bias %
$\eta_{12} = 0.70$	0.7002	0.7000, 0.7004	0.00022	—
$\eta_{13} = 0.60$	0.6005	0.6003, 0.6008	0.00021	0.08
$\eta_{14} = 0.35$	0.3501	0.3498, 0.3505	0.00021	—
$\eta_{23} = 0.60$	0.6000	0.5997, 0.6002	0.00022	—
$\eta_{24} = 0.60$	0.6002	0.5999, 0.6005	0.00021	—
$\eta_{34} = 0.40$	0.3984	0.3979, 0.3988	0.00035	–0.40

Specifically, the superiority that estimates of L-moment ratios (τ_3, τ_4) have over their corresponding conventional moment based counterparts (α_3, α_4) is obvious. For example, with samples of size $n = 25$ the estimates of skew and kurtosis for Distribution 4 (Table 7) were, on average, 85.65% and 74.74% of their associated population parameters, whereas the estimates of L-skew and L-kurtosis were 97.34% and 92.70% of their respective parameters. Similar results were also obtained for Distributions 1 and 3 and thus not reported. It is also evident from Tables 7 and 8 that L-skew and L-kurtosis are more efficient as their relative standard errors RSE = (standard error/estimate) × 100 are smaller than the conventional estimators of skew and kurtosis. For example, in terms of Distribution 4, inspection of Tables 7 and 8 ($n = 25$) indicates RSE measures of RSE$(\widehat{\alpha}_3) = 0.1533\%$ and RSE$(\widehat{\alpha}_4) = 0.3187\%$ compared with RSE$(\widehat{\tau}_3) = 0.0567\%$ and RSE$(\widehat{\tau}_4) = 0.1466\%$. This demonstrates that L-skew and L-kurtosis have more precision because they have less variance around their estimates.

Presented in Tables 9–12 are the results associated with the conventional Pearson and L-correlations. Inspection of Tables 9 and 10 indicates that the L-correlation is substantially superior to the Pearson correlation in terms of relative bias for small sample sizes. For example, in terms of a moderate correlation (Table 9, $n = 25$, $\rho^*_{12} = 0.40$) the relative bias for Distributions 3 and 4 was 23.95% for the Pearson correlation compared to 5.43% for the L-correlation (Table 10, $n = 25$, $\eta_{12} = 0.40$). For large sample sizes (Tables 11 and 12, $n = 1000$), both procedures performed adequately as their estimates were in close proximity with their respective population parameters.

In summary, the proposed L-moment-based S-D class of distributions is an attractive alternative to the conventional moment-based S-D system. In particular, the L-moment-based system has distinct advantages when leptokurtic distributions and small sample sizes are of concern. Finally, we would note that Mathematica Version 8.0 [20] source code is available from the authors for implementing the L-moment-based method.

Appendix

System of Conventional Moment-Based Equations for S-D Distributions

The moments ($\mu_{r=1,\ldots,4}$) associated with the S-D class of distributions in (1.1) can be determined from

$$\mu_r = \int_0^{\gamma_4} q(u,\gamma_1,\gamma_2,\gamma_3,\gamma_4)^r f(u)du + \int_{\gamma_4}^{1} q(u,\gamma_1,\gamma_2,\gamma_3,\gamma_4)^r f(u)du, \tag{A.1}$$

where $f(u) = 1$ is the zero-one uniform pdf. The mean, variance, skew, and kurtosis are defined in general as in [24]:

$$\begin{aligned} \alpha_1 &= \mu_1, \\ \alpha_2^2 &= \mu_2 - \mu_1^2, \\ \alpha_3 &= \frac{(\mu_3 - 3\mu_2\mu_1 + 2\mu_1^3)}{\alpha_2^3}, \\ \alpha_4 &= \frac{(\mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4)}{\alpha_2^4}. \end{aligned} \tag{A.2}$$

The moments associated with the location and scale parameters in (A.2) are

$$\begin{aligned} \mu_1 &= \gamma_1 + \frac{\gamma_2(1-\gamma_4)^{1+\gamma_3}}{(1+\gamma_3)} - \gamma_1\gamma_4 + \gamma_4\left(\gamma_1 - \frac{\gamma_2\gamma_4^{\gamma_3}}{(1+\gamma_3)}\right), \\ \mu_2 &= \frac{2\gamma_1\gamma_2(1-\gamma_4)^{1+\gamma_3}}{(1+\gamma_3)} + \frac{\gamma_2^2(1-\gamma_4)^{1+2\gamma_3}}{(1+2\gamma_3)} - \gamma_1^2(\gamma_4 - 1) \\ &\quad + \gamma_4\left(\gamma_1^2 + \gamma_2\gamma_4^{\gamma_3}\left(\frac{\gamma_2\gamma_4^{\gamma_3}}{(1+2\gamma_3)} - \frac{2\gamma_1}{(1+\gamma_3)}\right)\right). \end{aligned} \tag{A.3}$$

The moments associated with the shape parameters of skew and kurtosis in (A.2) are

$$\begin{aligned} \mu_3 &= \frac{3\gamma_1^2\gamma_2(1-\gamma_4)^{1+\gamma_3}}{(1+\gamma_3)} + 3\gamma_1\gamma_2^2\frac{(1-\gamma_4)^{1+2\gamma_3}}{(1+2\gamma_3)} - \gamma_2^3(1-\gamma_4)^{1+3\gamma_3} - \gamma_1^3(\gamma_4 - 1) \\ &\quad + \gamma_4\left(\gamma_1^3 + \gamma_2\gamma_4^{\gamma_3}\left(\frac{-3\gamma_1^2}{(1+1\gamma_3)} + \gamma_2\gamma_4^{\gamma_3}\left(\frac{3\gamma_1}{(1+2\gamma_3)} - \frac{\gamma_2\gamma_4^{\gamma_3}}{(1+3\gamma_3)}\right)\right)\right), \end{aligned}$$

$$\mu_4 = \frac{4\gamma_1^3\gamma_2(1-\gamma_4)^{1+\gamma_3}}{(1+\gamma_3)} + \frac{6\gamma_1^2\gamma_2^2(1-\gamma_4)^{1+2\gamma_3}}{(1+2\gamma_3)} + \frac{4\gamma_1\gamma_2^3(1-\gamma_4)^{1+3\gamma_3}}{(1+3\gamma_3)} + \frac{\gamma_2^4(1-\gamma_4)^{1+4\gamma_3}}{(1+4\gamma_3)} - \gamma_1^4(\gamma_4-1)$$
$$+ \gamma_4\left(\gamma_1^4 + \gamma_2\gamma_4^{\gamma_3}\left(\frac{-4\gamma_1^3}{(1+\gamma_3)} + \gamma_2\gamma_4^{\gamma_3}\left(\frac{6\gamma_1^2}{(1+2\gamma_3)} + \gamma_2\gamma_4^{\gamma_3}\left(\frac{-4\gamma_1}{(1+3\gamma_3)} + \frac{\gamma_2\gamma_4^{\gamma_3}}{(1+43\gamma_3)}\right)\right)\right)\right). \tag{A.4}$$

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