9-1-2020

Development of Frenet-Serret Frame and the Apollonian Window

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DEVELOPMENT OF FRENET-SERRET FRAME AND THE APOLLONIAN WINDOW

by

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M.Sc., Southern Illinois University, 2002

A Thesis
Submitted in Partial Fulfillment of the Requirements for the
Master of Science Degree

Department of Mathematics
in the Graduate School
Southern Illinois University Carbondale
August 2020

THESIS APPROVAL

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for the Degree of

Master of Science

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Approved by:

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Graduate School
Southern Illinois University Carbondale
June 23, 2020

AN ABSTRACT OF THE THESIS OF

Syeda Fareeza Karimushan, for the Master of Science degree in Mathematics, presented on June 23, 2020, at Southern Illinois University Carbondale.

TITLE: DEVELOPMENT OF FRENET-SERRET FRAME AND THE APOLLONIAN WINDOW

MAJOR PROFESSOR: Dr. Jerzy Kocik

The present study focuses on Frenet-Serret frame and the Apollonian Window. In the first part of the study Apollonian disks are generated for first four generations by developing visual basic codes in excel. For the second part of the study, three orthonormal basis vectors, namely, tangent, normal, and binormal vectors have been calculated for the tangent points of Apollonian discs for the first three generations. Equations of the normal, osculating and rectifying planes and Taylor series approximation have been calculated for specific θ. Because Apollonian Window consists of planar curves with constant curvature, torsion is nowhere present. The planar Frenet-Serret equations for the first three generations for the Apollonian Window is also shown.

ACKNOWLEDGMENTS

Foremost I would like to express my sincere gratitude to my adviser Dr. Jerzy Kocik for providing relentless support and guidance in my learning the areas of Differential Geometry and developing the research topic on Apollonian Window. Due to his immense knowledge, motivation, dedication and patience towards his students I was able to accomplish my research.

I am also extremely grateful to my other committee members, Dr. Bhaskar Bhattacharya and Dr. Seyed Yaser Samadi. Dr. Bhaskar has kindly and patiently provided me with the correct guidance when I needed in my research. I have also received proper guidance from Dr. Samadi in working out problems and solutions that has helped me immensely in my research.

I must also thank wholeheartedly our then Graduate Adviser Dr. Michael Sullivan, and Dr. Mary Wright who have been very patient in teaching me mathematics and has given me continuous support, guidance, and encouragement throughout the difficult journey of learning.

I must also thank all of other faculties and staff in the Department of Mathematics whose continuous hard work and dedication have helped to create the supportive environment in the department to go through this challenging journey. I am also very grateful to my classmates for their many help and encouragement.

Last but not least, I am very grateful to my husband Dr. Saiful Islam who were with me all along with suggestions during editing of several drafts of this thesis. My mother Professor Nilufer Rahman supported me throughly with all her affection and encouragement. My son Ilan Kashif Ahmed who encouraged and supported me in his own way. Also, I am grateful to many other family members and friends for their best wishes in completing this arduous journey.

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CHAPTER 1

INTRODUCTION TO APOLLONIAN WINDOW AND FRENET-SERRET FRAME

1.1 Apollonian Disk Packing

Apollonian disk packing involves the construction of the fourth disk from three initial mutually tangent disks on the Euclidean plane. The fourth or the solution disk is internally or externally tangent to the initial three disks.

Apollonian disk packing or Apollonian gasket is a version of the Apollonius' problem where patterns of disks are obtained from inscribing the disks in every ideal triangle which is formed from the initial three disks. An ideal triangle is an open connected region R in Euclidean plane. The boundary of R is made of arcs of the three mutually tangent disks [9]. This simple construction of the fourth disk from three mutually tangent disks is based on an old Theorem of Apollonius of Perga (262-190 BC)[10]:

Theorem 1. *Given three mutually tangent disks on a plane, there exist exactly two disks tangent to all three.*

There can be both inner and outer disks in an Apollonian disk packing. Therefore, there can be external and internal tangencies of three mutually tangent disks. Two disks maintain external tangency if they intersect in exactly one point and the intersection of their interior is empty. In case of internal tangency two disks intersect in exactly one point but the intersection of their interior is not empty. In an Apollonian disk packing all disk packings maintain external tangency[9].

1.1.1 Descartes Configuration

Descartes configuration is defined as quadruple of four mutually tangent disks. By Theorem 4.1 [9] the curvatures of four disks in Descartes configuration satisfying equation

$$2(A^2 + B^2 + C^2 + D^2) = (A + B + C + D)^2 \qquad (1.1)$$

is called the Descartes formula [9]. Any quadruple that satisfies the Descartes equation is a Descartes quadruple.

1.1.2 Integral Apollonian Disk Packing

In an Apollonian disk packing if the initial Descartes configuration has integer values then the entire packing has integral curvature and conversely [3]. In an integral Apollonian disk packing, there is an infinite number of disks with integer curvatures [9].

As defined by [9], in a Euclidean plane a two-dimensional disk is a region where the boundary of the disk is the circle. If the region (disk) is inner with respect to the circle the curvature of the disk is defined as a $= \frac{1}{r}$, where r is the radius of the circle. When the disk is external w.r.t. the circle, the curvature of the disk is a $= \frac{-1}{r}$.

1.1.3 Apollonian Window

Different types of integral Apollonian disk packings exist with different symmetric properties. One of these is the Apollonian Window. In an Apollonian window the two largest disks are congruent resulting in vertical and horizonal mirror symmetries [9]. The Apollonian Window is particularly important as its properties encompasses multiple areas of Mathematics, namely, number theory, algebra, geometry and Physics [9].

We can define a disk as a simple closed curve following the definition of [5]. A simple closed curve does not intersect anywhere other than joining up. Therefore, a disk is a simple closed curve in $\mathbb{R}$ with period δ with $\delta \in \mathbb{R}$.

Define regular curve $\alpha : \mathbb{R} \mapsto \mathbb{R}^2$ such that $\alpha(\theta) = \alpha(\lambda)$ if and only if $\theta - \lambda = k\delta$ for some integer k. Therefore, when θ increases by λ, $\alpha(\theta)$ returns to its starting point.

1.2 Frenet-Serret Frame

The Frenet-Serret frame is applied to non-degenerate curves, or curves that have non-zero curvature. Consider the curve $\alpha : I = (a, b) \mapsto \mathbb{R}^3$. It is arclength parameterized

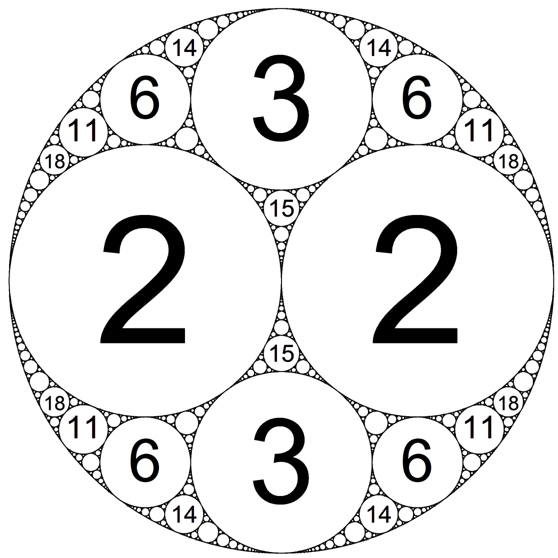

Figure 1.1: Apollonian Window

regular curve, or a unit speed curve, i.e., $\alpha'(\theta) \neq 0$ and arclength $\|\alpha'(\theta)\| = 1$. The curve is a differentiable map of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into a point $\alpha(\theta) = x(\theta),\ y(\theta),\ z(\theta) \in \mathbb{R}^3$. Differentiable means the function $\alpha(\theta)$ has derivatives of all orders at all points.

Definition 1. *If curve $\alpha : I = (a, b) \mapsto \mathbb{R}^3$ is a parametrized curve, then for any $a<\theta<b$ the arclength (of the trajectory of the particle) of α from a to θ is defined as $r(\theta) = \int_a^\theta \| \alpha'(\theta) \| \, d\theta$, where,*

$$\| \alpha'(\theta) \| = \sqrt{(x'(\theta))^2 + (y'(\theta))^2 + (z'(\theta))^2}$$

is the length of the vector $\alpha'(\theta)$. The arclength $s(\theta)$ of the trajectory of a particle is the distance the particle travels. This arclength is the integral of the particle's speed [11]. Because $\alpha'(\theta) \neq 0$, arclength $r(\theta)$ is a differentiable function of θ and $\frac{dr}{d\theta} = \| \alpha'(\theta) \|$ [1]. In other words, $\alpha(\theta)$ is a regular smooth curve with $\alpha'(\theta)$ continuous at all points for all θ. The variable θ is the parameter of the curve [1].

Lemma 1. *Suppose $f, g : (a, b) \mapsto \mathbb{R}^3$ are differentiable and satisfy $f(t) \cdot g(t) = const \quad \forall t$. Then $f'(t) \cdot g(t) = -f(t) \cdot g'(t)$. In particular, $|f(t)| = const$ if and only if $f(t) \cdot f'(t) = 0 \quad \forall t$.*

By the repeated application of the above Lemma it is possible to construct the Frenet Frame for suitable regular curves [11].

We can define the Frenet-Serret frame as a natural moving frame at each point of the curve α [11]. For each point in the curve, the Frenet-Serret frame consists of three vectors perpendicular to each other, namely, the tangent $\vec{T}(\theta)$, principal normal $\vec{N}(\theta)$, and binormal $\vec{B}(\theta)$. $\vec{T}(\theta),\ \vec{N}(\theta),\ \vec{B}(\theta)$ forms the orthonormal basis in $\mathbb{R}^3$. In other words, the Frenet-Serret trihedron consists these three vectors for the parameter value θ

Together $\vec{T}(\theta)$, $\vec{N}(\theta)$, $\vec{B}(\theta)$ are called the TNB Frame. The orientation of a particle along a curve is completely determined by the TNB Frame. We can measure the change in the moving frame [6] by differentiating $\vec{T}(\theta)$, $\vec{N}(\theta)$, $\vec{B}(\theta)$ as the particle moves along a curve.

$$\vec{T}(\theta) = \frac{\alpha'(\theta)}{\parallel \alpha'(\theta) \parallel} \tag{1.2}$$

$$\vec{N}(\theta) = \frac{1}{\kappa(\theta)}\vec{T}'(\theta) = \frac{\vec{T}'(\theta)}{\parallel \vec{T}'(\theta) \parallel} \tag{1.3}$$

$$\vec{B}(\theta) = \vec{T}(\theta) \times \vec{N}(\theta) \tag{1.4}$$

As can be seen from equation (2), $\vec{N}(\theta)$ is also a unit vector. Because $\vec{T}'(\theta) \cdot \vec{T}(\theta) = 0, \vec{T}(\theta)$ and $\vec{N}(\theta)$ are orthogonal to each other. Therefore, $\vec{B}(\theta) = \vec{T}(\theta) \times \vec{N}(\theta)$ is orthogonal to both $\vec{T}(\theta)$ and $\vec{N}(\theta)$.

The three planes, namely, the osculating, normal and rectifying planes associated with the TNB frame describe the geometry of the curve. The osculating plane is spanned by the vectors $\vec{T}(\theta)$ and $\vec{N}(\theta)$ and the normal plane is spanned by the vectors $\vec{N}(\theta)$ and $\vec{B}(\theta)$. The rectifying plane is spanned by the vectors $\vec{T}(\theta)$ and $\vec{B}(\theta)$.

1.3 Frenet-Serret Apparatus

The Frenet-Serret apparatus contains the above three unit vectors, curvature $\kappa(\theta)$ and torsion $\tau(\theta)$ of the parametric curve. Curvature, $\kappa(\theta)$ is a real-valued function of $\alpha(\theta)$

such that $\kappa(\theta) = \|\vec{T}'(\theta)\| \geqslant 0$. $\kappa(\theta)$ tells the rate at which the direction of $\vec{T}(\theta)$ changes [4]. Torsion tells to what degree a particle is twisting out of the osculating plane. The osculating plane is defined by the $\vec{T}(\theta)$ and $\vec{N}(\theta)$. In other words, torsion measures the rate of change of $\vec{B}(\theta)$.

To interpret curvature and torsion with regard to $\vec{N}(\theta)$, curvature measures how much $\vec{N}(\theta)$ changes in the direction tangent to a particular curve, and torsion measures how much $\vec{N}(\theta)$ changes in the direction orthogonal to the osculating plane of the curve.

$$\kappa(\theta) = \|\vec{T}'(\theta)\| \tag{1.5}$$

$$\vec{B}'(\theta) = -\tau(\theta)\vec{N}(\theta) \tag{1.6}$$

$$\tau(\theta) = -\vec{N}(\theta) \cdot \vec{B}'(\theta) \tag{1.7}$$

A plane curve is determined only by its curvature. A disk $(cos\theta, sin\theta)$ has a unit curvature everywhere and has zero torsion $\tau(\theta)$.

Theorem 2. *A space curve is planar if and only if its torsion is everywhere zero.*

Let α be unit speed non-linear curve, then α is planar curve if and only if $\tau = 0$. Following the proof given by Koch [6]

Suppose α lies in plane P, then:

$$\vec{T}(\theta) = \lim_{\delta \to 0} \frac{\alpha(\theta + \delta) - \alpha(\theta)}{\delta}$$

is parallel to P. Consequently

$$\kappa\vec{N}(\theta) = \lim_{\delta \to 0} \frac{\vec{T}(\theta + \delta) - \vec{T}(\theta)}{\delta}$$

is also parallel to P. Then $\vec{B}(\theta) = \vec{T}(\theta) \times \vec{N}(\theta)$ is perpendicular to P $\forall$ θ. $\vec{B}(\theta)$ has unit length and varies with θ, therefore, $\vec{B}(\theta)$ is constant and $\vec{B}'(\theta) = -\tau N\vec{(\theta)} = 0$

Conversely, suppose $\tau = 0$. Then $\vec{B}(\theta)$ is constant. Fix θ_0 and define plane Q:

$$Q = \{q | (q - \alpha(\theta_0)) \cdot \vec{B}(\theta) = 0\}$$

We want to claim α is in plane Q. Substitute α for q. Then $(\alpha(\theta) - \alpha(\theta_0)) \cdot \vec{B}(\theta)$ is constant since $(\alpha(\theta) - \alpha(\theta_0))' = \vec{T}(\theta) \cdot \vec{B}(\theta) = 0$. For $\theta = \theta_0$, this constant is always zero and $\alpha(\theta)$ always satisfy the equation of the plane. QED.

1.4 Frenet-Serret Equations

The Frenet-Serret set of equations contains a compact set of differential equations that can elegantly describe the intrinsic geometric properties of a curve in Euclidean space R^3. These three equations are expressed in terms of curvature and torsion.

$$\frac{dT}{d\theta} = \kappa N \tag{1.8}$$

$$\frac{dN}{d\theta} = -\kappa T + \tau B \tag{1.9}$$

$$\frac{dB}{d\theta} = -\tau N \tag{1.10}$$

1.5 Planar Frenet-Serret Equations

Let α be a regular smooth curve parametrized by arclength, therefore, $|\alpha'| = 1$. Suppose α has a continuous third derivative. Let $\vec{T}(\theta)$ and $\vec{N}(\theta)$ be the tangent and principal normal vectors of α respectively at parameter value θ_0. Then the planar Frenet-Serret equations are:

$$\frac{dT}{d\theta} = \kappa N \tag{1.11}$$

$$\frac{dN}{d\theta} = -\kappa T \tag{1.12}$$

1.6 Developing Frenet-Serret Structure for the Apollonian Window

The present research attempts to develop a Frenet-Serret structure for the Apollonian Window. As mentioned earlier, in an Apollonian Gasket, in this case, an Apollonian Window, a fourth disk is formed from three initial tangent disks with the fourth one being tangent to each of the three parent disks. This tangent disk formation is an infinite process.

Frenet-Serret formulas measure the rate of change of the three orthonormal vectors with respect to arc-length. Therefore, they are valid only for unit-speed curve [5]. Apollonian Disk Packing involves integer curvatures, therefore, we have non-unit speed curves.

1.7 Reparametrization of a Curve

When α is a regular curve but not unit-speed curve we can perform a reparametrization $\tilde{\alpha}$ [5].

Definition 2. *Let I and J be intervals. Define $\alpha : I \mapsto \mathbb{R}^3$ be a curve and r be a differentiable function. Then the composite function $\tilde{\alpha} = \alpha \circ r$ is called reparametrization of α by the differentiable function r.*

Following [5], for $\tilde{\alpha}$, the fundamental geometric meaning of the elements of Frenet-Serret apparatus remains the same as for α. At each time $\theta \in J$, the curve $\tilde{\alpha}$ is at the point $\tilde{\alpha}(\theta) = \tilde{\alpha}(s(\theta))$. If we take θ as the time parameter, $\tilde{\alpha}$ follows the same route as α, but it reaches a given point at a different time.

Again, if curve $\alpha : I = (a, b) \mapsto \mathbb{R}^3$ is a parametrized curve, then for any $a<\theta<b$ the arclength (of the trajectory of the particle) of α from a to θ, we define the parameter

$$s = L(\theta) = \int_a^\theta \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$$

which measures the length along the curve from $\alpha(a)$ to $\alpha(\theta)$.

Following Fundamental Theorem of Calculus we know:

$$\frac{ds}{d\theta} = L'(\theta) = \sqrt{(x'(\theta))^2 + (y'(\theta))^2 + (z'(\theta))^2} = \| \alpha'(\theta) \|$$

Because $\alpha'(\theta) \neq 0$, then $L'(\theta) > 0 \; \forall \; \theta$. This means as we move along the curve distance traveled increases. An increasing function means $s = L(\theta)$ is invertible. Thus $\theta(s) = L^{-1}(s)$.

Consider a circle with radius ρ. The parametric equation for the circle:

$$\alpha(\theta) =< \rho cos\theta, \rho sin\theta > \theta \in [0, 2\pi]$$

$$\alpha'(\theta) =< -\rho sin\theta, \rho cos\theta > \theta \in [0, 2\pi]$$

Then, $$s = L(\theta) = \int_a^\theta \parallel \alpha'(t) \parallel dt = \int_a^\theta \rho dt = \rho\theta$$

Solving θ in terms of s we get $\theta(s) = L^{-1}(s) = \frac{s}{\rho}$. Thus we have

$$\alpha(\theta(s)) = L^{-1}(s) =< \rho \cos\frac{s}{\rho}, \rho \sin\frac{s}{\rho} >, s \in [0, 2\pi s]$$

1.8 Example of Parametrization and Reparametrization of a disk in Apollonian Window

$$\begin{aligned}
\alpha &=< \frac{1}{6}\cos\theta, \frac{1}{6}\sin\theta > \\
\alpha' &=< -\frac{1}{6}\sin\theta, \frac{1}{6}\cos\theta > \\
\parallel \alpha' \parallel &= \frac{1}{6} \\
\tilde{\alpha}(s) &=< \frac{1}{6}\cos 6s, \frac{1}{6}\sin 6s > \\
\tilde{\alpha}'(s) &=< -\sin 6s, \cos 6s > \\
\parallel \tilde{\alpha}'(s) \parallel &= 1
\end{aligned}$$

Then, the unit tangent $\vec{T}(\theta)$, unit normal $\vec{N}(\theta)$ and $\widetilde{T}(s), \widetilde{N}(s)$ after reparametrization are:

$$\begin{aligned}
\vec{T}(\theta) &= \frac{< -\frac{1}{6}sin(\theta), \frac{1}{6}cos(\theta) >}{\frac{1}{6}} \\
&=< -sin(\theta), cos(\theta) > \\
\vec{N}(\theta) &= \frac{< -cos(\theta), -sin(\theta) >}{\sqrt{(-cos(\theta))^2 + (-sin(\theta))^2}} \\
&=< -cos(\theta), -sin(\theta) >
\end{aligned}$$

$$\widetilde{T}(s) = \frac{< -sin(6s), cos(6s) >}{\sqrt{(-sin(6s))^2 + (cos(6s))^2}}$$
$$=< -sin(6s), cos(6s) >$$
$$\widetilde{N}(s) = \frac{< -cos(6s), -sin(6s) >}{\sqrt{(-cos(6s))^2 + (-sin(6s))^2}}$$
$$=< -cos(6s), -sin(6s) >$$

Therefore, for the Apollonian Window, a parametric curve and a reparametrized curve will both produce unit tangent, normal and binormal vectors. This is because the Apollonian Window consists of all but one non-unit constant speed curves. In a constant speed curve, acceleration is orthogonal to velocity, thus $\tilde{\alpha} \cdot \tilde{\alpha}$ being constant is equivalent to $\tilde{\alpha}' \cdot \tilde{\alpha}' = 2\tilde{\alpha}' \cdot \tilde{\alpha}'' = 0$ [5].

Therefore, it is possible to give a Frenet-Serret structure to the Apollonian Window using parametric equation. Because infinite number of points of tangency for each disk can be generated, it will be interesting to see the evolution of the unit tangent, unit normal, and the unit binormal vectors in the Apollonian Window.

The Apollonian Window constitute of two lines of reflective symmetry. One line runs through the centers of the disks of radius two, and the other line passes through the center of the disk of radius three. Because these two lines are perpendicular, the Apollonian Window falls under the rotational symmetry of degree two. Therefore, if we develop the Frenet-Serret structure for the Apollonian Window of any one quadrant, we can to have the structure for the entire Window.

CHAPTER 2

METHODOLOGY

2.1 Generation of Curvatures in Apollonian Window

Because of the quadratic nature of Descartes formula it is difficult to calculate the curvatures of the subsequent disks. In 1974 Boyd solved the Descartes formula through the process of linearization [9]. Starting with three mutually tangent disks with curvatures a, b, c the two disks tangent to those three initial disks have curvatures d and d' satisfying the linear equation:

$$d + d' = 2\,(a + b + c) \tag{2.1}$$

If the initial Descartes configuration has disks of integral curvatures, then the entire packing is integral and vice versa, an observation made by Soddy (1937) [3].

The present study begins with three mutually tangent disks of the structure of an Apollonian Window. It starts with a disk A of radius and curvature one, and two disks, B and C, contained inside disk A. These three disks are externally tangent to each other. The radii of disks B and C are $\frac{1}{2}$, therefore curvatures of B and C are 2. When disks B and C are placed inside disk A, two ideal triangles are formed. This can be named as the zero generation consisting the set of curvatures $[-1, 2, 2, 3]$. Here we have assigned negative curvature to the bounding large disk A, so that all the other disks will have positive curvatures, maintaining the validity of the Descartes formula [10].

In the second step, two ideal triangles fit two tangent disks D and E as large as possible with curvatures 3. This results in the formation of six more ideal triangles inside disk A. Three disks with curvatures 6, 6, 15 fit both on the top and bottom ideal triangles respectively because of vertical and horizontal mirror symmetry of the Apollonian Window. This results in the formation of three sets of curvatures $[6, 3, -1, 14]$, $[2, 6, -1, 11]$ and $[2, 3, 6, 23]$, each on the four quadrants of the large disk A. We can name these three sets as the first generation curvatures.

In a similar process a second generation of 9 sets of curvatures are formed from 9 sets of mutually tangent disks. And in the third generation 27 sets of curvatures are generated from 27 groups of mutually tangent disks. This process of ideal triangle formation and disks fill the entire plane and it is an infinite process.

The following table provides a description of generations of curvatures and the number of ideal triangles and the sets of mutually tangent circles. Because of symmetric nature of Apollonian Window the table only describes the first quarter of larger disk of radius one.

Table 2.1: Number of Tangent Discs for First Three Generations

Generation	No. of Ideal Triangles	No. of Sets of Mutually Tangent Disks
0	1	1
1	3	3
2	9	9
3	27	27

2.1.1 Algorithm to Generate Curvatures for the Apollonian Window

The present study followed the algorithm [9] in order to generate curvatures of the disks in the ideal triangles:

$$a' = 2\,(a + b + c + d) - 3a \tag{2.2}$$

where a, b, c are the curvatures of the three initial tangent disks and d is the solution disk that is inscribed inside the ideal triangle formed by disks of curvatures a, b and c. a' is the curvature of the disk that results from the three tangent disks with curvatures b, c and d. For the next generation, disk a is replaced by disk d of the previous generation whereas disk b and c remain unchanged for the same branch. At every generation a set of three children (sets of curvatures) is produced from each branch due to the formation of three ideal triangles at each branch. Through this recurrence formula it is possible to produce infinite number of disks in the Apollonian Window.

It should be noted that it is possible to generate an infinite number of curvatures for an infinite number of generations starting from generation zero if the following generations are from the same branch, e.g., disks b and c remain unchanged for all the generations.

The present research generated 1000 data of curvatures for generation one, two and three. This data is produced in excel using visual basic code.

Standard Parametrization of a Circle: There exists a natural parametrization of a circle by the very definition of trigonometric functions [11]:

$$\alpha(\theta) = \rho(cos\theta, sin\theta)$$

$$= \rho cos(\theta), \rho sin(\theta) \quad 0 \leq \theta \leq 2\pi$$

Standard Parametrization of a circle with radius ρ

$$\alpha(\theta) = (\rho \cos\theta, \rho \sin\theta), \theta \in [0, 2\pi]$$

$$\alpha'(\theta) = (-\rho \sin\theta, \rho \cos\theta)$$

$$\parallel \alpha'(\theta) \parallel = \rho$$

2.2 Frenet-Serret Structure for the Apollonian Window

In order to develop a Frenet-Serret structure for the Apollonian Window, the first step is to show the tangent points of the disks. The following tree diagram has been created to show the tangent points for each generation. For example, the tangent points of the disks with curvatures 2, 3 and -1 for generation zero are the points $(2, 3), (2, -1)$ and $(3, -1)$. The pair of tangent points of the disks with curvatures $(2, 3)$ and $(3, -1)$ produces tangent point at $(3, 6)$. Similarly, $(2, 3)$ and $(2, -1)$ produces tangent point at $(2, 6)$ for generation one. The following tree diagram shows the tangent points upto the fourth generation.

The next step calculates the elements of the Frenet-Serret apparatus to develop a **Frenet-Serret structure for the Apollonian Window.**

- $(2,3,-1)$
  - $(2,3)(3,-1) \to (3,6)$
    - $(3,6)(3,-1) \to (3,14)$
      - $(3,14)(3,-1) \to (3,26)$
        - $(3,26)(3,-1) \to (3,42)$
        - $(3,26)(3,14) \to (3,87)$
      - $(3,14)(3,6) \to (3,47)$
        - $(3,47)(3,6) \to (3,98)$
        - $(3,47)(3,14) \to (3,122)$
    - $(3,6)(2,3) \to (3,23)$
      - $(3,23)(2,3) \to (3,50)$
        - $(3,50)(2,3) \to (3,87)$
        - $(3,50)(3,23) \to (3,150)$
      - $(3,23)(3,6) \to (3,62)$
        - $(3,62)(3,6) \to (3,119)$
        - $(3,62)(3,23) \to (3,170)$
  - $(2,3)(2,-1) \to (2,6)$
    - $(2,6)(2,-1) \to (2,11)$
      - $(2,11)(2,-1) \to (2,18)$
        - $(2,18)(2,-1) \to (2,95)$
        - $(2,18)(2,11) \to (2,63)$
      - $(2,11)(2,6) \to (2,39)$
        - $(2,39)(2,6) \to (2,83)$
        - $(2,39)(2,11) \to (2,98)$
    - $(2,6)(2,3) \to (2,23)$
      - $(2,23)(2,3) \to (2,50)$
        - $(2,50)(2,3) \to (2,87)$
        - $(2,50)(2,23) \to (2,147)$
      - $(2,23)(2,6) \to (2,59)$
        - $(2,59)(2,6) \to (2,111)$
        - $(2,59)(2,23) \to (2,182)$

Figure 2.1: Tree Diagram of Tangent Points

We can also derive equations of the three planes, namely, the osculating, normal and rectifying plane for the Apollonian Win-dow.

To understand how curvature influence the shape of $\alpha(\theta)$ Taylor Polynomial will be calculated for different θ.

$$\alpha(\theta) \approx \alpha(\theta_0) + \theta\alpha'(\theta_0) + \frac{\theta^2}{2}\alpha''(\theta_0) + \frac{\theta^3}{6}\alpha'''(\theta_0) + ... \tag{2.3}$$

Denote

$$\alpha(\theta_0) = \alpha, \kappa(\theta_0) = \kappa, \tau(\theta_0) = \tau, \vec{T}(\theta_0) = T, \vec{N}(\theta_0) = N, \vec{B}(\theta_0) = B$$

It should be noted

$$\begin{aligned} \alpha' &= T \\ \alpha'' &= T' = \kappa N \\ \alpha''' &= (\kappa N)' = \kappa' N + \kappa N' \\ &= \kappa' N + \kappa(-\kappa T + \tau B) \end{aligned}$$

Then we can write

$$\alpha(\theta) \approx \alpha(\theta_0) + \theta T_0 + \frac{\theta^2}{2}\kappa_0 N_0 + \frac{\theta^3}{6}\kappa_0\tau_0 B_0 \tag{2.4}$$

The first two terms in Equation 4 denote linearization and the third term is the second order approximation with respect to arclength. The fourth term involves torsion and because we only have planar curve (i.e., $\vec{B}$ is constant) for the Apollonian Window, the fourth term vanishes.

CHAPTER 3

CALCULATIONS AND FINDINGS

For the first part of the research, 1000 curvature data have been generated for generation one, two and three of the Apollonian Window.

In this chapter the elements of the TNB apparatus for the Apollonian Window have been calculated. Equations of planes for specific θs are also shown. The following table provides the elements of TNB apparatus for the tangent points of the first three generations. As can be seen, for the first two parametric equations of generation zero, $\vec{T}(\theta)$ and $\vec{N}(\theta)$ are $< -sin(\theta), cos(\theta), 0 >$ and $< -cos(\theta), -sin(\theta), 0 >$ respectively. For the parametric equation $< -sin(\theta), -cos(\theta), 0 >$, $\vec{T}(\theta)$and $\vec{N}(\theta)$ are $< sin(\theta), -cos(\theta), 0 >$ and $< cos(\theta), sin(\theta), 0 >$ respectively. As can be seen from the table, the parametric equations vary only by a scalar value. Therefore, $\vec{T}(\theta)$ and $\vec{N}(\theta)$ are the same for the entire Apollonian Window. Because planar curve has constant $\vec{B}(\theta)$ we see that $\vec{B}(\theta) = \hat{k}$ for the entire Window irrespective of the curvature of the disks for different generations.

Table 3.1 provides the parametric equation of the discs at different tangent points for the first three generations. It also shows the unit tangent, normal and biromal vectors as well as curvatures for the first three generations. Table 3.2 gives the equations of the osculating, normal and rectifying planes for specific θ.

Table 3.1: Tangent, Normal and Binormal Vectors at Tangent Points for First Three Generations

Generation	Tangent Points	Parametric Equation of Discs at Tangent Points	Unit Tangent	Unit Normal	Unit Binormal	Curvature
		$< \frac{1}{2}cos(\theta), \frac{1}{2}sin(\theta), 0 >$	$< -sin(\theta), cos(\theta), 0 >$	$< -cos(\theta), -sin(\theta), 0 >$	$\hat{k}$	2
		$< \frac{1}{3}cos(\theta), \frac{1}{3}sin(\theta), 0 >$	$< -sin(\theta), cos(\theta), 0 >$	$< -cos(\theta), -sin(\theta), 0 >$	$\hat{k}$	3
0	$(2,3,-1) \rightarrow (2,3)(2,-1)(3,-1)$	$< -cos(\theta), -sin(\theta), 0 >$	$< sin(\theta), -cos(\theta), 0 >$	$< cos(\theta), sin(\theta), 0 >$	$\hat{k}$	-1
	$(2,3)(2,-1) \rightarrow (2,6)$	$< \frac{1}{6}cos(\theta), \frac{1}{6}sin(\theta), 0 >$	$< -sin(\theta), cos(\theta, 0 >$	$< -cos(\theta), -sin(\theta), 0 >$	$\hat{k}$	6
1	$(2,3)(3,-1) \rightarrow (3,6)$	$< \frac{1}{6}cos(\theta), \frac{1}{6}sin(\theta), 0 >$	$< -sin(\theta), cos(\theta), 0 >$	$< -cos(\theta), -sin(\theta), 0 >$	$\hat{k}$	6
	$(2,6)(2,3) \rightarrow (2,23)$	$< \frac{1}{23}cos(\theta), \frac{1}{23}sin(\theta), 0 >$	$< -sin(\theta), cos(\theta), 0 >$	$< -cos(\theta), -sin(\theta), 0 >$	$\hat{k}$	23
	$(2,6)(2,-1) \rightarrow (2,11)$	$< \frac{1}{11}cos(\theta), \frac{1}{11}sin(\theta), 0 >$	$< -sin(\theta), cos(\theta), 0 >$	$< -cos(\theta), -sin(\theta), 0 >$	$\hat{k}$	11
	$(3,6)(2,3) \rightarrow (3,23)$	$< \frac{1}{23}cos(\theta), \frac{1}{23}sin(\theta), 0 >$	$< -sin(\theta), cos(\theta), 0 >$	$< -cos(\theta), -sin(\theta), 0 >$	$\hat{k}$	23
2	$(3,6)(3,-1) \rightarrow (3,14)$	$< \frac{1}{14}cos(\theta), \frac{1}{14}sin(\theta), 0 >$	$< -sin(\theta), cos(\theta), 0 >$	$< -cos(\theta), -sin(\theta), 0 >$	$\hat{k}$	14

Table 3.2: Equation of the Three Planes for Specific θ

θ	$\vec{T}(\theta) \perp$ Normal Plane	$\vec{B}(\theta) \perp$ Osculating Plane	$\vec{N}(\theta) \perp$ Rectifying Plane
0	$y = 0$	$z = 0$	$x = \frac{1}{6}$
$\frac{\pi}{6}$	$x = \sqrt{3}y$	$z = 0$	$y = \frac{1}{3} - \sqrt{3}x$
$\frac{\pi}{4}$	$x = y$	$z = 0$	$y = \frac{1}{3\sqrt{2}} - x$
$\frac{\pi}{3}$	$x = \frac{y}{\sqrt{3}}$	$z = 0$	$y = \frac{1}{3\sqrt{3}} - \frac{x}{\sqrt{3}}$
$\frac{\pi}{2}$	$x = 0$	$z = 0$	$y = \frac{1}{6}$

Table 3.3 provides the planar Frenet-Serret equations for the first three generations of the Apollonian Window.

Table 3.3: Planar Frenet-Serret Equations for the First Three Generations

Generation	$\vec{T}'(\theta)$	$\vec{N}'(\theta)$
0	$< -2cos(\theta), -2sin(\theta) >$ $< -3cos(\theta), -3sin(\theta) >$ $< -sin(\theta), -cos(\theta) >$	$< 2sin(\theta), -2cos(\theta) >$ $< 3sin(\theta), -3cos(\theta) >$ $< -cos(\theta), -sin(\theta) >$
1	$< -6cos(\theta), -6sin(\theta) >$	$< 6sin(\theta), -6cos(\theta) >$
2	$< -23cos(\theta), -23sin(\theta) >$ $< -11cos(\theta), -11sin(\theta) >$ $< -14cos(\theta), -14sin(\theta) >$	$< 23sin(\theta), -23cos(\theta) >$ $< 11sin(\theta), -11cos(\theta) >$ $< 14sin(\theta), -14cos(\theta) >$

Table 3.4 provides Taylor series approximation for specific θs for the parametric curve $< \frac{1}{6}\, cos(\theta),\ \frac{1}{6}\, sin(\theta),\ 0 >$.

Table 3.4: Taylor Series Approximation for a Parametric Curve

θ	Taylor Series Approximation for Parametric Curve $< \frac{1}{6}cos(\theta), \frac{1}{6}sin(\theta), 0 >$
0	$< \frac{2-\theta^2}{12}, \frac{\theta}{6}, 0 >$
$\frac{\pi}{6}$	$< \frac{2\sqrt{3}-2\theta-\sqrt{3}\theta^2}{24}, \frac{2+2\sqrt{3}\theta-\theta^2}{24}, 0 >$
$\frac{\pi}{4}$	$< \frac{2\sqrt{2}-2\sqrt{2}\theta-\sqrt{2}\theta^2}{24}, \frac{2\sqrt{2}+2\sqrt{2}\theta-\sqrt{2}\theta^2}{24}, 0 >$
$\frac{\pi}{3}$	$< \frac{2-2\sqrt{3}\theta-\theta^2}{24}, \frac{2\sqrt{3}+2\theta-\sqrt{3}\theta^2}{24}, 0 >$
$\frac{\pi}{2}$	$< \frac{-\theta}{6}, \frac{2-\theta^2}{12}, 0 >$

3.1 Summary and Conclusion

The present research had twofold purpose. The first one was to generate Apollonian data. By using the algorithm provided in [9] 1000 curvature data have been produced for first four generations. This was done by writing Visual Basic code in Excel.

The second part of the thesis focused on the Frenet-Serret structure and the Apollonian Window. A tree diagram is formed to show the tangent points of the disks for the first five generations. Parametric equations for the disks of the first three generations of Apollonian Window is shown. Based on these equations, the three orthnormal basis vectors, namely, tangent $\vec{T}(\theta)$, principal normal $\vec{N}(\theta)$ and binormal $\vec{B}(\theta)$ vectors, curvature κ

and torsion τ of the disks are calculated. The values for the curvatures are in agreement with the curvatures originally given in the Apollonian Window as expected.

The research also calculates the equations of the three planes, namely, the normal, osculating and rectifiying planes for specific θ. The planar Frenet-Serret equations for the first three generations are shown. Taylor series approximation for different values of θ for a parametric curve is also calculated.

Because we have planar curves in the Apollonian Window, the curve α never leaves the osculating plane. This property is shown by the finding that $\vec{B}(\theta)$ is constant and torsion $\tau = 0$.

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