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Approximation Of Continuously Distributed Delay Differential Equations

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APPROXIMATION OF CONTINUOUSLY DISTRIBUTED DELAY DIFFERENTIAL EQUATIONS

by

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We establish a theorem on the approximation of the solutions of delay differential equations with continuously distributed delay with solutions of delay differential equations with discrete delays. We present numerical simulations of the trajectories of discrete delay differential equations and the dependence of their behavior for various delay amounts. We further simulate continuously distributed delays by considering discrete approximation of the continuous distribution.

DEDICATION

This is dedication to my parents and my beloved sister for supporting me all the way.

ACKNOWLEDGMENTS

I would like to express my sincere appreciation to my supervisor Dr. Harry Randolph Hughes for his constant guidance, encouragement, and taking time to talk with me on many occasions. Also my sincere thanks goes to Dr. Michael Sullivan and Dr. David Olive the members of my graduate committee.

A special thanks also to J. K. Hale [3] whose text book is the main reference of this thesis.

PREFACE

Delay differential equations (DDE) or simply the system of differential equations with a time lag is widely used in applied mathematics areas such as population dynamics, prey-predator system analysis, the study of epidemics, automation and other areas in biology and engineering. The delay differential equation concept is widely used among the modern day researchers due to rapid progress in the understanding and applicability of delay differential equations and systems.

In this thesis, the convergence of systems of delay differential equations is discussed according to the continuous dependency on the parameters of the system of delay differential equations and through the graphical interpretations of how the behavior of the system of delay differential equations changes according to the time lag of the system of delay differential equations using the MATLAB programming.

In the study of the convergence of the systems of delay differential equations, I used the book "Functional Differential Equations" by Jack K. Hale [3] as my main reference. The proofs of some lemmas and theorems in this book are slightly elaborated upon to make them easier to understand.

The organization of this thesis is as follows. In the first chapter, delay differential equations are introduced. Further, a few general examples of delay differential equations are briefly discussed in this chapter. Chapter 2 contains a few important theoretical concepts of the existence, uniqueness and continuous dependence of solutions of delay differential equations which are taken from Hale's book [3].

In Chapter 3, results concerning the solutions of systems of delay differential equations are established using a continuous dependence theorem. The results in this chapter are developed by using the theoretical concepts in Chapter 2. In the first part of this chapter, it is shown that solutions of certain systems with discrete delays converge to solutions of systems with uniformly distributed continuous delays. In the second part, the convergence of solutions of systems with discrete delays to systems with more general distributions of continuous delay are discussed.

Chapter 4 concerns the numerical simulations and investigation of the trajectories of a few delay differential equations according to the delay (time lag). The graphical simulations were obtained by using the MATLAB routine "dde23" which was designed to solve delay differential equations.

Chapter 5 discusses modifications of the few examples in Chapter 4 which illustrates the trajectory changes relative to equilibria when averages of multiple discrete delay times are used to approximate uniformly distributed continuous delays. The graphical simulations are obtained by using MATLAB programming.

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CHAPTER 1

BASIC CONCEPTS

In this section, we review the definition and basic properties of delay differential equations. Our main reference is "Functional Differential Equations" by J. K. Hale (1997) [3].

1.1 DELAY DIFFERENTIAL EQUATION (DDE)

Definition. A differential equation in which the derivative of the unknown function is given at a certain time in terms of the values of the function at the previous times, is called a delay differential equation ("DDE").

The general form of the time dependent DDE for the function $x(t)$ in $\mathbb{R}^n$ is given by:

$$\frac{dx(t)}{dt} = \dot{x}(t) = f(t, x(t), x_t);$$

where $f : \mathbb{R} \times \mathbb{R}^n \times C(\mathbb{R}, \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ and $x_t = x(\Gamma)$ for $\Gamma \leq t$; (x_t :=the trajectory of the solution in the past).

Let's identify a few examples which illustrate the types of Delay Differential equations.

Example 1.1. Continuous DDE

The continuous delay differential equation has the form as:

$$\dot{x}(t) = f\left(t, x(t), \int_{-\infty}^{0} x(t+\Gamma) d\mu(\Gamma)\right),$$

where μ is a distribution.

Example 1.2. Discrete DDE

The discrete delay differential equation has the form as:

$$\dot{x}(t) = f(t, x(t), x(t - \Gamma_1), ..., x(t - \Gamma_m)) \text{ for } \Gamma_1 > ... > \Gamma_m \geq 0.$$

Example 1.3. Linear with discrete DDE

The linear discrete delay differential equation has the form as:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \Gamma_1) + ... + A_m x(t - \Gamma_m),$$

for $\Gamma_1 > ... > \Gamma_m \geq 0$ and $A_0, ..., A_m \in \mathbb{R}^{n \times n}$.

1.2 DELAY LOGISTIC EQUATIONS

Delay logistic equations are the simplest type of delay differential equations where the time lag is associated only with the state variable but not with the derivative of the state variable, the so called retarded delay differential equation or retarded functional differential equation. The delay logistic equations can be divided into two main categories as:

(1) Discrete delay logistic equations

(2) Distributed delay logistic equations.

Let's discuss briefly these two types of delay logistic equations.

1.2.1 Discrete Delay Logistic Equations

The general form of the delay logistic equations with a single discrete delay can be written as $\dot{x}(t) = F(t, x(t), x(t - \tau))$, where τ represents the time lag.

In 1948, Hutchinson used this discrete delayed logistic concept to make sense of the biological mechanism which is known as Hutchinson's equations [4]. It is a special case of the following equation:

$$\dot{N}(t) = \gamma N(t) \left(1 - \frac{N(t - \tau)}{K}\right) \tag{1.1}$$

where the parameter γ represents the intrinsic growth while K represents the carrying capacity. This Hutchinson's equation is also known as Wright's equation or delayed logistic equation with a discrete delay.

In application to the probability distribution of prime numbers, Lord Cherwell also had used this concept of delayed logistic equation given by the equation (1.1) with a single discrete delay [12].

1.2.2 Distributed Delay Logistic Equations

The general logistic delay differential equation with distributed delay is given by the following integro-differential equation.

$$\dot{x}(t) = \alpha x(t)\left(1 - \int_{-\infty}^{t} x(s)g(s,t)ds\right) \tag{1.2}$$

$$x(t) = \phi(t); \qquad t \leq 0 \text{ for a given initial function } \phi(t),$$

where $g(s,t)$ represents the weight function and α represents a given positive constant. The following integro-differential equation is an example of a distributed delay logistic equation which was used in the parasite population growth model by MacDonald [6]:

$$\dot{N}(t) = \gamma N(t)\left(1 - \frac{N(t)}{K} - \int_{0}^{t} N(s)g(t-s)ds\right) \tag{1.3}$$

where the parameter γ represents the intrinsic growth while K represents the carrying capacity of the population.

CHAPTER 2

BASIC THEORY OF DELAY DIFFERENTIAL EQUATIONS

Let's consider the following notation throughout this chapter:

1. $\mathbb{R}^n :=$ $n-$dimensional real Euclidean space with norm $|.|$.

2. $\mathbb{R} :=$ $1-$dimensional Euclidean space.

3. $C([a,b],\mathbb{R}^n)$:=the Banach space of continuous function from the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence.

4. $\|\phi\| = \sup_{a\leq\theta\leq b}|\phi(\theta)|$; where $\phi \in C([a,b],\mathbb{R}^n)$.

5. $C = C([-r,0],\mathbb{R}^n)$, the special case where $[a,b] = [-r,0]$.

2.1 EXISTENCE

Definition. Suppose D is a subset of $\mathbb{R} \times C$ and the function $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$ where $(t, x_t) \in D$ for $\sigma \in \mathbb{R}$, $A \geq 0$ and $t \in [\sigma, \sigma + A)$.
Here x_t is defined as $x_t(\theta) = x(t+\theta); \theta \in [-r,0]$. Then consider the function f such that $f : D \longrightarrow \mathbb{R}^n$. Then,

$$\dot{x}(t) = f(t, x_t) \tag{2.1}$$

is known as a retarded delay differential equation (RDDE(f)) or a retarded functional differential equation (RFDE(f)) on D.

A function x is called a solution of equation (2.1) through (σ, φ) if the function $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$ satisfies $(t, x_t) \in D$ for $\sigma \in \mathbb{R}, A \geq 0$ and $t \in [\sigma, \sigma + A)$, satisfies the equation (2.1), and $x_\sigma = \varphi$.

The following Schauder's theorem is an important theorem which shows the existence of a fixed point for a given continuous function. This theorem is widely used in most of the proofs in this chapter.

Theorem 2.1. *Schauder's fixed point theorem* [2, *p*.161]
Let X be a separated locally convex topological vector space and let K be a non-void, compact and convex subset of the space X. Then given any continuous function $f : K \longrightarrow K$ there exists a point $x \in K$ such that $f(x) = x$.

Then Schauder's fixed point theorem implies the existence of a solution of the retarded delay differential equation (2.1). The following existence theorem is taken from the Hale's book [3, p.13]. The proof is elaborated upon in order to make it easier to understand.

Theorem 2.2. *Suppose D is an open subset in $\mathbb{R} \times C$ and $f : D \longrightarrow \mathbb{R}^n$ is continuous. If $(\sigma, \varphi) \in D$ then the equation* (2.1) *has a solution through (σ, φ).*

Proof. For any real numbers α, β let's define:

$$I_\alpha = \{t : 0 \leq t \leq \alpha\},$$

$$B_\beta = \{\varphi \in C : \mid \varphi \mid \leq \beta\}.$$

Suppose f is continuous on D. If $\mid f(\sigma, \varphi) \mid < M$, then by using the continuity of f, there are α, β such that $\mid f(\sigma + t, \varphi + \psi) \mid \leq M$ for $(t, \psi) \in I_\alpha \times B_\beta$. Now let

$$\mathcal{A}(\overline{\alpha}, \overline{\beta}) = \left\{\eta \in C([-r, \overline{\alpha}], \mathbb{R}^n) : \eta_0 = 0, \eta_t \in B_{\overline{\beta}}, t \in I_{\overline{\alpha}}\right\} \tag{2.2}$$

for any non negative reals $\overline{\alpha}, \overline{\beta}$. Let's define a function $\widetilde{\varphi} \in C([\sigma - r, \sigma + \overline{\alpha}], \mathbb{R}^n)$ such that

$$\widetilde{\varphi}_\sigma = \varphi,$$
$$\widetilde{\varphi}(t + \sigma) = \varphi(0) \text{ ; for } t \in I_{\overline{\alpha}}.$$

Suppose $\overline{\beta} < \beta$ and choose $\overline{\alpha} < \alpha$ so that $\mid \widetilde{\varphi}_{\sigma+t} - \varphi \mid < \beta - \overline{\beta}$ for $t \in I_{\overline{\alpha}}$ and $M\overline{\alpha} \leq \overline{\beta}$.

Then for $t \in I_{\overline{\alpha}}$ and $\eta \in \mathcal{A}(\overline{\alpha}, \overline{\beta})$

$$
\begin{aligned}
| \eta_t + \widetilde{\varphi}_{t+\sigma} - \varphi | &\leq | \eta_t | + | \widetilde{\varphi}_{t+\sigma} - \varphi | \\
&\leq \overline{\beta} + \beta - \overline{\beta} = \beta,
\end{aligned}
$$

and thus $| \; f(\sigma + t, \eta_t + \widetilde{\varphi}_{t+\sigma}) \; | \leq M$ for $t \in I_{\overline{\alpha}}$ and $\eta \in \mathcal{A}(\overline{\alpha}, \overline{\beta})$. Let's consider the transformation, $T : \mathcal{A}(\overline{\alpha}, \overline{\beta}) \longrightarrow C([-r, \overline{\alpha}], \mathbb{R}^n)$ such that

$$
\begin{aligned}
(T\eta)(t) &= \int_0^t f(\sigma + s, \eta_s + \widetilde{\varphi}_{s+\sigma}) ds; \qquad t \in I_{\overline{\alpha}}, \\
(T\eta)_0 &= 0.
\end{aligned}
$$

Suppose η is a fixed point of T, then x is a solution of equation (2.1) through (σ, φ) if it is related by $x_{\sigma+t} = \widetilde{\varphi}_{\sigma+t} + \eta_t$. Then finding the solution of the equation (2.1) is equivalent to finding fixed points of the transformation T in $\mathcal{A}(\overline{\alpha}, \overline{\beta})$. The existence of fixed points of T can be proved by using Schauder's fixed point theorem. The set $\mathcal{A}(\overline{\alpha}, \overline{\beta})$ is a closed, bounded and convex subset of $C([-r, \overline{\alpha}], \mathbb{R}^n)$.

Claim: $T\mathcal{A}(\overline{\alpha}, \overline{\beta})$ belongs to a compact subset of $C([-r, \overline{\alpha}], \mathbb{R}^n)$.

Consider, for $t \in I_{\overline{\alpha}}$,

$$
\begin{aligned}
|(T\eta)(t)| &= \left| \int_0^t f(\sigma + s, \eta_s + \widetilde{\varphi}_{s+\sigma}) ds \right| \\
&\leq \int_0^t |f(\sigma + s, \eta_s + \widetilde{\varphi}_{s+\sigma})| \, ds \\
&\leq Mt \leq M\overline{\alpha} \leq \overline{\beta}.
\end{aligned}
$$

This implies that $T(\mathcal{A}(\overline{\alpha}, \overline{\beta})) \subset \mathcal{A}(\overline{\alpha}, \overline{\beta})$. Since

$$
\begin{aligned}
| \; (T\eta)(t) - (T\eta)(\overline{t}) \; | &= \left| \int_{\overline{t}}^t f(\sigma + s, \eta_s + \widetilde{\varphi}_{s+\sigma}) ds \right| \\
&\leq \int_{\overline{t}}^t | \; f(\sigma + s, \eta_s + \widetilde{\varphi}_{s+\sigma}) \; | \, ds \\
&\leq M \; | \; t - \overline{t} \; |; \qquad \forall t, \overline{t} \in I_{\overline{\alpha}},
\end{aligned}
$$

$T\mathcal{A}(\overline{\alpha},\overline{\beta})$ is equicontinuous and $T\mathcal{A}(\overline{\alpha},\overline{\beta})$ belongs to a compact subset of $C([-r,\overline{\alpha}],\mathbb{R}^n)$.

Claim: T is continuous on $\mathcal{A}(\overline{\alpha},\overline{\beta})$.

Let $\{\eta_k\}$ be a sequence in $\mathcal{A}(\overline{\alpha},\overline{\beta})$ which converges to $\eta \in \mathcal{A}(\overline{\alpha},\overline{\beta})$. Then there is a subsequence $\{\eta_{k_i}\}$ of $\{\eta_k\}$ such that $T_{\eta_{k_i}} \longrightarrow \Upsilon$ as $i \longrightarrow \infty$, since $T\mathcal{A}(\overline{\alpha},\overline{\beta})$ belongs to a compact subset of $C([-r,\overline{\alpha}],\mathbb{R}^n)$.

Then, $f(\sigma+s,(\eta_{k_i})_s+\widetilde{\varphi}_{s+\sigma}) \longrightarrow f(\sigma+s,\eta_s+\widetilde{\varphi}_{s+\sigma})$, as $i \longrightarrow \infty$ for $s \in I_{\overline{\alpha}}$,

and f is bounded on $\mathcal{A}(\overline{\alpha},\overline{\beta})$).

Thus by using the Lebesgue dominated convergence theorem,

$$\lim_{i\to\infty}\left[\int_0^t f(\sigma+s,(\eta_{k_i})_s+\widetilde{\varphi}_{s+\sigma})ds\right] = \int_0^t f(\sigma+s,\eta_s+\widetilde{\varphi}_{s+\sigma})ds = (T\eta)(t). \tag{2.3}$$

Thus we have $\Upsilon(t) = (T\eta)(t)$. That is the limit of any convergent subsequence is independent of the subsequence. This implies T is continuous on $\mathcal{A}(\overline{\alpha},\overline{\beta})$.

Therefore T satisfies all the conditions of Schauder's theorem and hence T has a fixed point on $\mathcal{A}(\overline{\alpha},\overline{\beta})$. This means that the equation (2.1) has a solution through (σ,ϕ).

□

2.2 UNIQUENESS

Definition. Let $f : D \longrightarrow \mathbb{R}^n$ where D is an open subset in $\mathbb{R}\times C$ and K be a compact set in $\mathbb{R}\times C$. Then $f(t,\varphi)$ is said to be Lipschitz in φ in K if there is a constant k (>0) such that for any $(t,\varphi_i) \in K, (i=1,2)$,

$$\|f(t,\varphi_1)-f(t,\varphi_2)\| \le k\|\varphi_1-\varphi_2\|. \tag{2.4}$$

The uniqueness of the solution of the retarded delay differential equation (2.1) follows by the following uniqueness theorem which is taken from Hale's book [3, p.22]. Here also the proof is slightly elaborated upon to understand it easily.

Theorem 2.3. *Suppose D is an open subset in $\mathbb{R} \times C$, $f : D \longrightarrow \mathbb{R}^n$ is continuous and $f(t, \varphi)$ is Lipschitz in φ in each compact set in D. If $(\sigma, \varphi) \in D$ then the equation* (2.1) *has a unique solution through* (σ, φ).

Lemma 2.4. *If $f(t, \varphi)$ is a continuous function, $\varphi \in C$ and $\sigma \in \mathbb{R}$ then solving equation (2.1) is equivalent to solving the following integral equation*

$$x(t) = \varphi(0) + \int_{\sigma}^{t} f(s, x_s) ds, t \geq \sigma, x_\sigma = \varphi. \tag{2.5}$$

Proof. (Theorem 2.3)

For any real numbers α, β let's define:

$$I_\alpha = \{t : 0 \leq t \leq \alpha\}$$

$$B_\beta = \{\varphi \in C : \mid \varphi \mid \leq \beta\}.$$

Now suppose x, y are the solutions of equation (2.1) on $[\sigma - r, \sigma + \alpha]$ with $x_\sigma = \varphi = y_\sigma$. Then by Lemma (2.3) we have

$$x(t) = \varphi(0) + \int_{\sigma}^{t} f(s, x_s) ds; \qquad t \geq \sigma$$
$$y(t) = \varphi(0) + \int_{\sigma}^{t} f(s, y_s) ds; \qquad t \geq \sigma.$$

This implies

$$x(t) - y(t) = \int_{\sigma}^{t} (f(s, x_s) - f(s, y_s)) ds; \qquad t \geq \sigma$$

$$x_\sigma - y_\sigma = 0.$$

Let k be the Lipschitz constant of $f(t, \varphi)$ in any compact set $W \subseteq D$ such that points $(t, x_t) \in W$ and $(t, y_t) \in W$ for $t - \sigma \in I_\alpha$. Then we have

$$\mid f(t, x_t) - f(t, y_t) \mid \leq k \mid x_t - y_t \mid. \tag{2.6}$$

Choose $\bar{\alpha}$ so that $k\bar{\alpha} < 1$. Then for $t - \sigma \in I_{\bar{\alpha}}$, we have

$$\begin{aligned}
| x(t) - y(t) | &= \left| \int_\sigma^t (f(s, x_s) - f(s, y_s))ds \right| \\
&\leq \int_\sigma^t | (f(s, x_s) - f(s, y_s)) |ds \\
&\leq \int_\sigma^t k | x_s - y_s |ds \\
&\leq k(t - \sigma) \sup_{\sigma \leq s \leq t} | x_s - y_s | \\
&\leq k\bar{\alpha} \sup_{\sigma \leq s \leq t} | x_s - y_s |; \qquad t - \sigma \in I_{\bar{\alpha}}.
\end{aligned}$$

This implies $x(t) = y(t)$ for $t - \sigma \in I_{\bar{\alpha}}$. Thus the equation (2.1) has a unique solution through (σ, φ).

□

2.3 CONTINUOUS DEPENDENCE

In the sections (2.1) and (2.2) we discussed the existence and uniqueness of the solution of the delay differential equation (2.1). Now, let's discuss the continuous dependence of the system of delay differential equations. The following continuous dependence theorem is taken from the Hale's book [3, p.21]. Here the proof of the continuous dependence theorem is slightly elaborated upon to understand it easily.

Theorem 2.5. *Suppose D is an open subset in $\mathbb{R} \times C$, $(\sigma, \varphi) \in D$ and $f, f^k : D \longrightarrow \mathbb{R}^n$ for $k = 1, 2, 3, ...$ are continuous functions on D. Let $W \subseteq D$ be the compact set defined by*

$$W = \{(t, x_t) : t \in [\sigma, b]\},$$

and let V be a neighborhood of W. Let's take the equation

$$\dot{x}(t) = f^k(t, x_t); \qquad k = 1, 2, ... \tag{2.7}$$

Suppose;

(a) x is a solution of the equation (2.1) through (σ, φ) which exists and is unique on

$[\sigma - r, b]$, $b > \sigma$.

(b) f and f^k are bounded on V for each $k = 1, 2, ...$

(c) If $(\sigma^k, \varphi^k, f^k); k = 1, 2, ...,$ satisfies $\sigma^k \longrightarrow \sigma, \varphi^k \longrightarrow \varphi$ and $f^k \longrightarrow f$ on V as $k \longrightarrow \infty$,

then there is a K such that, for $k \geq K$, each solution $x^k = x^k(\sigma^k, \varphi^k, f^k)$ through (σ^k, φ^k) of equation (2.7) exists on $[\sigma^k - r, b]$ and $x^k \longrightarrow x$ uniformly on $[\sigma - r, b]$.

Proof. Let $W \subseteq D$ be the compact set defined by $W = \{(t, x_t), \sigma \leq t \leq b\}$. Then by using the hypotheses on f and f^k, there is an open neighborhood V of W and a constant $M > 0$ such that $| f^k(t, \psi) | \leq M$ for $(t, \psi) \in V$ and $k \geq 0$ where $f^0 = f$. We also define $\varphi^0 = \varphi$ and $\sigma^0 = \sigma$. For any real numbers α, β let's define:

$$I_\alpha = \{t : 0 \leq t \leq \alpha\},$$

$$B_\beta = \{\varphi \in C :| \varphi | \leq \beta\},$$

$$\mathcal{A}(\alpha, \beta) = \{\eta \in C([-r, \alpha], \mathbb{R}^n) : \eta_0 = 0, \eta_t \in B_\beta, t \in I_\alpha\}.$$

Let's define a function $\widetilde{\varphi}^k \in C([\sigma^k - r, \sigma^k + \alpha], \mathbb{R}^n)$ such that

$$\widetilde{\varphi}^k_{\sigma^k} = \varphi^k$$

$$\widetilde{\varphi}^k(t + \sigma^k) = \varphi^k(0); \qquad t \in I_\alpha.$$

There is an open neighborhood U of W with $U \subset V$ and positive α, β so that

$(t + \widehat{t}, \eta + \psi) \in V$ for any $(t, \eta) \in I_\alpha \times B_\beta$, $(\widehat{t}, \psi) \in U$.

Choose $\overline{\alpha}, \overline{\beta}$ such that $\overline{\alpha} < \frac{\alpha}{2}$, $\overline{\beta} < \frac{\beta}{2}$, $M\overline{\alpha} < \beta$ and $| \widetilde{\varphi}^0_{t+\sigma} - \varphi^0 | < \frac{\beta}{2}$ for $t \in I_{\overline{\alpha}}$. Then the set of functions $\widetilde{\varphi}^k_{\sigma^k}, k \geq 0$ forms a compact set in $C([-r, \alpha], \mathbb{R}^n)$ and $\widetilde{\varphi}^k \longrightarrow \widetilde{\varphi}^0 = \widetilde{\varphi}$ as $k \longrightarrow \infty$, since $\varphi^k, k \geq 0$ forms a compact set in C. Thus, there is a $K \geq 0$ such that $| \sigma^k - \sigma | < \overline{\alpha}$, $| \widetilde{\varphi}^k_t - \widetilde{\varphi}^k | < \overline{\beta}$ for $t \in I_{\overline{\alpha}}$, $k \geq K$. Thus $| f^k(\sigma^k + t, \eta_t + \widetilde{\varphi}^k_t) | \leq M$ for $t \in I_{\overline{\alpha}}$ and $\eta \in \mathcal{A}(\overline{\alpha}, \overline{\beta})$; $k = 0, 1, 2, ...$ Now define the operators

$$T_k : \mathcal{A}(\overline{\alpha}, \overline{\beta}) \longrightarrow C([0, \overline{\alpha}], \mathbb{R}^n) \text{ such that}$$

$$(T_k\eta)(t) = \int_0^t f^k(\sigma^k + s, \eta_s + \widetilde{\varphi}^k_{s+\sigma^k})ds; \qquad t \in I_{\overline{\alpha}},$$

$$(T_k\eta)_0 = 0.$$

Since $\widetilde{\varphi}^k \longrightarrow \widetilde{\varphi}$ as $k \longrightarrow \infty$ and the hypotheses imply that $f^k(t, \psi) \longrightarrow f(t, \varphi)$ and $\sigma^k \longrightarrow \sigma$ as $k \longrightarrow \infty$ and $\psi \longrightarrow \varphi$. Also, f^k and f are uniformly bounded on V. Thus by using the Lebesgue dominated convergence theorem for the function f^k:

$$\lim_{k\to\infty} \left[\int_0^t f^k(\sigma^k + s, \eta_{k_s} + \widetilde{\varphi}^k_{s+\sigma^k})ds \right] = \int_0^t f(\sigma + s, \eta_s + \varphi_{s+\sigma})ds = (T_0\eta)(t). \tag{2.8}$$

Thus $T_k\eta \longrightarrow T_o\eta$ for each $\eta \in \mathcal{A}(\overline{\alpha}, \overline{\beta})$. Further the set $\mathcal{A}(\overline{\alpha}, \overline{\beta})$ is a closed, bounded and convex subset of $C([0, \overline{\alpha}], \mathbb{R}^n)$. Then consider for $t \in I_{\overline{\alpha}}$,

$$\begin{aligned}
\mid (T_k\eta)(t) \mid &= \left| \int_0^t f^k(\sigma^k + s, \eta_s + \widetilde{\varphi}^k_{s+\sigma^k})ds \right| \\
&\leq \int_0^t \mid f^k(\sigma^k + s, \eta_s + \widetilde{\varphi}^k_{s+\sigma^k}) \mid ds \\
&\leq Mt \leq M\overline{\alpha} \leq \overline{\beta}.
\end{aligned}$$

This implies that $(T_k)(\mathcal{A}(\overline{\alpha}, \overline{\beta})) \subset \mathcal{A}(\overline{\alpha}, \overline{\beta})$. Then consider

$$\begin{aligned}
\mid (T_k\eta)(t) - (T_k\eta)(\overline{t}) \mid &= \left| \int_{\overline{t}}^t f^k(\sigma^k + s, \eta_s + \widetilde{\varphi}^k_{s+\sigma^k})ds \right| \\
&\leq \int_{\overline{t}}^t \mid f^k(\sigma^k + s, \eta_s + \widetilde{\varphi}^k_{s+\sigma^k}) \mid ds \\
&\leq M \mid t - \overline{t} \mid; \qquad \forall t \in I_{\overline{\alpha}}.
\end{aligned}$$

Thus $T_k\mathcal{A}(\overline{\alpha}, \overline{\beta})$ belongs to a compact subset of $C([-r, \overline{\alpha}], \mathbb{R}^n)$.

Thus, by using the Schauder's theorem, there is a fixed point $\eta^k \in \mathcal{A}(\overline{\alpha}, \overline{\beta})$ of T_k. Since the family of functions $\eta^k = T_k\eta^k \in \mathcal{A}(\overline{\alpha}, \overline{\beta})$ are equicontinuous and uniformly bounded, there is a subsequence $< \eta^k >$ such that $\eta^k \longrightarrow \widehat{\eta}$ as $k \longrightarrow \infty$. Since $T_k\eta \longrightarrow T_o\eta$ for each $\eta \in \mathcal{A}(\overline{\alpha}, \overline{\beta})$ this gives that $T_0\widehat{\eta} = \widehat{\eta}$. Then every convergent subsequence of η^k must converge to $\widehat{\eta}$. This gives that $\eta^k \longrightarrow \widehat{\eta}$ as $k \longrightarrow \infty$. Thus due to the compactness of the set $\{(t, x_t), \sigma \leq t \leq b\}$, and the the relation $x^k_{\sigma^k+t} = \widetilde{\varphi}^k_{\sigma^k+t} + {\eta^k}_t$ for $t \in I_{\overline{\alpha}}$, the proof can be completed by successively stepping the intervals of length $\overline{\alpha}$. $\square$

CHAPTER 3

CONVERGENCE CONCEPTS OF THE SYSTEM OF SOLUTIONS OF DDE

In Chapter 2, we discussed the existence, uniqueness and continuous dependence of the solutions of the delay differential equations. Now, in the first half of this chapter, it is shown solutions of certain systems with discrete delays converge to solutions of systems with uniformly distributed continuous delays. Then in the second half of this chapter, it is shown the solutions of systems with discrete delays converge to systems with more general distributions of continuous delays.

3.1 UNIFORMLY DISTRIBUTED DELAY

Let's discuss an application of the uniformly distributed delay by using the continuous dependence as follows.

Example 3.1. Let's consider the following delay differential equations:

$$\dot{x}(t) = h\left(x(t), \frac{1}{\tau}\sum_{j=1}^{k} x_t\left(\frac{-j\tau}{k}\right)\cdot\frac{\tau}{k}\right) = h\left(x(t), \frac{1}{k}\sum_{j=1}^{k} x_t\left(\frac{-j\tau}{k}\right)\right) = f^k(t, x_t) \tag{3.1}$$

$$\dot{x}(t) = h\left(x(t), \frac{1}{\tau}\int_{-\tau}^{0} x_t(s)ds\right) = f(t, x_t) \tag{3.2}$$

where the continuous functions $x^k, x : [\sigma - \tau, b] \longrightarrow \mathbb{R}^n$ are solutions of these equations through (σ, φ) if they satisfies the equations (3.1) and (3.2) respectively for $t \in [\sigma, b]$ and $x^k_\sigma = \varphi$ and $x_\sigma = \varphi$ respectively.

Consider a function $h(x, y)$, Lipschitz continuous in x and $y \in \mathbb{R}^n$ with the Lipschitz constant λ. Now, we shall show that the solutions of (3.1) converge to solutions of (3.2) as $k \longrightarrow \infty$.

This example can be analyzed by using Theorems $2.2, 2.3$, and 2.5 in Chapter 2. Let C and D be defined as in Chapter 2.

(i) Suppose that $\sigma_k = \sigma$ and $\varphi_k = \varphi$ for $k = 1, 2, 3, ...$

(ii) Let's show that the function $f : D \longrightarrow \mathbb{R}^n$ is Lipschitz continuous.

Note that in this example the function $f(t, \psi)$ only depends on $\psi \in C$.

Take any (t, ψ_1) and (t, ψ_2) in D then we have

$$\begin{aligned}
\mid f(t, \psi_1) - f(t, \psi_2) \mid &= \left| h(\psi_1(0), \frac{1}{\tau}\int_{-\tau}^{0} \psi_1(s)ds) - h(\psi_2(0), \frac{1}{\tau}\int_{-\tau}^{0} \psi_2(s)ds) \right| \\
&\leq \lambda \left(\|\psi_1(0) - \psi_2(0)\|_{\mathbb{R}^n} + \left\| \frac{1}{\tau}\int_{-\tau}^{0} (\psi_1(s) - \psi_2(s))ds \right\|_{\mathbb{R}^n} \right) \\
&\leq \lambda \left(\|\psi_1(0) - \psi_2(0)\|_{\mathbb{R}^n} + \frac{1}{\tau}\int_{-\tau}^{0} sup_{s\in[-\tau,0]} \mid \psi_1(s) - \psi_2(s) \mid ds \right) \\
&\leq \lambda \left(1 + \frac{\tau}{\tau}\right) \|\psi_1 - \psi_2\|_C \\
&\leq 2\lambda \|\psi_1 - \psi_2\|_C.
\end{aligned}$$

This implies that $f(t, \psi)$ is Lipschitz continuous in ψ. Then by using Theorem 2.2 and Theorem 2.3, the function f has a unique solution through (σ, φ).

Now consider the remaining hypotheses in Theorem 2.5 as follows.

(iii) Since the solution of (3.2) $x : [\sigma - \tau, b] \longrightarrow \mathbb{R}^n$ is a continuous function on a closed bounded interval it is bounded. I. e., $\exists M_1 > 0$ such that $\mid x(s) \mid \leq M_1; \forall s \in [\sigma - \tau, b]$.

Considering the compact set $W = \{(t, x_t) : t \in [\sigma, b]\}$, it can be shown that there is an open cover of W which is formed by the open rectangles.

Choose δ_t such that $B((t, x_t); \delta_t) \subset D$ where

$$B((t, x_t); \delta_t) = \{(s, \psi) : \|\psi - x_s\|_C < \delta_t \text{ for } \mid s - t \mid < \delta_t\}.$$

Then the set of open rectangles $B((t, x_t) : \delta_t)$ give an open cover of the compact set W. Thus the compactness of W implies the existence of a finite sub-cover as $B((t_i, x_{ti}) : \delta_i)$ for $i = 1, 2, 3, ..., N$.

Now let's take $\delta = \min_i(\delta_i) > 0$.

We have that $V = \{(s, \psi) : \|\psi - x_s\|_C < \delta \text{ for } s \in [\sigma, b]\} \subset D$ is an open neighborhood of

W. Then, we show that $f(t,\psi)$ is bounded on V. For any $(t,\psi) \in V$ consider :

$$\left| h\left(x(t), \frac{1}{\tau}\int_{-\tau}^{0} x_t(s)ds\right) - h\left(\psi(0), \frac{1}{\tau}\int_{-\tau}^{0} \psi(s)ds\right)\right| \leq 2\lambda \|x-\psi\|_C$$

$$\leq 2\lambda\delta.$$

Then consider,

$$\begin{aligned} &| f(t,\psi) |_{\mathbb{R}^n} \\ &= \left| h\left(\psi(0), \frac{1}{\tau}\int_{-\tau}^{0} \psi(s)ds\right) - h\left(x(t), \frac{1}{\tau}\int_{-\tau}^{0} x_t(s)ds\right) + h\left(x(t), \frac{1}{\tau}\int_{-\tau}^{0} x_t(s)ds\right)\right|_{\mathbb{R}^n} \\ &\leq 2\lambda\delta + \sup_{t\in[\sigma,b]} \left| h(x(t), \frac{1}{\tau}\int_{-\tau}^{0} x_t(s)ds)\right|. \end{aligned}$$

Now by using the fact that the function h is continuous we have that:

$\sup_{|x|\leq M_1, |y|\leq M_1} | h(x,y) | \leq M_2$ for some $M_2 > 0$.

Thus we have:

$$| f(t,\psi) |_{\mathbb{R}^n} \leq 2\lambda\delta + M_2$$

$$= M.$$

We need to show that $f^k(t,\psi)$ is bounded on V for all $k = 1, 2, ...$ For any $(t,\psi) \in V$ consider :

$$\left| h\left(x(t), \frac{1}{k}\sum_{j=1}^{k} x_t\left(\frac{-j\tau}{k}\right)\right) - h\left(\psi^k(0), \frac{1}{k}\sum_{j=1}^{k} \psi^k\left(\frac{-j\tau}{k}\right)\right)\right| \leq 2\lambda\|x-\psi^k\|_C$$

$$\leq 2\lambda\delta.$$

Then consider:

$$
\begin{aligned}
&\mid f(t,\psi^k) \mid_{\mathbb{R}^n} \\
&= \left| h\left(\psi^k(0), \frac{1}{k}\sum_{j=1}^{k}\psi^k(\frac{-j\tau}{k})\right) - h\left(x(t), \frac{1}{k}\sum_{j=1}^{k}x_t(\frac{-j\tau}{k})\right) + h\left(x(t), \frac{1}{k}\sum_{j=1}^{k}x_t(\frac{-j\tau}{k})\right)\right|_{\mathbb{R}^n} \\
&\leq 2\lambda\delta + \sup_{t\in[\sigma,b]}\left| h\left(x(t), \frac{1}{k}\sum_{j=1}^{k}x_t\left(\frac{-j\tau}{k}\right)\right)\right| \\
&\leq 2\lambda\delta + M_2 = M.
\end{aligned}
$$

Then both f and f^k have a common bound $M(>0)$.

(iv) It then needs to be shown that $f^k(t,\psi^k) \longrightarrow f(t,\psi)$ for all $(t,\psi)\in V$ as $k\longrightarrow\infty$ and $\psi^k \longrightarrow \psi$ for any $(t,\psi^k)\in V$. Consider:

$$
\begin{aligned}
\mid f^k(t,\psi^k) - f(t,\psi) \mid_{\mathbb{R}^n} &= \mid f^k(t,\psi^k) - f^k(t,\psi) + f^k(t,\psi) - f(t,\psi) \mid_{\mathbb{R}^n} \\
&\leq \mid f^k(t,\psi^k) - f^k(t,\psi) \mid_{\mathbb{R}^n} + \mid f^k(t,\psi) - f(t,\psi) \mid_{\mathbb{R}^n}. \qquad (3.3)
\end{aligned}
$$

Consider

$$
\begin{aligned}
\mid f^k(t,\psi^k) - f^k(t,\psi) \mid_{\mathbb{R}^n} &= \left| h\left(\psi^k(0), \frac{1}{k}\sum_{j=1}^{k}\psi^k(\frac{-j\tau}{k})\right) - h\left(\psi(0), \frac{1}{k}\sum_{j=1}^{k}\psi(\frac{-j\tau}{k})\right)\right|_{\mathbb{R}^n} \\
&\leq 2\lambda\|\psi^k - \psi\|_C \\
&\longrightarrow 0;\text{ since } \psi^k \longrightarrow \psi \text{ in } C \text{ as } k\longrightarrow\infty. \qquad (3.4)
\end{aligned}
$$

Let's take the approximation $\lim_{k\to\infty}\frac{1}{\tau}\sum_{j=1}^{k}\psi(\frac{-j\tau}{k})\cdot\frac{\tau}{k} = \frac{1}{\tau}\int_{-\tau}^{0}\psi(s)ds$ by using the fact that the Riemann sum converges to the integral.

Then consider:

$$\begin{aligned}
&| \, f^k(t,\psi) - f(t,\psi) \, |_{\mathbb{R}^n} \\
&= \left| h\left(\psi(0), \frac{1}{k}\sum_{j=1}^{k} \psi\left(\frac{-j\tau}{k}\right)\right) - h\left(\psi(0), \frac{1}{\tau}\int_{-\tau}^{0} \psi(s)ds\right)\right|_{\mathbb{R}^n} \\
&\leq \lambda \left| \frac{1}{k}\sum_{j=1}^{k} \psi\left(\frac{-j\tau}{k}\right) - \frac{1}{\tau}\int_{-\tau}^{0} \psi(s)ds \right|_{\mathbb{R}^n} \\
&\qquad \longrightarrow 0 \text{ as } k \longrightarrow \infty.
\end{aligned} \tag{3.5}$$

Then by using inequalities (3.4) and (3.5) in the inequality (3.3), we have $f^k(t,\psi^k) \longrightarrow f(t,\psi)$ as $k \longrightarrow \infty$ and $\psi^k \longrightarrow \psi$ for all $(t,\psi) \in V$ and for $(t,\psi^k) \in V$. Since all of the hypotheses of Theorem 2.5 are satisfied, the solutions of the equation (3.1) converge uniformly to the solution of the equation (3.2).

3.2 MORE GENERAL DISTRIBUTION OF CONTINUOUS DELAY

Let's consider the convergence of solutions of systems with discrete delays to systems with more general distributions of continuous delay by establishing the following theorem.

Theorem 3.1. *Suppose $D \subset \mathbb{R} \times C$ is open and let the function $g(t, \psi)$, $g : D \longrightarrow \mathbb{R}^n$ be continuous, Lipschitz continuous in ψ with the Lipschitz constant α uniformly in t, for $(t, \psi) \in D$. Suppose the function $h(x, y)$ is Lipschitz continuous in x and $y \in \mathbb{R}^n$ with the Lipschitz constant λ. Then consider the following delay differential equations:*

$$\begin{aligned}\dot{x}(t) &= h\left(x(t), \frac{1}{\tau}\sum_{j=1}^{k} g\left(t - \frac{j\tau}{k}, x_t\left(\frac{-j\tau}{k}\right)\right) \cdot \frac{\tau}{k}\right) \\ &= h\left(x(t), \frac{1}{k}\sum_{j=1}^{k} g\left(t - \frac{j\tau}{k}, x_t\left(\frac{-j\tau}{k}\right)\right)\right) = f^k(t, x_t) \qquad (3.6)\end{aligned}$$

$$\dot{x}(t) = h\left(x(t), \frac{1}{\tau}\int_{-\tau}^{0} g(s, x_t(s))ds\right) = f(t, x_t). \qquad (3.7)$$

Then the solutions of (3.6) *converge to solution of* (3.7) *as $k \longrightarrow \infty$.*

Proof. The theorem can be proved by using Theorems 2.2, 2.3, and 2.5 in Chapter 2.

(i) Suppose that $\sigma_k = \sigma$ and $\varphi_k = \varphi$ for $k = 1, 2, 3, ...$

(ii) For any (t, ψ_1) and (t, ψ_2) in D we have

$$\begin{aligned}&\mid f(t, \psi_1) - f(t, \psi_2) \mid \\ &= \left| h\left(\psi_1(t), \frac{1}{\tau}\int_{-\tau}^{0} g(s, \psi_1(s))ds\right) - h\left(\psi_2(t), \frac{1}{\tau}\int_{-\tau}^{0} g(s, \psi_2(s))ds\right) \right| \\ &\leq \lambda\left(\|\psi_1(t) - \psi_2(t)\|_{\mathbb{R}^n} + \left\| \frac{1}{\tau}\int_{-\tau}^{0} (g(s, \psi_1(s)) - g(s, \psi_2(s)))ds \right\|_{\mathbb{R}^n} \right) \\ &\leq \lambda(\|\psi_1(t) - \psi_2(t)\|_{\mathbb{R}^n} + \alpha\|\psi_1 - \psi_2\|_C) \\ &\leq \lambda(1 + \alpha)\|\psi_1 - \psi_2\|_C.\end{aligned}$$

This implies that $f(t,\psi)$ is Lipschitz continuous in ψ when $g(t,\psi)$ is Lipschitz continuous on D. Then by using Theorem 2.2 and Theorem 2.3 in Chapter 2, the function f has a unique solution through (σ,ψ).

(iii) Considering the compact set $W = \{(t, x_t) : t \in [\sigma, b]\}$, it can be shown that there is an open cover of W which is formed by the open rectangles.

Choose δ_t such that $B((t,x_t);\delta_t) \subset D$ where

$$B((t,x_t);\delta_t) = \{(s,\psi) : \|\psi - x_s\|_C < \delta_t for \mid s - t \mid < \delta_t\}.$$

Then the set of open rectangles $B((t,x_t):\delta_t)$ give an open cover of the compact set W. Thus the compactness of W implies the existence of a finite sub-cover as $B((t_i,x_{ti}):\delta_i)$ for $i = 1,2,3,...,N$.

Now let's take $\delta = \min_i(\delta_i) > 0$.

We have that $V = \{(s,\psi) : \|\psi - x_s\|_C < \delta \text{ for } s \in [\sigma,b]\} \subset D$ is an open neighborhood of W. We then, need to show that $f(t,\psi)$ is bounded on V. For any $(t,\psi) \in V$ consider :

$$\left| h\left(x(t), \frac{1}{\tau}\int_{-\tau}^{0} g(s, x_t(s))ds\right) - h\left(\psi(t), \frac{1}{\tau}\int_{-\tau}^{0} g(s,\psi(s))ds\right)\right|$$

$$\leq \lambda(1+\alpha)\|x_t - \psi\|_C$$

$$\leq \lambda(1+\alpha)\delta$$

$$\mid f(t,\psi) \mid_{\mathbb{R}^n} \leq \lambda(1+\alpha)\delta + \sup_{t\in[\sigma,b]} \left| h\left(x(t), \frac{1}{\tau}\int_{-\tau}^{0} g(s,x_t(s))ds\right)\right|.$$

Now by using the fact that the function h is a continuous function we have that; $\sup_{|x|\leq M_1, |y|\leq M_2} \mid h(x,y) \mid \leq M_3$ for some $M_3 > 0$, where M_2 is the bound of the function g. Thus, $\mid f(t,\psi) \mid_{\mathbb{R}^n} \leq \lambda(1+\alpha)\delta + M_3 = M$ gives the bound of the function $f(t,\psi)$ on V. We then need to show that $f^k(t,\psi)$ is bounded on V for all $k = 1,2,....$

For any $(t,\psi) \in V$ consider :

$$\left| h\left(x(t), \frac{1}{\tau} \sum_{j=1}^{k} g\left(t - \frac{j\tau}{k}, x_t(\frac{-j\tau}{k}) \right) \right) - h\left(\psi^k(t), \frac{1}{k} \sum_{j=1}^{k} g\left(t - \frac{j\tau}{k}, \psi^k(\frac{-j\tau}{k}) \right) \right) \right|$$

$$\leq \lambda(1+\alpha)\|x - \psi^k\|_C$$

$$\leq \lambda(1+\alpha)\delta.$$

Then consider:

$$| f(t,\psi^k) |_{\mathbb{R}^n} \leq \lambda(1+\alpha)\delta + \sup_{t\in[\sigma,b]} \left| h\left(x(t), \frac{1}{k} \sum_{j=1}^{k} g\left(t - \frac{j\tau}{k}, x_t(\frac{-j\tau}{k}) \right) \right) \right|$$

$$\leq \lambda(1+\alpha)\delta + M_3 = M.$$

Then for both f and f^k have a common bound $M(>0)$.

(iv) It then needs to be shown that $f^k(t,\psi^k) \longrightarrow f(t,\psi)$ for all $(t,\psi) \in V$ as $k \longrightarrow \infty$ and $\psi^k \longrightarrow \psi$ for $(t,\psi^k) \in V$. Consider:

$$| f^k(t,\psi^k) - f(t,\psi) |_{\mathbb{R}^n} = | f^k(t,\psi^k) - f^k(t,\psi) + f^k(t,\psi) - f(t,\psi) |_{\mathbb{R}^n}$$

$$\leq | f^k(t,\psi^k) - f^k(t,\psi) |_{\mathbb{R}^n} + | f^k(t,\psi) - f(t,\psi) |_{\mathbb{R}^n}. \tag{3.8}$$

Consider:

$$| f^k(t,\psi^k) - f^k(t,\psi) |_{\mathbb{R}^n}$$

$$= \left| h\left(\psi^k(t), \frac{1}{k} \sum_{j=1}^{k} g(t - \frac{j\tau}{k}, \psi^k(\frac{-j\tau}{k}) \right) - h\left(\psi(t), \frac{1}{k} \sum_{j=1}^{k} g(t - \frac{j\tau}{k}, \psi(\frac{-j\tau}{k}) \right) \right|_{\mathbb{R}^n}$$

$$\leq \lambda(1+\alpha)\|\psi^k - \psi\|_C$$

$$\longrightarrow 0; \text{ since } \psi^k \longrightarrow \psi \text{ in } C \text{ as } k \longrightarrow \infty. \tag{3.9}$$

Let's take the approximation $\lim_{k\to\infty} \frac{1}{\tau}\sum_{j=1}^{k} g(t-\frac{j\tau}{k}, x_t(\frac{-j\tau}{k}))\cdot\frac{\tau}{k} = \frac{1}{\tau}\int_{-\tau}^{0} g(s, x_t(s))ds$ by using the fact that the Riemann sum converges to the integral. Then consider:

$$
\begin{aligned}
& \mid f^k(t,\psi) - f(t,\psi) \mid_{\mathbb{R}^n} \\
&= \left| h\left(\psi(t), \frac{1}{k}\sum_{j=1}^{k} g\left(t-\frac{j\tau}{k}, \psi(\frac{-j\tau}{k})\right)\right) - h\left(\psi(t), \frac{1}{\tau}\int_{-\tau}^{0} g(s, x_t(s))ds\right)\right|_{\mathbb{R}^n} \\
&\leq \lambda \left| \frac{1}{k}\sum_{j=1}^{k} g\left(t-\frac{j\tau}{k}, \psi\left(\frac{-j\tau}{k}\right)\right) - \frac{1}{\tau}\int_{-\tau}^{0} g(s, x_t(s))ds \right|_{\mathbb{R}^n} \\
&\longrightarrow 0 \text{ as } k \longrightarrow \infty. \qquad (3.10)
\end{aligned}
$$

Then by using inequalities (3.9) and (3.10) in the inequality (3.8), we have $f^k(t,\psi^k) \longrightarrow f(t,\psi)$ as $k \longrightarrow \infty$ and $\psi^k \longrightarrow \psi$ for all $(t,\psi) \in V$ and for $(t,\psi^k) \in V$. Since all of the hypotheses of Theorem 2.5 are satisfied, the solutions of the equation (3.6) converge uniformly to solution of the equation (3.7) and that completes the proof of the theorem.

□

CHAPTER 4

NUMERICAL SIMULATION AND INVESTIGATION OF THE TRAJECTORIES OF DDE

4.1 DELAYED LOTKA VOLTERRA PREDATOR-PREY SYSTEMS

In 1928, Volterra investigated the predator-prey systems where x and y represent the prey and predators respectively [10]. The Volterra predator-prey equations with delay are given by:

$$\begin{aligned} \dot{x}(t) &= x(t)\left(a - bx(t) - \int_{-\tau}^{0} F_1(\theta)y(t+\theta)d\theta\right) \\ \dot{y}(t) &= y(t)\left(-\delta + cx(t) + \int_{-\tau}^{0} F_2(\theta)x(t+\theta)d\theta\right), \end{aligned} \tag{4.1}$$

where all the constants and functions are nonnegative.

In 1957, Wangersky and Cunningham [11] used the predator-prey model given by the following form:

$$\begin{aligned} \dot{x}(t) &= ax(t)\left[1 - \frac{x(t)}{K}\right] - bx(t)y(t), \\ \dot{y}(t) &= -cy(t) + dx(t-\tau)y(t-\tau), \end{aligned} \tag{4.2}$$

where the constants a, b, c, d and K are nonnegative constants.

The equilibrium of the system is given by $\left[\frac{c}{d}, \frac{a}{b}\left(1 - \frac{c}{Kd}\right)\right]$.

4.2 EXAMPLES FOR TRAJECTORY CHANGES ACCORDING TO DELAY

The following examples show how the trajectories of the predator-prey systems change with respect to the delay term changes. These examples consist of discrete delay differential equations (DDE) with a single delay and DDE with two delay terms which are solved by using MATLAB programs.

In 2001, Annik Martin and Shigui Ruan [7] also have discussed these types of examples in their investigations of "Predator-Prey Models With Delay And Prey Harvesting".

The MATLAB codes and all the figures for the following examples are given in the section 4.3 and the initial condition of each predator-prey system is chosen close to the non-trivial equilibrium point in order to obtain a clear graphical interpretation. In all examples, the initial segment is chosen to be a constant function.

Example 4.1. Consider the following predator-prey model:

$$\dot{x}(t) = 15x(t)\left[1 - \frac{x(t)}{25}\right] - 0.5x(t)y(t),$$

$$\dot{y}(t) = -40y(t) + 2.5x(t-\tau)y(t-\tau).$$

This is an example of the predator-prey model that was used by Wangersky and Cunningham [11] which is given by equation (4.2).

The non-trivial equilibrium point of this system is given by $(16, 10.8)$. Now, let's consider the behavior of the trajectory according to the delay $\tau \geq 0$ by taking the initial segment with constants $x = 15$ and $y = 11$ in DDE systems.

When the delay τ is zero the predator-prey system becomes an ODE system and the predator-prey system spirals around the equilibrium point (Figure 4.1). When the delay $\tau = 0.01$ and $\tau = 0.04$, the plot of predator-prey solution (Figure 4.2 and 4.3) of this model shows that the system spirals around the nontrivial equilibrium and the trajectory behavior is similar to the ODE system trajectory behavior.

But when the time delay becomes $\tau = 0.043$ (Figure 4.4), the trajectory behavior near the nontrivial equilibrium point is different from the trajectory of the ODE system. Then as the delay term increases to $\tau = 0.1$ and $\tau = 0.5$ the system shows the existence of self-intersections which can't be seen in the ODE systems as shown in Figure 4.5 and Figure 4.6 respectively.

Example 4.2. Consider the following generalized Gauss Type predator-prey model [7]:

$$\dot{x}(t) = x(t)\left[4 - \frac{x(t-\tau)}{5} - \frac{y(t)}{x(t)+15}\right],$$
$$\dot{y}(t) = y(t)\left[\frac{2x(t)}{x(t)+15} - \frac{2}{3}\right].$$

In this example the time lag is only associated with the prey system and there is no time lag associated with the predator system. Let's look at the trajectory behavior of the system around the equilibrium point $(7.5, 56.25)$ according to the delay τ (≥ 0) and let $(10, 56)$ be the values for the initial segment of the DDE system. When the delay $\tau = 0$ the predator-prey system is an ODE system and the trajectory spirals around the non-trivial equilibrium point as shown in Figure 4.7.

Figure 4.8 shows that the predator-prey system spirals around the equilibrium point even when $\tau = 0.2$ and the trajectory behavior is similar to the trajectory of ODE system. But the trajectory of the system changes as the delay term increases which can be identified by the subsequent figures.

Figures 4.9 and 4.10 show that the equilibrium point of the predator-prey system is the limit point of the trajectory when $\tau = 0.45$ and $\tau = 0.5$ respectively and it gives a different trajectory than the ODE system. But Figure 4.11 shows that trajectory of the predator-prey system converges to a periodic orbit when $\tau = 0.5755$ and Figure 4.12 also shows the existence of a periodic orbit when $\tau = 0.7$ where the trajectory is totally different from the trajectory of the ODE system. Thus, in this prey-predator example the trajectory of the solution changes from spiraling around and converging to the equilibrium and then to periodic orbits.

Example 4.3. Consider the following predator-prey model [7]:

$$\dot{x}(t) = x(t)\left[4 - \frac{x(t)}{5} - \frac{y(t)}{x(t-\tau)+15}\right],$$
$$\dot{y}(t) = y(t)\left[\frac{2x(t)}{x(t-\tau)+15} - \frac{2}{3}\right].$$

In this model the delay term is associated with the prey term but not with the predator term. Now, let's consider the trajectory behavior around the equilibrium point $(7.5, 56.25)$ according to the variation of delay τ (≥ 0) and let $(10, 56)$ be the values for the initial segment of the DDE system.

Figure 4.7 shows that the predator-prey system spirals around the non-trivial equilibrium point when $\tau = 0$ where the system is an ODE system.The system shows the spiral behavior until the delay term increases up to $\tau = 0.9$ (Figures $4.13, 4.14$) which is similar to the trajectory behavior of the ODE system. After that the system shows the existence of self-intersections of trajectory. Figures 4.15 and 4.16 show that the spiral behavior of the system changes and self-intersections occur when $\tau = 2.3$ and $\tau = 3.5$.

Example 4.4. Consider the following generalized Gauss Type predator-prey model [7]:

$$\dot{x}(t) = x(t)\left[4 - \frac{x(t)}{5} - \frac{y(t)}{x(t)+15}\right],$$
$$\dot{y}(t) = y(t-\tau)\left[\frac{2x(t)}{x(t-\tau)+15}\right] - y(t)\frac{2}{3}.$$

In this generalized Gauss Type model the delay term is associated with both prey and predator terms. Now, let's consider the trajectory behavior of the system around the non-trivial equilibrium point $(7.5, 56.25)$ according to the variation of delay τ (≥ 0) and let $(10, 56)$ be the values for the initial segment of the DDE system.

As in the previous examples, Figure 4.7 shows that the predator-prey system spirals around the equilibrium point when the delay $\tau = 0$ when system is an ODE system. When $\tau = 0.2$, $\tau = 0.4001$, and $\tau = 0.5$ (Figures 4.17, 4.18, 4.19) the trajectory shows a spiral behavior around the non-trivial equilibrium as in the ODE system. But the following Figure 4.20 shows at $\tau = 6.5$ the trajectory converges to the non-trivial equilibrium with self-intersections which can't be seen in the ODE systems.

Example 4.5. Consider the following predator-prey model:

$$\dot{x}(t) = x(t)\left[4 - \frac{x(t-\tau_1)}{5} - \frac{y(t)}{x(t)+15}\right],$$
$$\dot{y}(t) = y(t-\tau_2)\left[\frac{2x(t-\tau_1)}{x(t)+15}\right] - y(t)\frac{2}{3}.$$

When considering this predator-prey model the delay is associated with both prey and predator terms with different delay values. Let's consider the trajectory behavior around the equilibrium point $(7.5, 56.25)$ according to the variation of delay τ_1 (≥ 0) and τ_2 (≥ 0) and let $(10, 56)$ be the values for the initial segment of the DDE system.

When $\tau_1 = \tau_2 = \tau$ the system can be written as follows:

$$\dot{x}(t) = x(t)\left[4 - \frac{x(t-\tau)}{5} - \frac{y(t)}{x(t)+15}\right],$$
$$\dot{y}(t) = y(t-\tau)\left[\frac{2x(t-\tau)}{x(t)+15}\right] - y(t)\frac{2}{3}.$$

In this case, Figure 4.7 shows that the predator-prey system spirals around the equilibrium point when the delay $\tau = 0$ (ODE system) .

The following Figures 4.21 and 4.22 show at $\tau = 0.1$ and $\tau = 0.3144$ the trajectory spiraling towards the equilibrium point of the predator-prey system. But Figure 4.23 shows that when $\tau = 0.4$ the predator-prey system indicates the presence of a limit cycle and Figure 4.24 shows the existence of a periodic orbit when delay term $\tau = 0.6$ which give different trajectories from the ODE system.

Let's consider the behavior of the equilibrium point according to different delay terms in prey and predator terms.

Figure 4.7 shows that the predator-prey system spiral around the equilibrium point when the delay $\tau = 0$ which gives the trajectory of the ODE system. Further the following Figure 4.25 and Figure 4.26 show at $[\tau_1, \tau_2] = [0.3, 0.2]$ and $[\tau_1, \tau_2] = [0.3334, 0.2]$ the trajectory spiraling towards the equilibrium point of the predator-prey system. Figure 4.27 gives that when $[\tau_1, \tau_2] = [0.4, 0.2]$ the predator-prey system indicates the presence of a limit cycle with a periodic solution, and Figure 4.28 shows a periodic orbit with self-intersections when $[\tau_1, \tau_2] = [0.5, 0.2]$.

Conclusions from the experimental results

When the delay term $\tau = 0$, the trajectory for the above examples gives the trajectory of an ODE system. As shown in the above examples, for some small delay values the trajectories of delay differential equations are similar to the trajectory of the ODE system. But as the delay values increase the trajectories show the existence of limit cycles, periodic orbits, and self-intersections which can not be seen in the trajectory of the ODE system.

4.3 MATLAB CODES AND FIGURES OF THE EXAMPLES

The MATLAB codes of the above examples are given below. In simulating the delay differential equations, the MATLAB routine "dde23" is used. The "dde23" has the form:

```
sol = dde23(ddefile,lags,history,tspan);
```

Here the "ddefile" is the name of a function which numerically solves the delay differential equations. The delay terms of the DDE is provided as the vector "lags". The term "history" is the argument which is the name of a function that evaluates the solution at the input value of "t" and returns it as a column vector. Quite often the history arguments are given by a constant vector. The last term "tspan" is the interval of the integration or the interval for which the solution should be evaluated.

The MATLAB codes of the above examples can be written as follows. All the codes are written in terms of a general form. In all these codes $\tau = k$ for different k values give the different figures as shown in following figures.

MATLAB code of the example 4.1

```
function example1
clear all; close all; clc;
% solving ode
[t,y]=ode15s(@ex1a,[0 100],[40 16]);
figure(1);
plot(y(:,1),y(:,2));
xlabel('x (preys)');
ylabel('y (predators)');
title('Plot of a prey-predator solution');
%axis([0 50 0 50]);
% solving dde for tau = k
```

```
sol = dde23(@ex1dde,k,[15 11],[0 40]);

figure(2);

plot(sol.y(1,:),sol.y(2,:));

xlabel('x (preys)');

ylabel('y (predators)');

title('Plot of a prey-predator solution when tau = k');

%axis([0 50 0 50]);

function dxdt = ex1a(t,x)

dxdt=zeros(2,1);

dxdt(1) = 15*x(1)*(1-(x(1)/25))-0.5*x(2)*x(1);

dxdt(2) = -40*x(2)+2.5*x(1)*x(2);

function dxdt = ex1dde(t,x,Z)

xlag1 = Z(:,1);

dxdt = [ 15*x(1)*(1-(x(1)/25))-0.5*x(2)*x(1)

          -40*x(2)+2.5*xlag1(1)*xlag1(2)                   ];
```

MATLAB code of the example 4.3

```
function example3

clear all; close all; clc;

% solving ode

[t,y]=ode15s(@ex3a,[10 50],[50 50]);

figure(1);

plot(y(:,1),y(:,2));

xlabel('x (preys)');

ylabel('y (predators)');

title('Plot of a prey-predator solution');

% solving dde for tau = k
```

```
sol = dde23(@ex3dde,k,[10 56],[0 100]);
figure(2);
plot(sol.y(1,:),sol.y(2,:));
xlabel('x (preys)');
ylabel('y (predators)');
title('Plot of a prey-predator solution with delay k');
function dxdt = ex3a(t,x)
dxdt=zeros(2,1);
dxdt(1) = x(1)*((4-(x(1)/5))-(x(2)/(x(1)+15))) ;
dxdt(2) = x(2)*(2*(x(1)/(x(1)+15))-(2/3));
function dxdt = ex3dde(t,x,Z)
xlag1 = Z(:,1);
dxdt = [ x(1)*((4-(x(1)/5))-(x(2)/(xlag1(1)+15)))
         x(2)*(2*(x(1)/(xlag1(1)+15))-(2/3))                    ];
```

MATLAB code of the example 4.4

```
function example4
clear all; close all; clc;
% solving ode
[t,y]=ode15s(@ex4a,[10 50],[50 50]);
figure(1);
plot(y(:,1),y(:,2));
xlabel('x (preys)');
ylabel('y (predators)');
title('Plot of a prey-predator solution');
% solving dde for tau = k
```

```
sol = dde23(@ex4dde,k,[10 56],[0 100]);
figure(2);
plot(sol.y(1,:),sol.y(2,:));
xlabel('x (preys)');
ylabel('y (predators)');
title('Plot of a prey-predator solution with delay k');
function dxdt = ex4a(t,x)
dxdt=zeros(2,1);
dxdt(1) = x(1)*((4-(x(1)/5))-(x(2)/(x(1)+15))) ;
dxdt(2) = x(2)*(2*(x(1)/(x(1)+15))-(2/3));
function dxdt = ex4dde(t,x,Z)
xlag1 = Z(:,1);
dxdt = [ x(1)*((4-(x(1)/5))-(x(2)/(x(1)+15)))
         xlag1(2)*2*(x(1)/(xlag1(1)+15))-x(2)*(2/3)                    ];
```

<u>MATLAB code of the example 4.5</u>

```
function example5
clear all; close all; clc;
% solving ode
[t,y]=ode15s(@ex5a,[10 50],[50 50]);
figure(1);
plot(y(:,1),y(:,2));
xlabel('x (preys)');
ylabel('y (predators)');
title('Plot of a prey-predator solution');
% solving dde for [tau1 tau2] = [k1 k2]
```

```
sol = dde23(@ex5dde,[k1 k2],[10 56],[0 100]);

figure(2);

plot(sol.y(1,:),sol.y(2,:));

xlabel('x (preys)');

ylabel('y (predators)');

title('Plot of a prey-predator solution with delay [k1 k2]');

function dxdt = ex5a(t,x)

dxdt=zeros(2,1);

dxdt(1) = x(1)*((4-(x(1)/5))-(x(2)/(x(1)+15))) ;

dxdt(2) = x(2)*(2*(x(1)/(x(1)+15))-(2/3));

function dxdt = ex5dde(t,x,Z)

xlag1 = Z(:,1);

xlag2 = Z(:,2);

dxdt = [ x(1)*((4-(xlag1(1)/5))-(x(2)/(x(1)+15)))
         xlag2(2)*2*(xlag1(1)/(x(1)+15))-x(2)*(2/3)                   ];
```

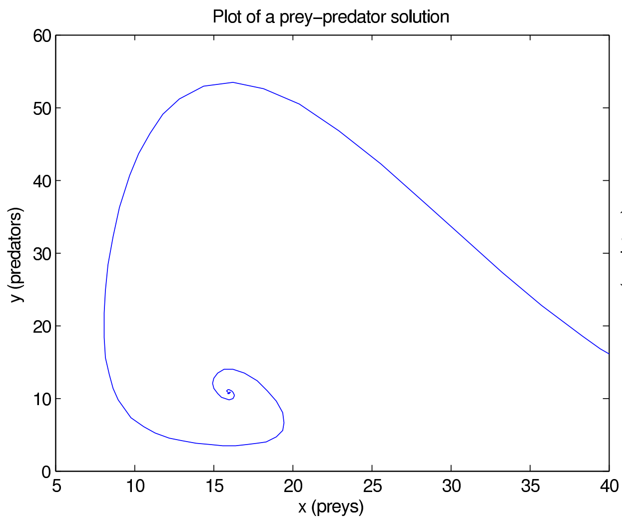

Figure 4.1. Trajectory behavior of Example 4.1 with $\tau = 0$.

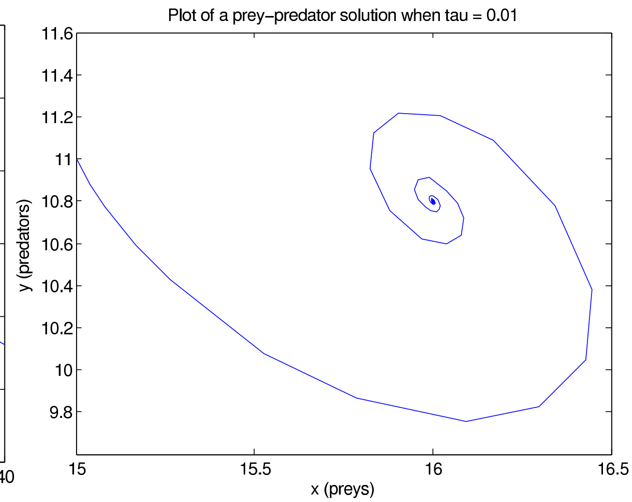

Figure 4.2. Trajectory behavior of Example 4.1 with $\tau = 0.01$.

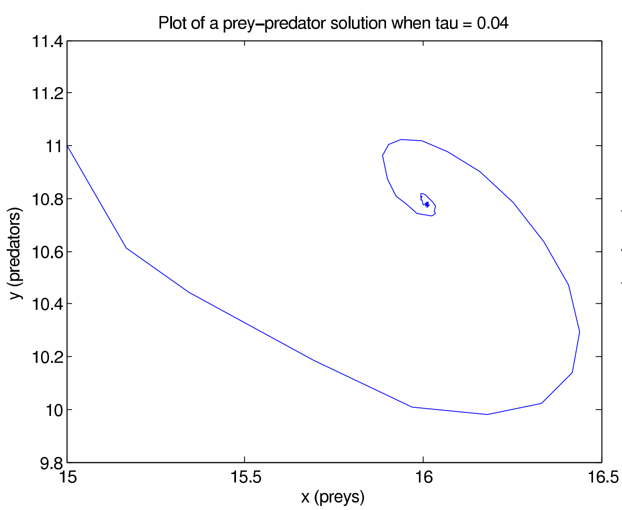

Figure 4.3. Trajectory behavior of Example 4.1 with $\tau = 0.04$.

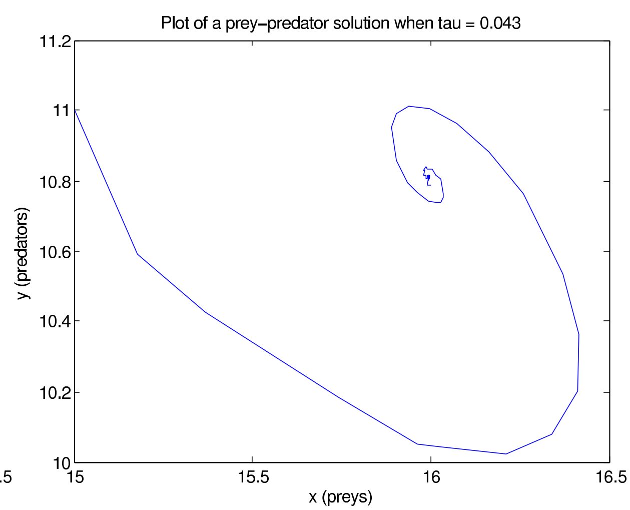

Figure 4.4. Trajectory behavior of Example 4.1 with $\tau = 0.043$.

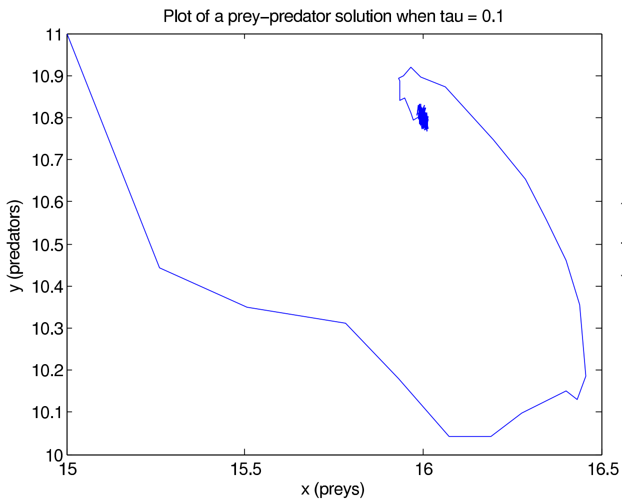

Figure 4.5. Trajectory behavior of Example 4.1 with $\tau = 0.1$.

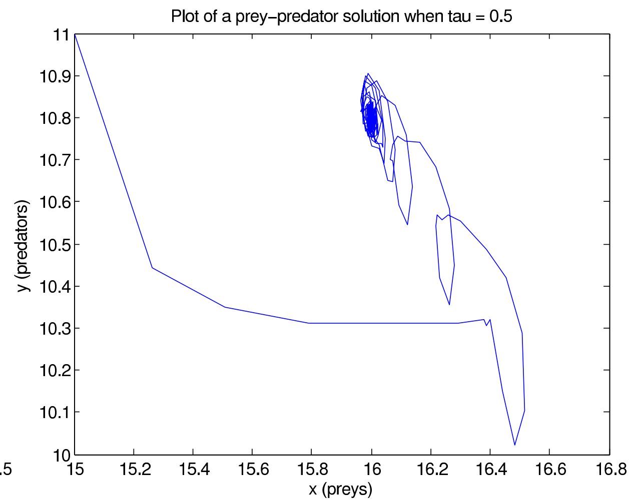

Figure 4.6. Trajectory behavior of Example 4.1 with $\tau = 0.5$.

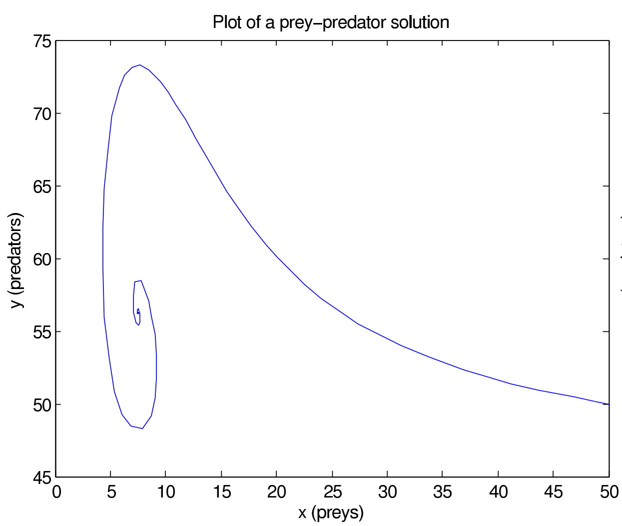

Figure 4.7. Trajectory behavior of Example 4.2 with $\tau = 0$.

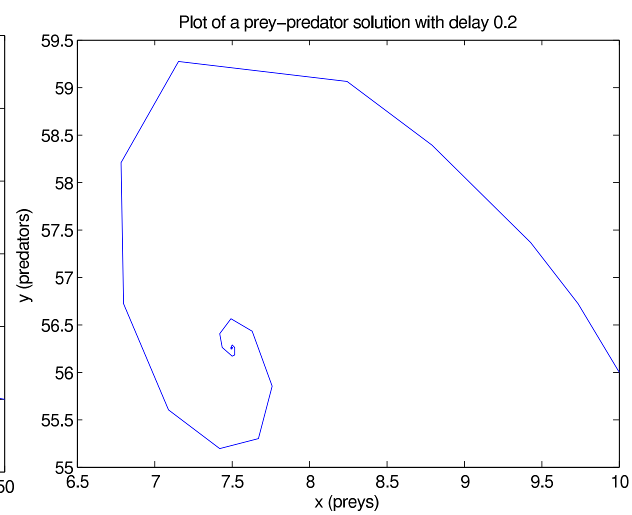

Figure 4.8. Trajectory behavior of Example 4.2 with $\tau = 0.2$.

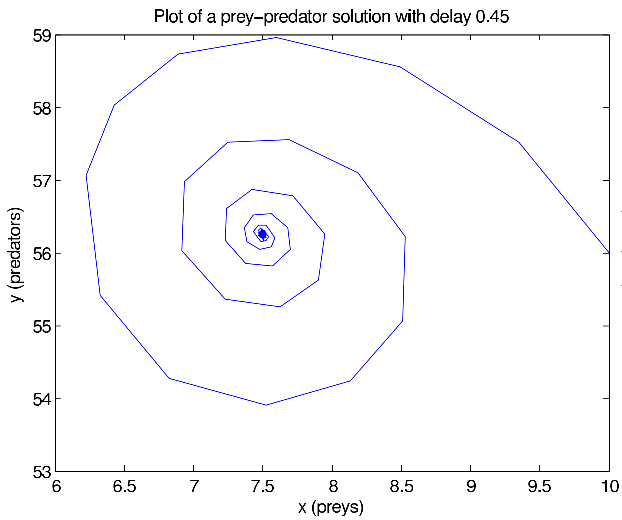

Figure 4.9. Trajectory behavior of Example 4.2 with $\tau = 0.45$.

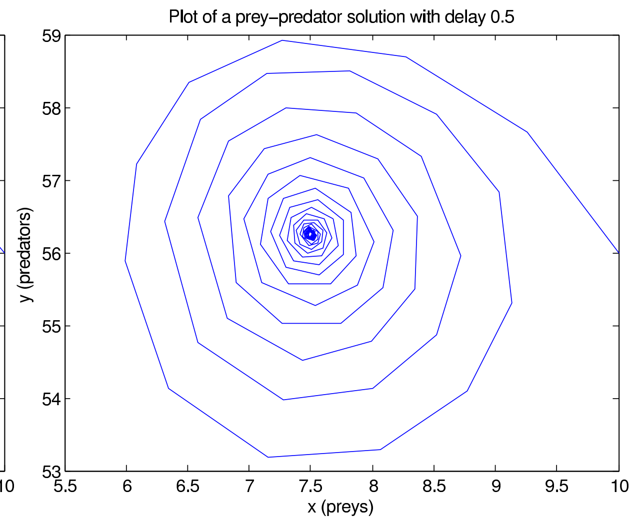

Figure 4.10. Trajectory behavior of Example 4.2 with $\tau = 0.5$.

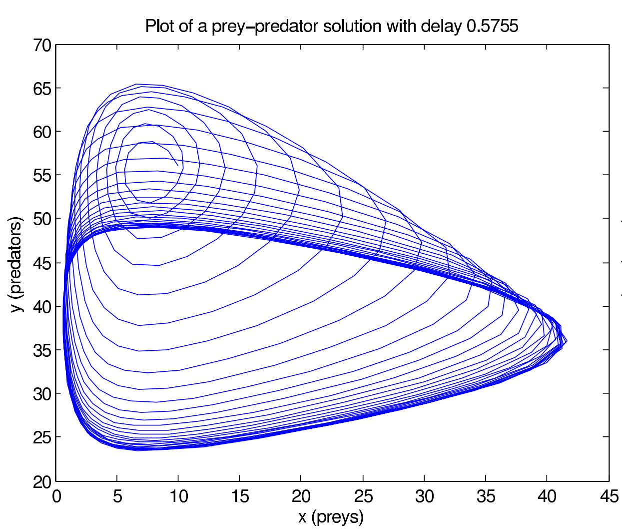

Figure 4.11. Trajectory behavior of Example 4.2 with $\tau = 0.5755$.

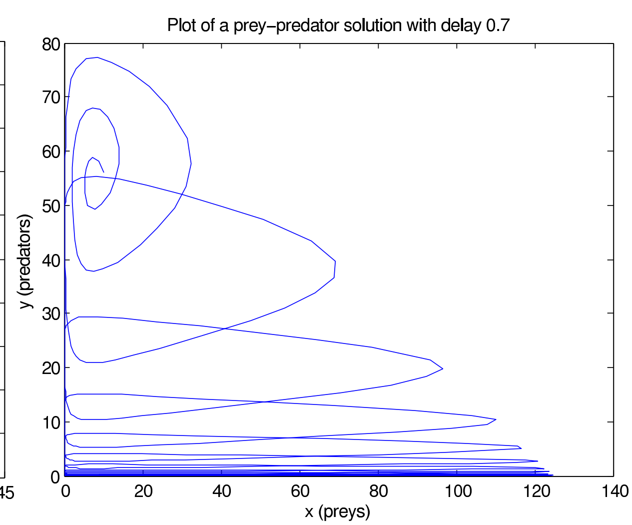

Figure 4.12. Trajectory behavior of Example 4.2 with $\tau = 0.7$.

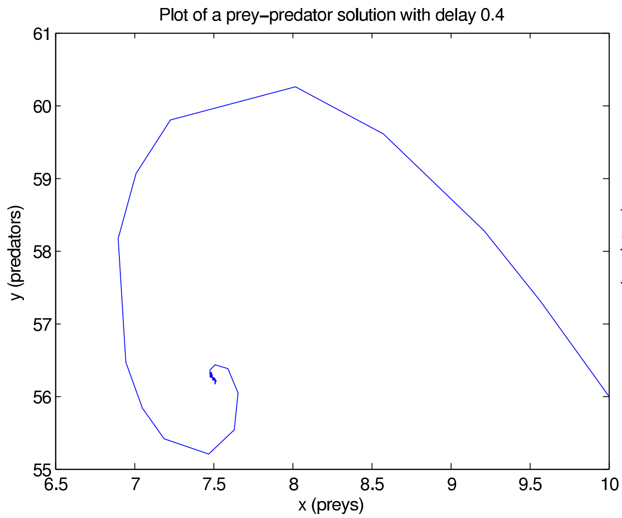

Figure 4.13. Trajectory behavior of Example 4.3 with $\tau = 0.4$.

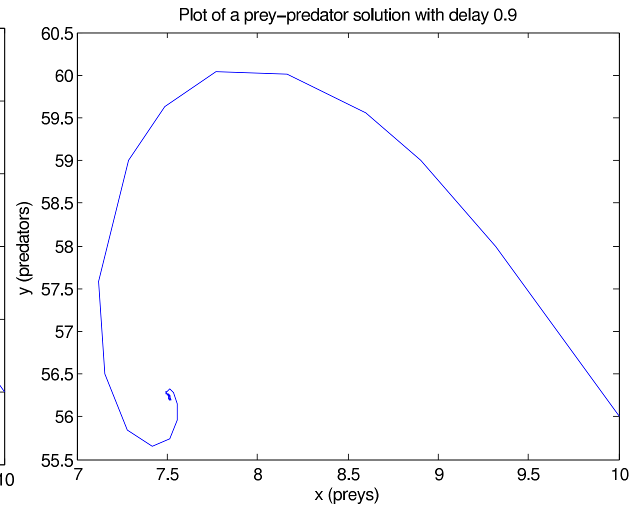

Figure 4.14. Trajectory behavior of Example 4.3 with $\tau = 0.9$.

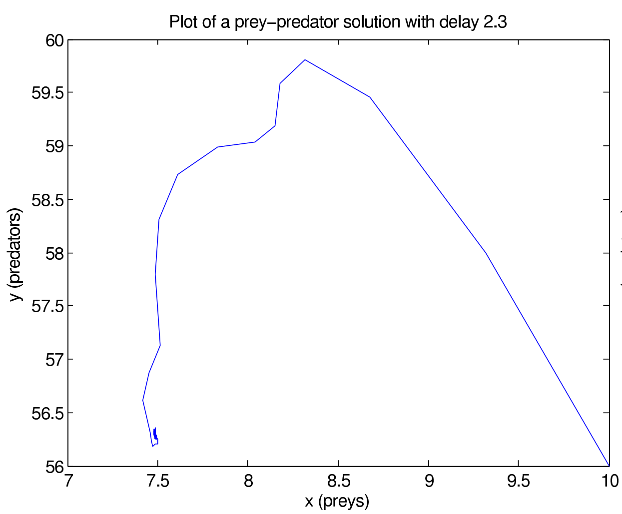

Figure 4.15. Trajectory behavior of Example 4.3 with $\tau = 2.3$.

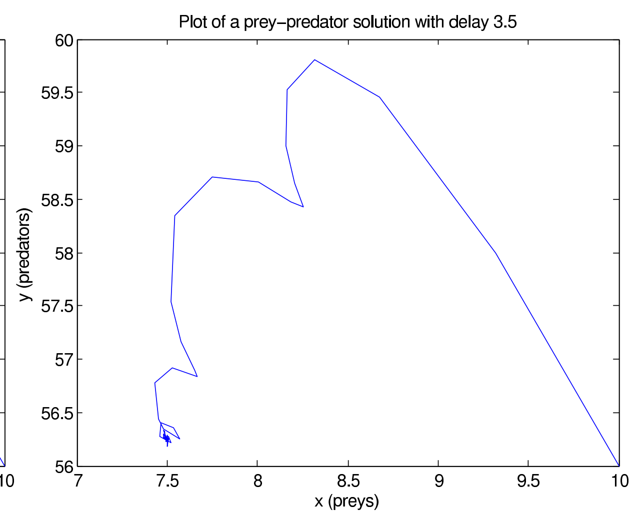

Figure 4.16. Trajectory behavior of Example 4.3 with $\tau = 3.5$.

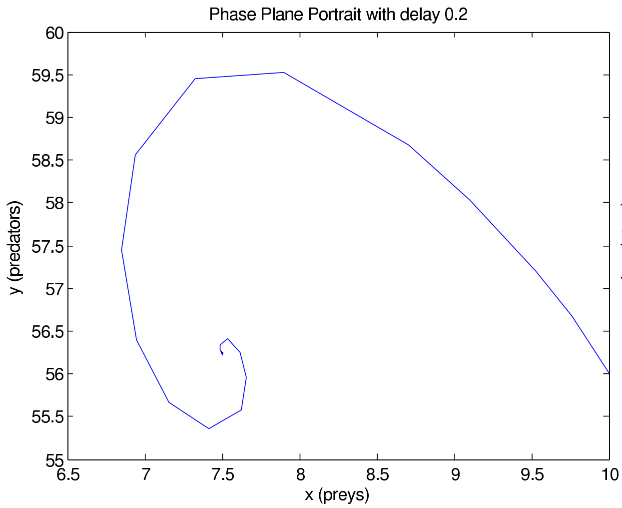

Figure 4.17. Trajectory behavior of Example 4.4 with $\tau = 0.2$.

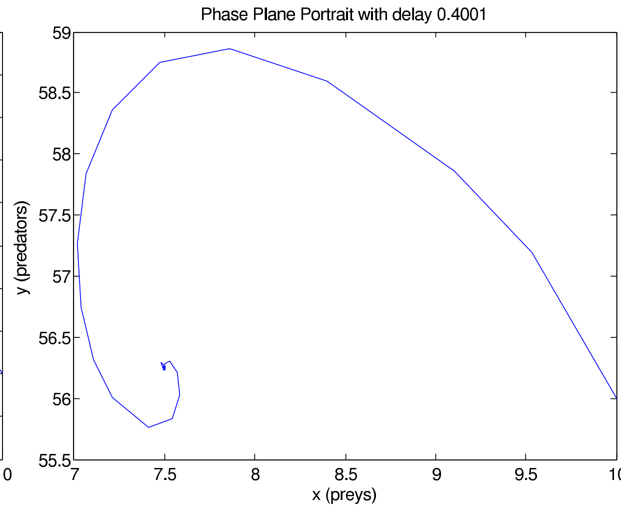

Figure 4.18. Trajectory behavior of Example 4.4 with $\tau = 0.4001$.

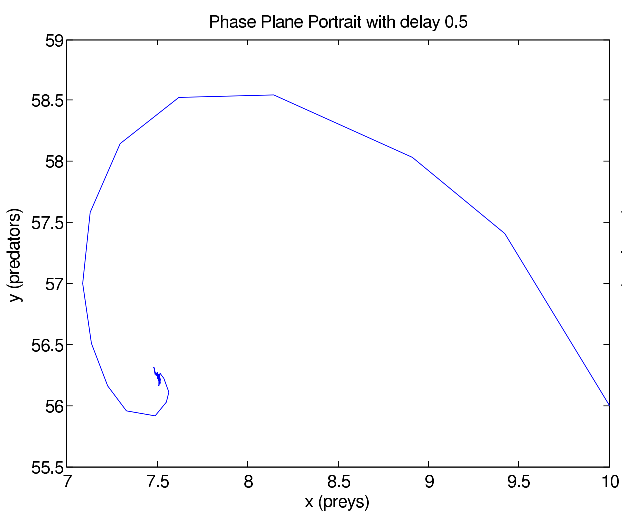

Figure 4.19. Trajectory behavior of Example 4.4 with $\tau = 0.5$.

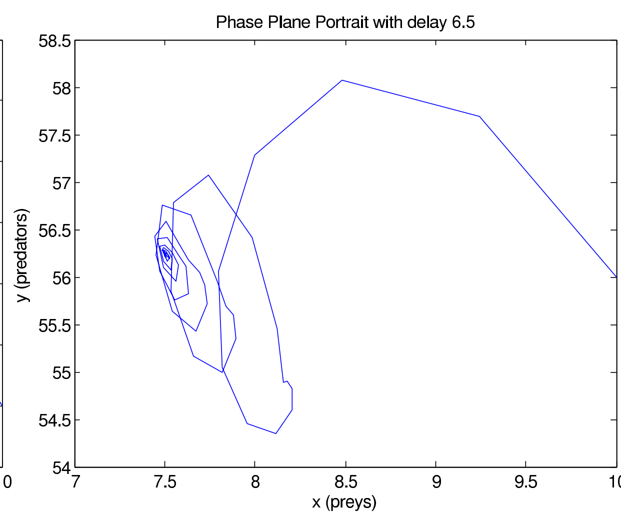

Figure 4.20. Trajectory behavior of Example 4.4 with $\tau = 6.5$.

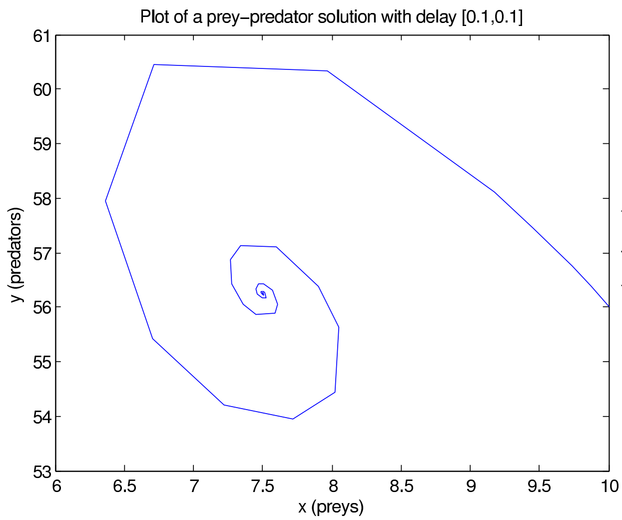

Figure 4.21. Trajectory behavior of Example 4.5 with $\tau = 0.1$.

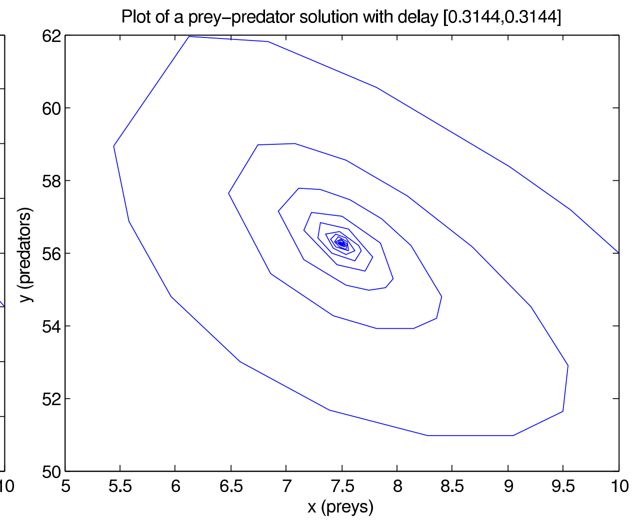

Figure 4.22. Trajectory behavior of Example 4.5 with $\tau = 0.3144$.

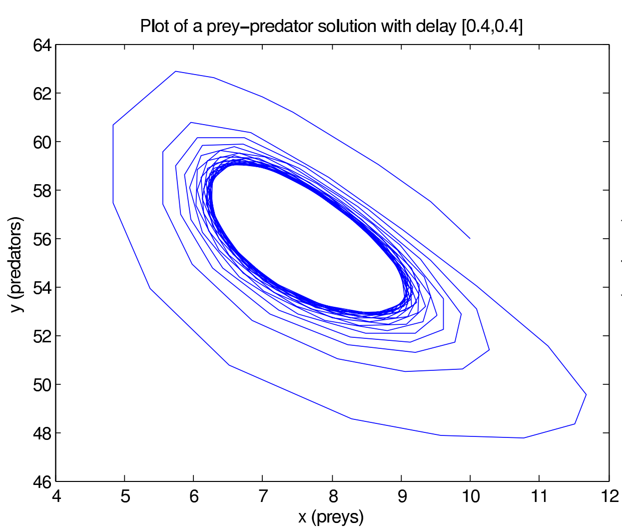

Figure 4.23. Trajectory behavior of Example 4.5 with $\tau = 0.4$.

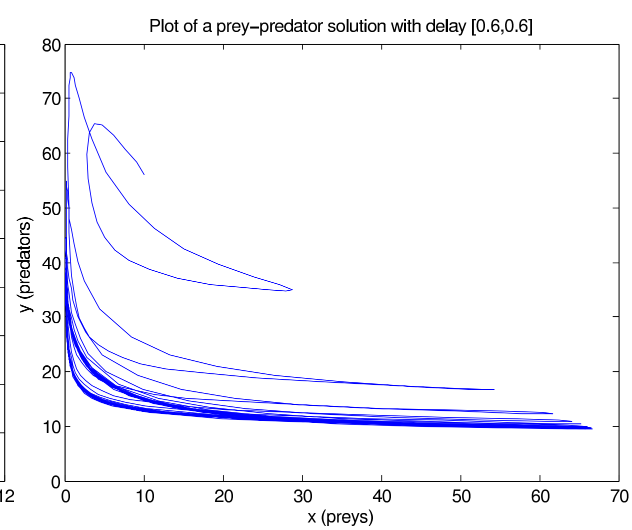

Figure 4.24. Trajectory behavior of Example 4.5 with $\tau = 0.6$.

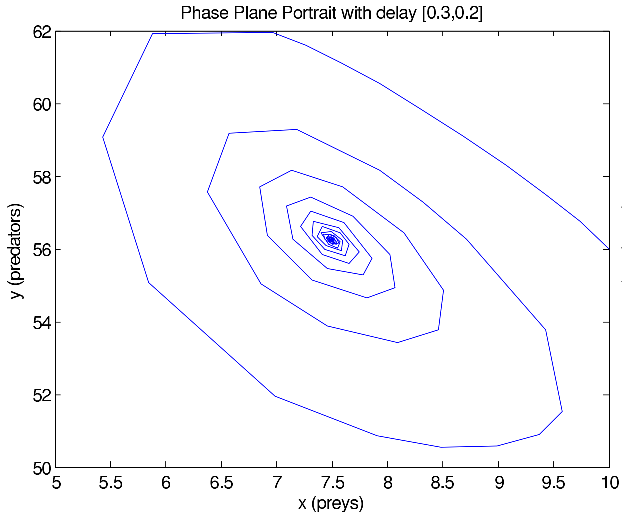

Figure 4.25. Trajectory behavior of Example 4.5 with $[\tau_1, \tau_2] = [0.3, 0.2]$.

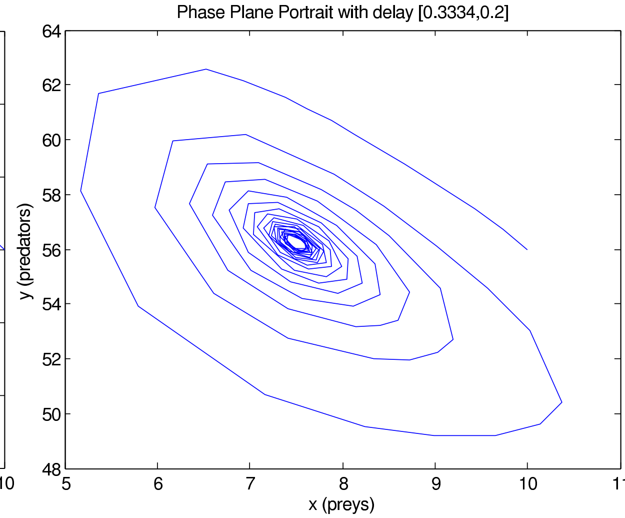

Figure 4.26. Trajectory behavior of Example 4.5 with $[\tau_1, \tau_2] = [0.334, 0.2]$.

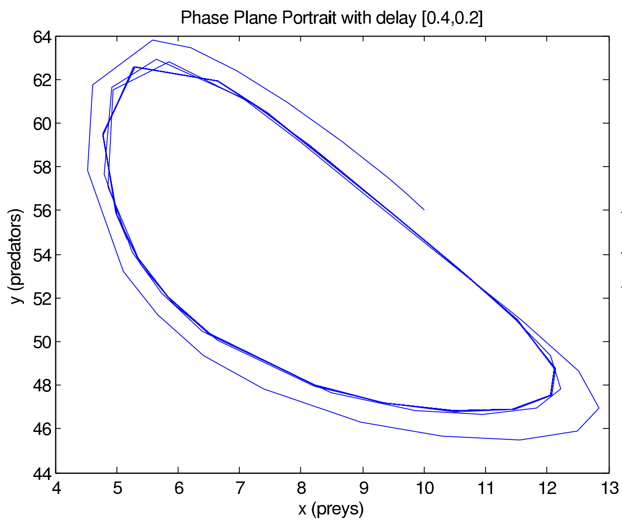

Figure 4.27. Trajectory behavior of Example 4.5 with $[\tau_1, \tau_2] = [0.4, 0.2]$.

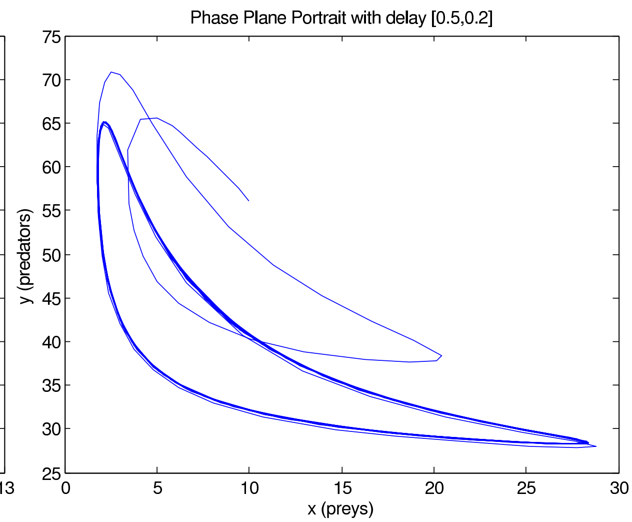

Figure 4.28. Trajectory behavior of Example 4.5 with $[\tau_1, \tau_2] = [0.5, 0.2]$.

CHAPTER 5

NUMERICAL SIMULATIONS AND INVESTIGATION OF THE TRAJECTORIES OF DDE CONSIDERING THE AVERAGE

In this chapter, let's discuss the trajectory behavior of the system of delay differential equations by considering the average for multiple discrete delays as a Riemann sum approximation for uniformly distributed continuous delays. We use some of the examples from Chapter 4. The stability change of the non-trivial equilibria in each example is considered with respect to the sample size n of the Riemann sum approximation of the integral over the interval $[t-\tau, t]$ for fixed τ. All of the figures of the examples are shown at the end of the chapter and the initial segment in the examples are chosen close to the non-trivial equilibrium point.

5.1 EXAMPLES FOR TRAJECTORY CHANGES ACCORDING TO THE SAMPLE SIZE

Example 5.1. Consider the following predator-prey model:

$$\dot{x}(t) = 15x(t)\left[1 - \frac{x(t)}{25}\right] - 0.5x(t)y(t),$$
$$\dot{y}(t) = -40y(t) + \frac{2.5}{n^2}\sum_{i,j=1}^{n}\left[x\left(t - \frac{i\tau}{n}\right)y\left(t - \frac{j\tau}{n}\right)\right].$$

The sample size $n = 1$ (Figure 5.1) case is discussed in Example 4.1. Let's consider the trajectory around the non-trivial equilibrium point $(16, 10.8)$ when $n = 1, 2, ..., 8$ at the fixed interval length $\tau = 0.1$.

When $n = 1$ (Figure 5.1) the uniform spiral stability behavior vanishes, but when $n = 6$ (Figure 5.6) the spiral becomes more uniform than the $n < 6$ cases and becomes spiral around the non-trivial equilibrium. When $n = 7$ (Figure 5.7) and $n = 8$ (Figure 5.8) the spiral behavior of trajectories of the DDE system becomes more uniform than

$n = 1, 2, ..., 5$. Thus, as the sample size increases, the trajectory of DDE systems converge to the trajectory behavior of ODE systems.

Example 5.2. Consider the following predator-prey model:

$$\dot{x}(t) = x(t)\left[4 - \frac{\sum_{i=1}^{n} x(t - \frac{i\tau}{n})}{5n} - \frac{y(t)}{x(t) + 15}\right],$$
$$\dot{y}(t) = y(t)\left[\frac{2x(t)}{x(t) + 15} - \frac{2}{3}\right].$$

Now consider the trajectory around the non-trivial equilibrium point $(7.5, 56.25)$ at fixed delay $\tau = 0.7$. As in Example 4.2 in Chapter 4, when $n = 1$ (Figure 5.9) the system shows the existence of a periodic orbit. But when $n = 2$ the predator-prey system indicates the existence of slow convergence to the equilibrium with self-intersections as shown in Figure 5.10.

When the sample size $n = 3, 4, ..., 6$ the trajectories converge to the non-trivial equilibrium without self-intersections as shown in the following Figures $5.11, 5.12, ..., 5.16$. When $n = 7$ and 8 (Figures 5.15 and 5.16) it shows a similar behavior as in $n = 6$ (Figure 5.14). Thus, it can be considered that the stable focus behavior is not changing until $n = 8$. However the system shows that the trajectories of DDE systems are converging to the trajectories of ODE systems as the sample size increases.

Example 5.3. Consider the following predator-prey model:

$$\dot{x}(t) = x(t)\left[4 - \frac{\sum_{i=1}^{n} x(t - \frac{i\tau_1}{n})}{5n} - \frac{y(t)}{x(t) + 15}\right],$$
$$\dot{y}(t) = \left[\frac{2\sum_{i,j=1}^{n} x(t - \frac{i\tau_1}{n})y(t - \frac{j\tau_2}{n})}{n^2(x(t) + 15)}\right] - y(t)\frac{2}{3}.$$

Let's look at this example when $\tau_1 = \tau_2 = \tau$ as in Example 4.5 in the Chapter 4 and consider the fixed delay as $\tau = 0.6$. As shown in the Figure 5.17, when $n = 1$ the predator-prey system shows the existence of a periodic orbit. The system converges in to a limit cycle when $n = 2$ (Figure 5.18). But as $n = 3$ and 4 (Figures 5.19 and 5.20) the limit cycles converge in to stable focus behaviors around the non-trivial equilibrium point. While it doesn't show a rapid trajectory change when $n = 5, 6, ..., 8$ (Figures $5.21, 5.22, 5.23, 5.24$), the trajectories converge to the trajectory behavior of the ODE system as the sample size increases.

Let's consider the behavior of the equilibrium point according to different delay terms in prey and predator terms as discussed in Example 4.5 in Chapter 4. In this situation let's take the fixed delay values as $[\tau_1, \tau_2] = [0.4, 0.2]$. Here when $n = 1$ (Figure 5.25) the predator-prey system shows the existence of a limit cycle around the non-trivial equilibrium point which gives a different trajectory behavior than in the trajectory of the ODE system.

When the sample size $n = 2$ (Figure 5.26) the predator-prey system becomes a stable focus around the non-trivial equlibrium point as shown in the Figure. There is not rapid trajectory behavior change after $n = 2$ to $n = 8$ (Figure 5.32), but the Figure 5.29 and Figure 5.30 show that trajectories of the predator-prey indicate a faster convergence to the non-trivial equilibrium as the sample size increases. We can see that trajectories are more similar to trajectories with small delay when sample size increases.

Conclusions from the experimental results

Illustrating the above examples we have that the trajectories of the delay differential equations converge faster to the non-trivial equilibrium when the sample size of the multiple discrete delay times are increased by considering the average for multiple discrete delays which approximate uniformly distributed continuous delays.

MATLAB Code

The MATLAB codes of the above examples can be obtained by changing the codes in Chapter 4. Now, let's consider delays associated with Riemann sums as discrete delay values in numerical simulations. Then the general form of the MATLAB codes can be written as follows. In all these codes $\tau = \tau_i$ where $\tau_i = \frac{ik}{n}$ for $i = 1, 2, ..., n$ with fixed delay value k. Consider the MATLAB code of Example 5.1 as a reference.

MATLAB code of the example 5.1

```
function example1
clear all; close all; clc;
% solving ode
[t,y]=ode15s(@ex1a,[0 100],[40 16]);
figure(1);
plot(y(:,1),y(:,2));
xlabel('x (preys)');
ylabel('y (predators)');
title('Plot of a prey-predator solution');
% solving dde for tau = k with sample size n
sol = dde23(@ex1dde,[tau1 tau2 tau3 tau4],[15 11],[0 40]);
figure(2);
plot(sol.y(1,:),sol.y(2,:));
xlabel('x (preys)');
ylabel('y (predators)');
title('Plot of a prey-predator solution when tau = k');
function dxdt = ex1a(t,x)
dxdt=zeros(2,1);
dxdt(1) = 15*x(1)*(1-(x(1)/25))-0.5*x(2)*x(1);
```

```
dxdt(2) = -40*x(2)+2.5*x(1)*x(2);

function dxdt = ex1dde(t,x,Z)

xlag1 = Z(:,1);

xlag2 = Z(:,2);

xlag3 = Z(:,3);

xlag4 = Z(:,4);

dxdt = [ 15*x(1)*(1-(x(1)/25))-0.5*x(2)*x(1)

    -40*x(2)+2.5*((xlag1(1)+...+xlag4(1))/4)*((xlag1(2)+...+xlag4(2))/4)];
```

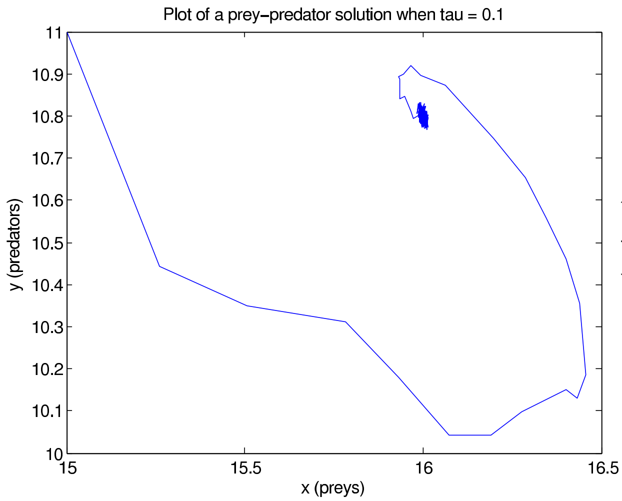

Figure 5.1. Trajectory behavior of Example 5.1 at $n = 1,\tau = 0.1$.

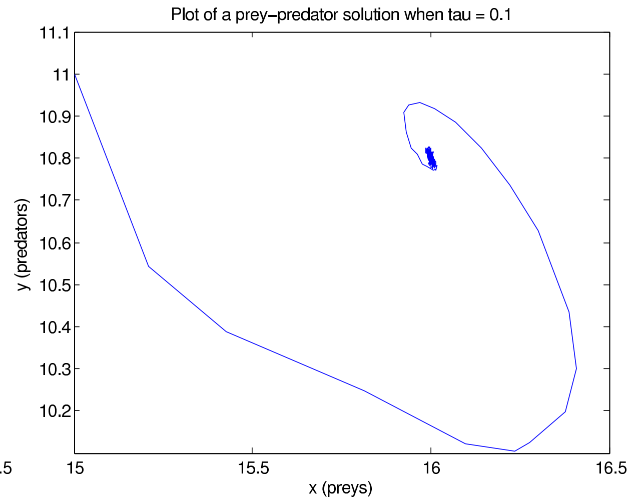

Figure 5.2. Trajectory behavior of Example 5.1 at $n = 2,\tau = 0.1$.

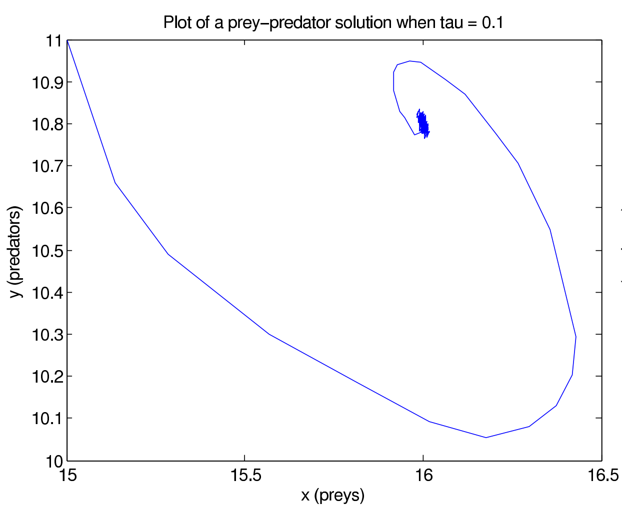

Figure 5.3. Trajectory behavior of Example 5.1 at $n = 3,\tau = 0.1$.

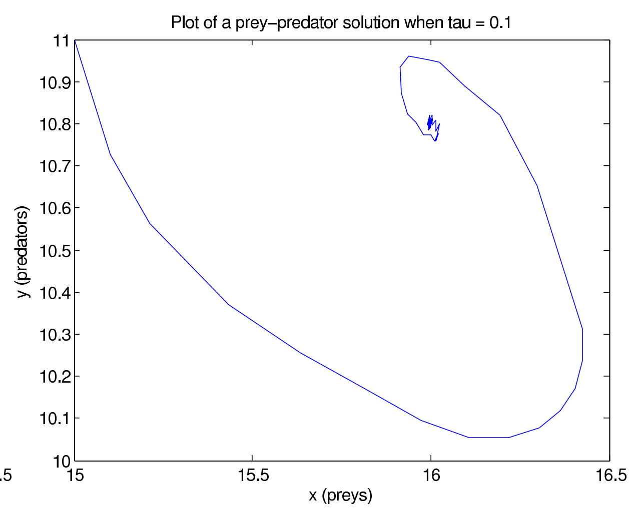

Figure 5.4. Trajectory behavior of Example 5.1 at $n = 4,\tau = 0.1$.

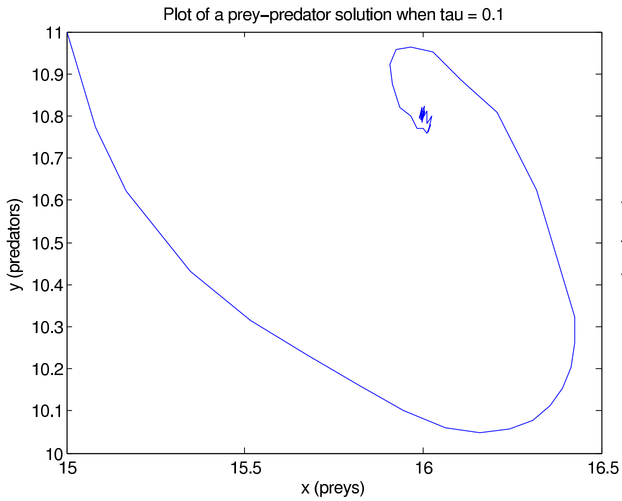

Figure 5.5. Trajectory behavior of Example 5.1 at $n = 5, \tau = 0.1$.

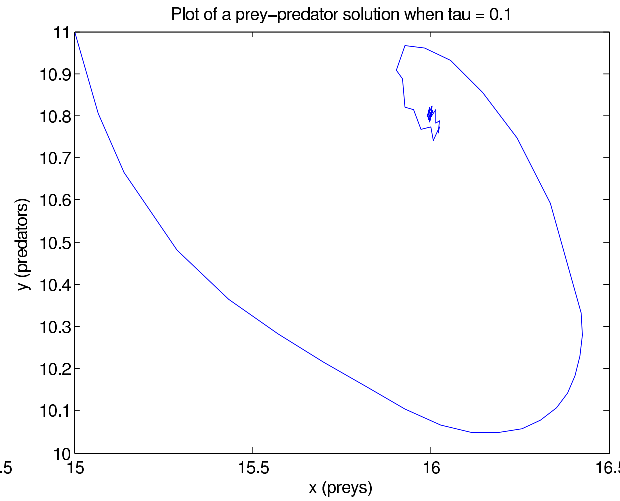

Figure 5.6. Trajectory behavior of Example 5.1 at $n = 6, \tau = 0.1$.

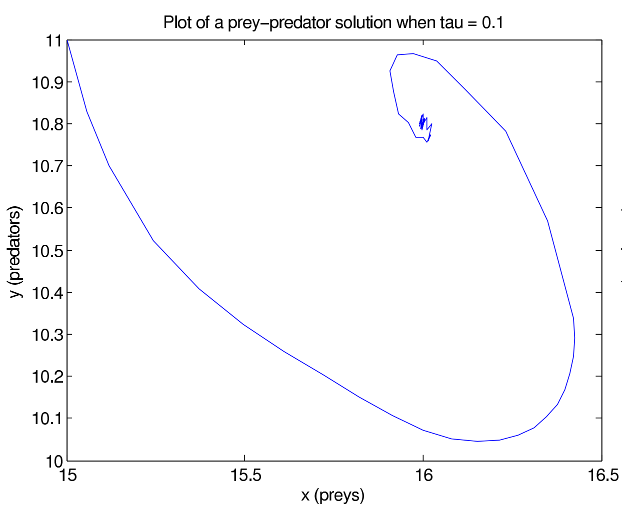

Figure 5.7. Trajectory behavior of Example 5.1 at $n = 7, \tau = 0.1$.

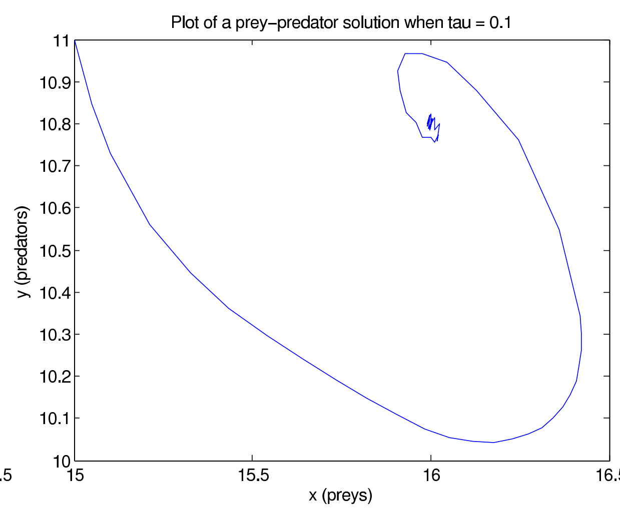

Figure 5.8. Trajectory behavior of Example 5.1 at $n = 8, \tau = 0.1$.

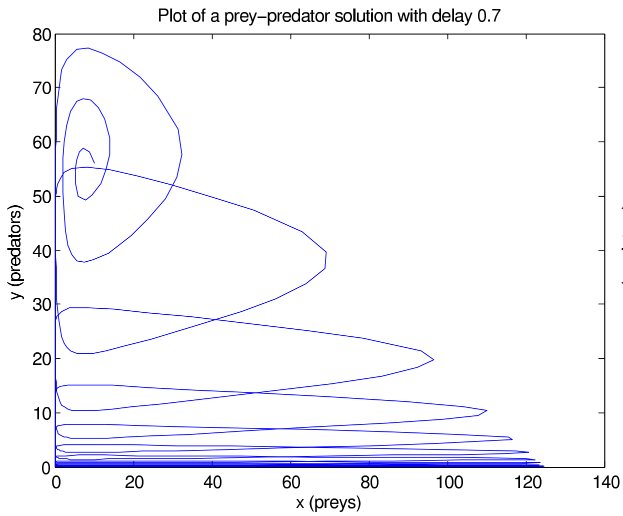

Figure 5.9. Trajectory behavior of Example 5.2 at $n = 1, \tau = 0.7$.

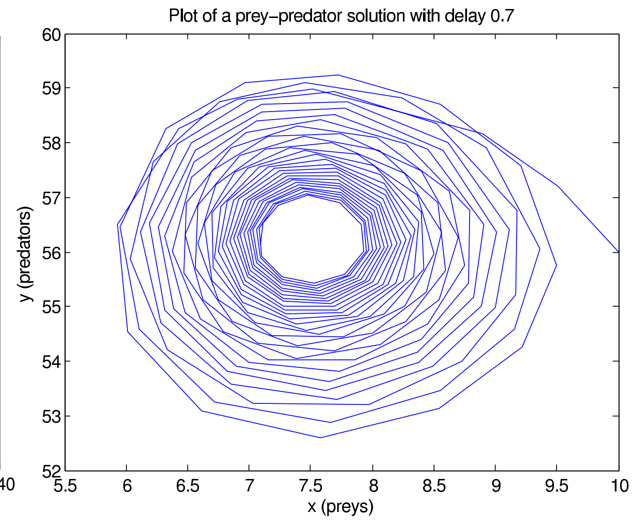

Figure 5.10. Trajectory behavior of Example 5.2 at $n = 2, \tau = 0.7$.

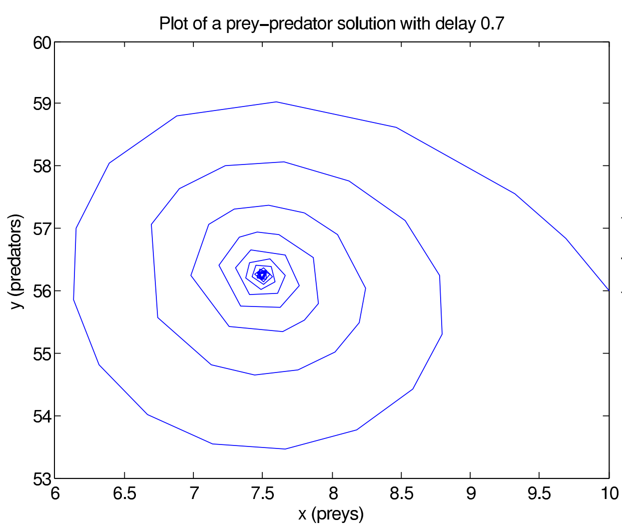

Figure 5.11. Trajectory behavior of Example 5.2 at $n = 3, \tau = 0.7$.

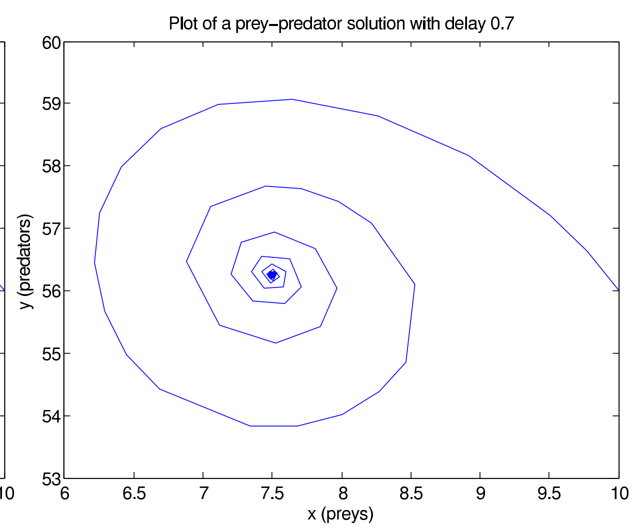

Figure 5.12. Trajectory behavior of Example 5.2 at $n = 4, \tau = 0.7$.

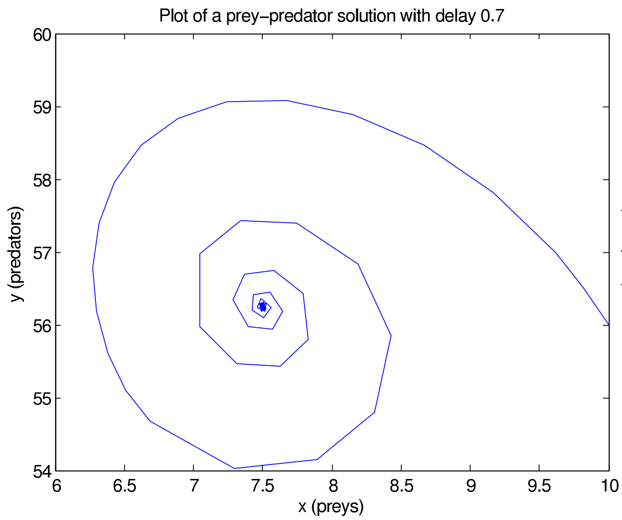

Figure 5.13. Trajectory behavior of Example 5.2 at $n = 5, \tau = 0.7$.

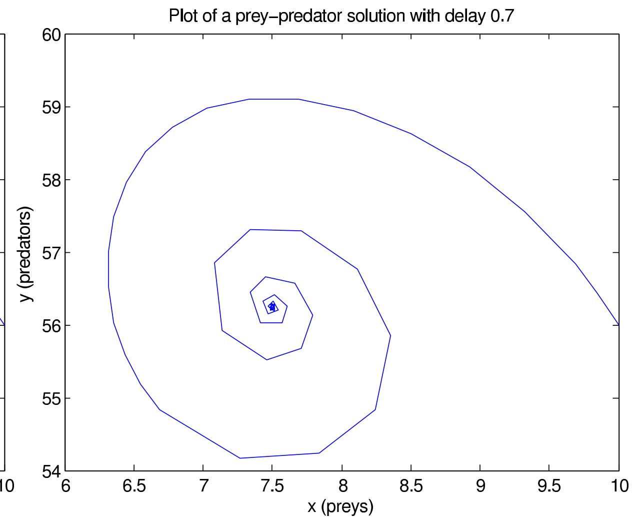

Figure 5.14. Trajectory behavior of Example 5.2 at $n = 6, \tau = 0.7$.

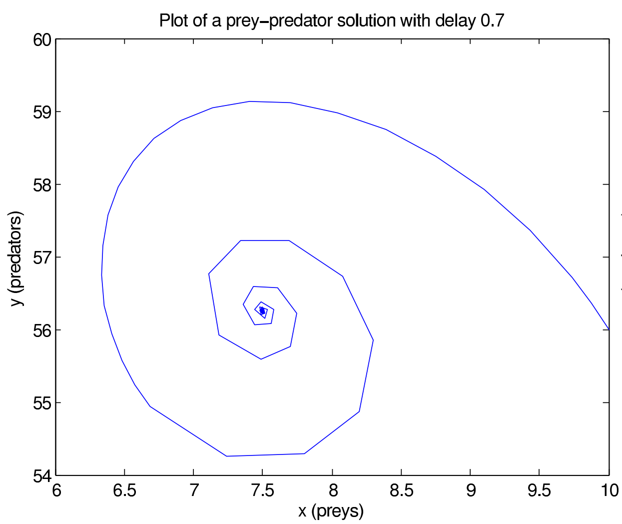

Figure 5.15. Trajectory behavior of Example 5.2 at $n = 7, \tau = 0.7$.

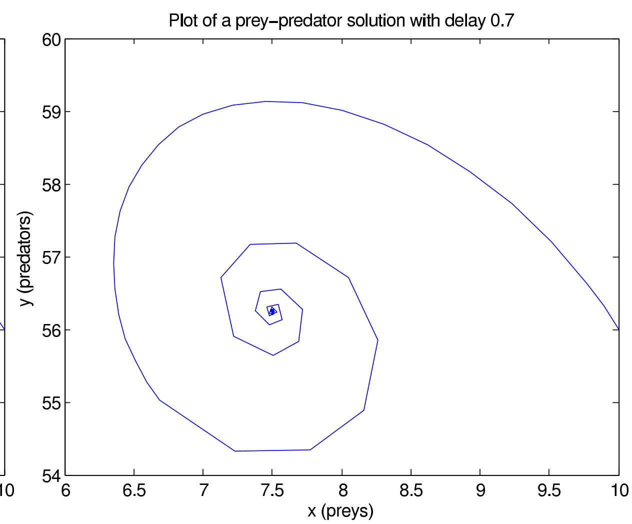

Figure 5.16. Trajectory behavior of Example 5.2 at $n = 8, \tau = 0.7$.

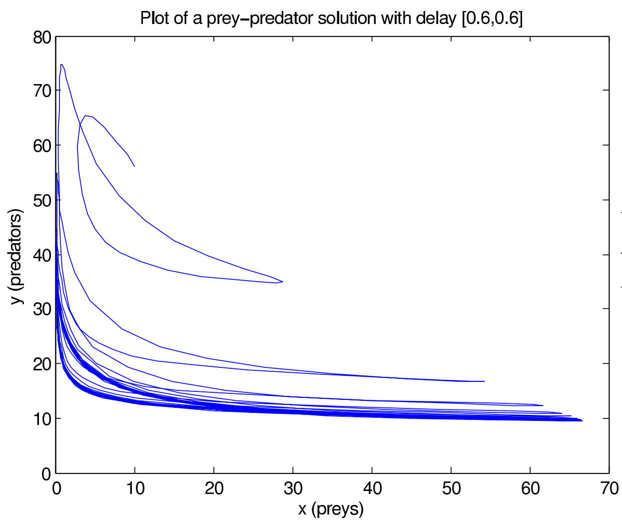

Figure 5.17. Trajectory behavior of Example 5.3 at $n = 1, \tau = 0.6$.

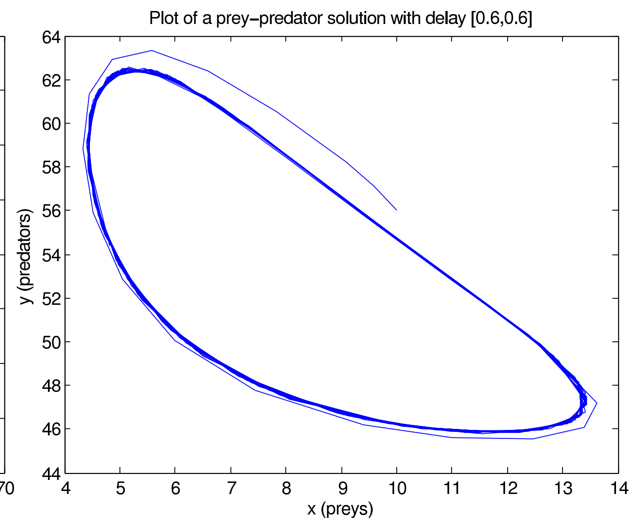

Figure 5.18. Trajectory behavior of Example 5.3 at $n = 2, \tau = 0.6$.

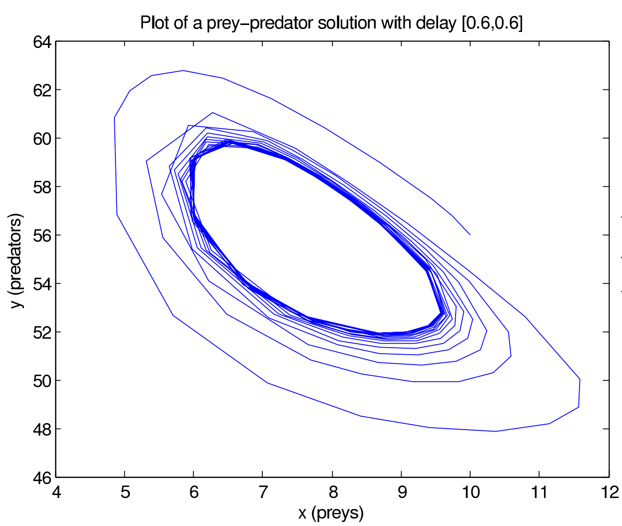

Figure 5.19. Trajectory behavior of Example 5.3 at $n = 3, \tau = 0.6$.

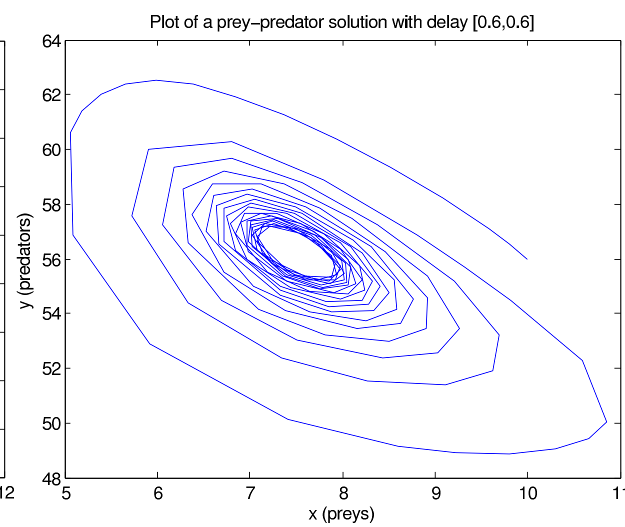

Figure 5.20. Trajectory behavior of Example 5.3 at $n = 4, \tau = 0.6$.

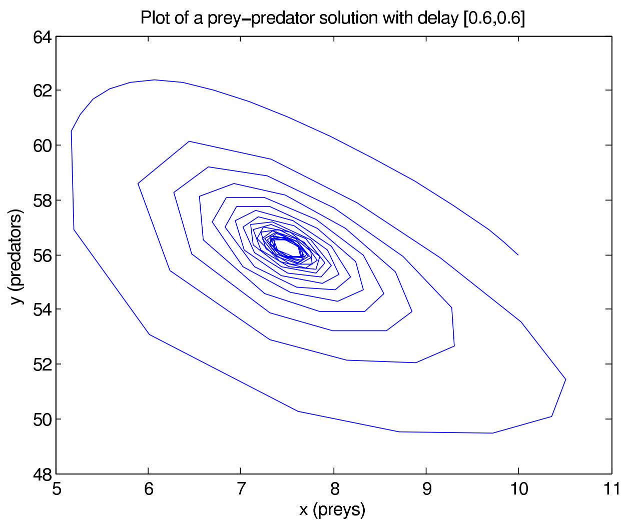

Figure 5.21. Trajectory behavior of Example 5.3 at $n = 5, \tau = 0.6$.

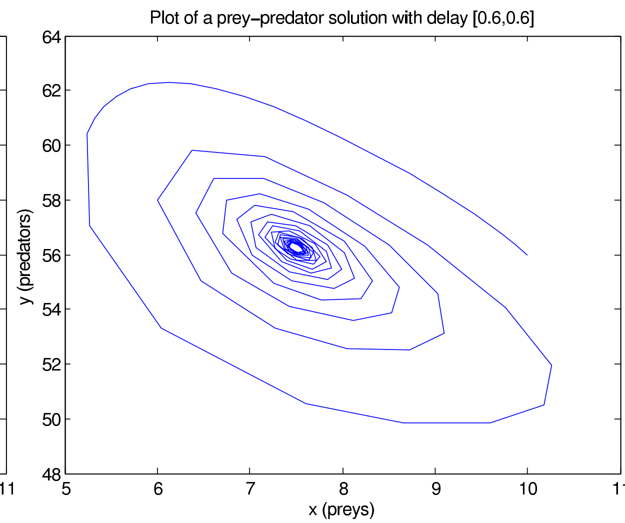

Figure 5.22. Trajectory behavior of Example 5.3 at $n = 6, \tau = 0.6$.

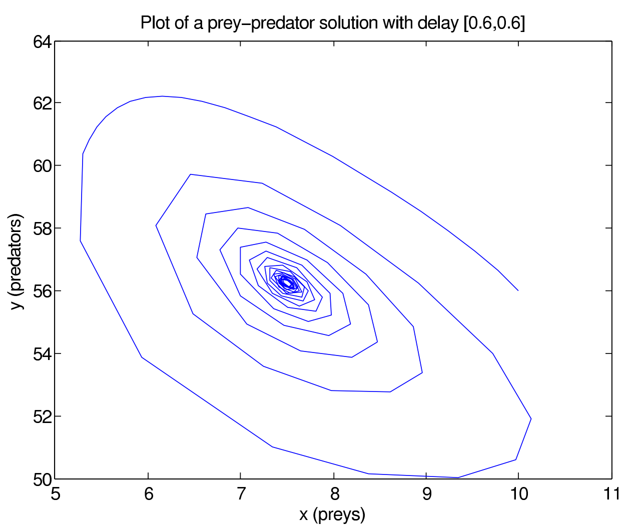

Figure 5.23. Trajectory behavior of Example 5.3 at $n = 7, \tau = 0.6$.

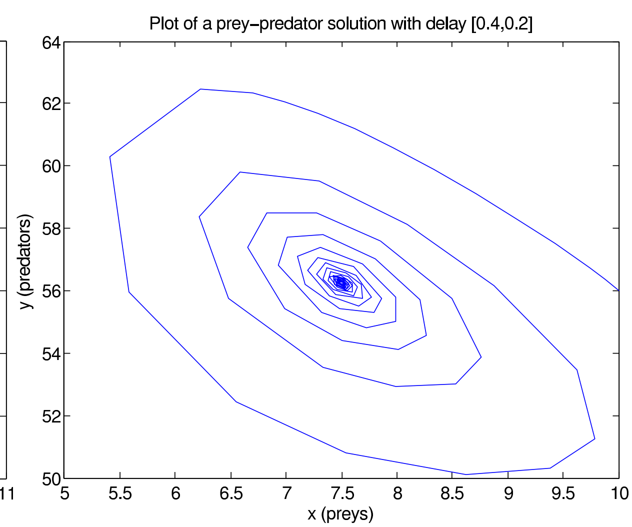

Figure 5.24. Trajectory behavior of Example 5.3 at $n = 8, \tau = 0.6$.

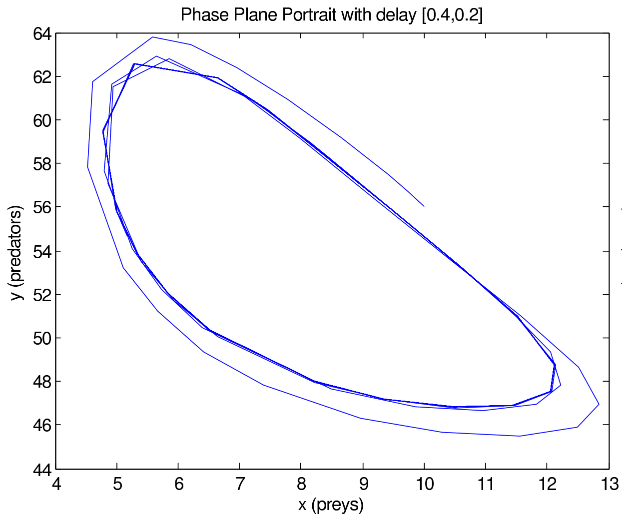

Figure 5.25. Trajectory behavior of Example 5.3 at $n = 1$, $[\tau_1, \tau_2] = [0.4, 0.2]$.

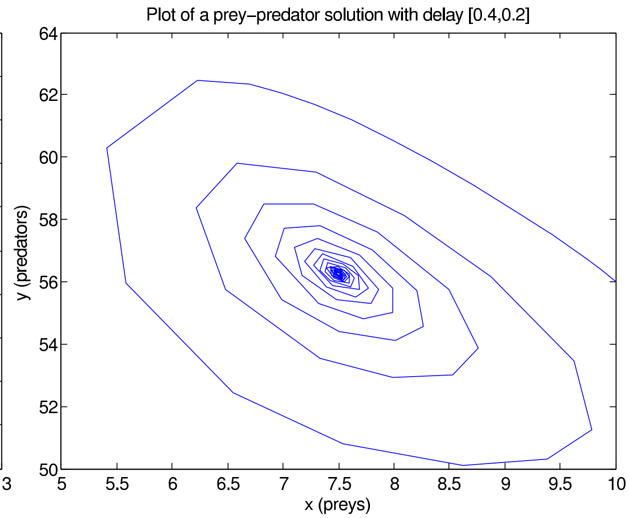

Figure 5.26. Trajectory behavior of Example 5.3 at $n = 2$, $[\tau_1, \tau_2] = [0.4, 0.2]$.

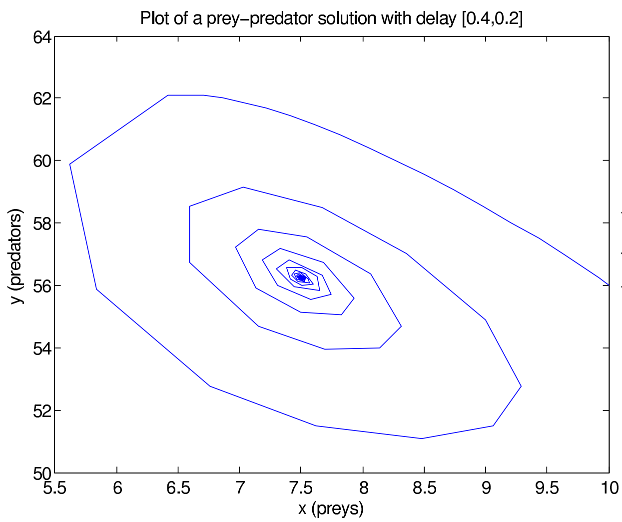

Figure 5.27. Trajectory behavior of Example 5.3 at $n = 3$, $[\tau_1, \tau_2] = [0.4, 0.2]$.

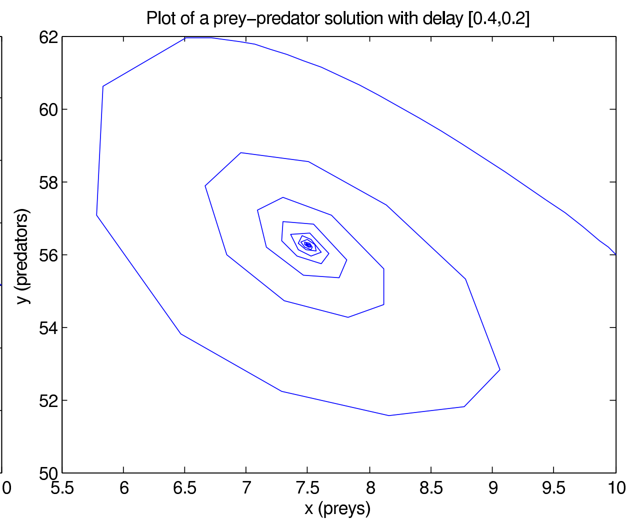

Figure 5.28. Trajectory behavior of Example 5.3 at $n = 4$, $[\tau_1, \tau_2] = [0.4, 0.2]$.

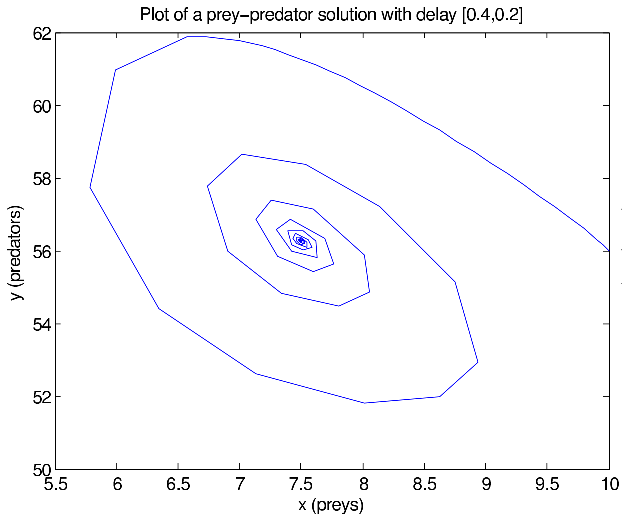

Figure 5.29. Trajectory behavior of Example 5.3 at $n = 5$, $[\tau_1, \tau_2] = [0.4, 0.2]$.

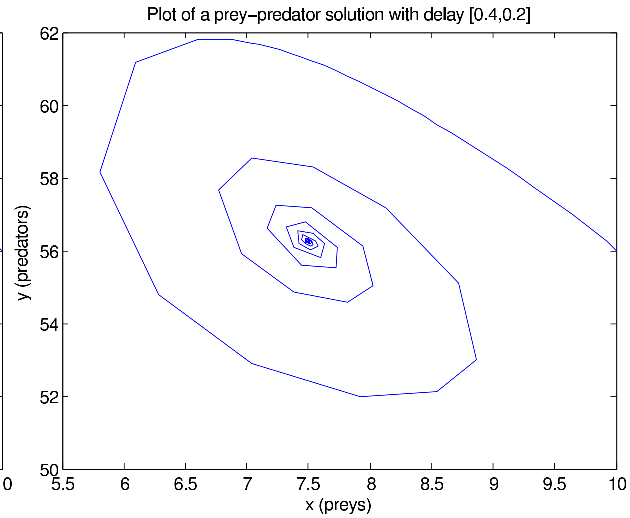

Figure 5.30. Trajectory behavior of Example 5.3 at $n = 6$, $[\tau_1, \tau_2] = [0.4, 0.2]$.

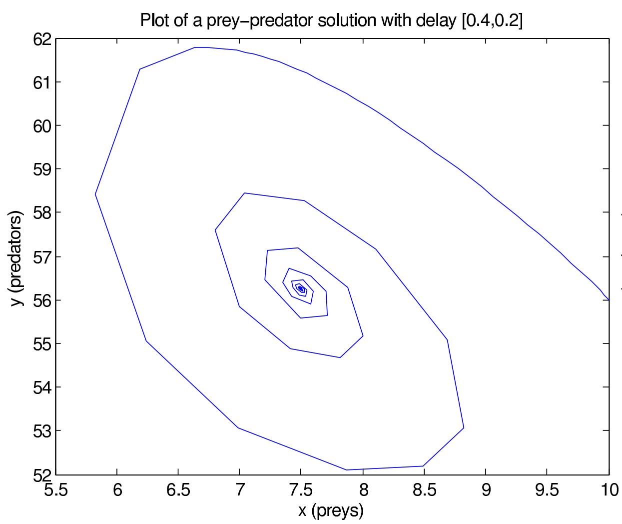

Figure 5.31. Trajectory behavior of Example 5.3 at $n = 7$, $[\tau_1, \tau_2] = [0.4, 0.2]$.

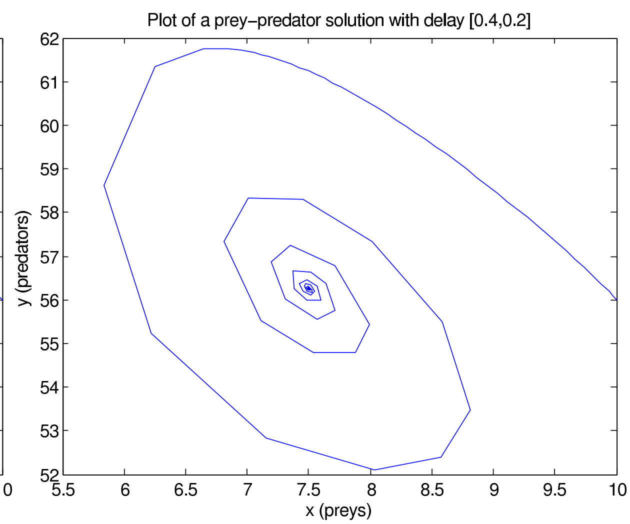

Figure 5.32. Trajectory behavior of Example 5.3 at $n = 8$, $[\tau_1, \tau_2] = [0.4, 0.2]$.

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