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TRADE-OFFS IN DISTINGUISHING TWO-QUBIT STATE PREPARATIONS USING ONE-WAY LOCC

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by

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A Thesis
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Approved by:

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Quantum state discrimination is a fundamental problem in quantum information science. We investigate the optimal distinguishability of orthogonal two-qubit (bipartite) quantum states. The scenario consists of three parties: Alice, Bob, and Charlie. Charlie prepares one of two orthogonal states and sends one qubit to Alice and the other to Bob. Their goal is to correctly identify which state Charlie sent. In most state discrimination scenarios, it is assumed that Alice and Bob can freely communicate with one another so as to collectively agree on the best guess.

In this research, we consider a more restricted setting where only one-way classical communication is possible from Alice to Bob. Under this setting, we study two figures of merit (i) Alice's optimal probability, P, of identifying the state , and (ii) Alice's optimal probability, $P^{\perp}$, of identifying the state along with helping Bob identify the state perfectly. We show that in general $P \neq P^{\perp}$ and we prove a theorem for when $P = P^{\perp}$. We also found that the maximum of $P - P^{\perp}$ can arbitrarily approach $1/2$.

TABLE OF CONTENTS

ABSTRACT i

LIST OF FIGURES iii

CHAPTERS

CHAPTER 1 – Introduction 1

1.1 – Dirac Notation 2

1.2 – Quantum State 3

1.3 – Measurement 5

CHAPTER 2 – Problem Statement 7

CHAPTER 3 – General Result 9

CHAPTER 4 – Quantifying the Gap 17

4.1 – Form of $P^{\perp}$ 19

4.2 – Calculating $|\eta_i'\rangle$ and $|\nu_i'\rangle$ 20

4.3 – Restrictions on the Unitary 22

CHAPTER 5 – Conclusions 25

REFERENCES 27

VITA 28

LIST OF FIGURES

FIGURE PAGE

Figure 1 Problem Figure 7

Figure 2 LOCC Protocol 8

Figure 3 One-way LOCC 9

CHAPTER 1

INTRODUCTION

In classical computers, the fundamental unit of information is the bit, which can take the value of zero or one. However, another fundamental unit of information exists in the form of a quantum bit (qubit). The nature of qubits are such that it can be in a zero state or the one state, but it can also be in a superposition (a linear combination) of the two.

Additionally, measuring a qubit generally changes its state and the measurement outcome is probabilistic and completely random. If we measure a qubit to check if it is in the zero state or one state, the measurement result will be zero with some probability P and one with probability $1 - P$. This also means that if we have identical copies of a qubit and we perform the same measurement on each qubit, it is generally not guaranteed that we will get the same measurement result.

The analysis of measurement probabilities for determining the state of a quantum system is known as quantum state discrimination. In general, it is not possible to perfectly distinguish one quantum state of a qubit from another (unless the two states are orthogonal). Thus, the analysis generally comes down to the optimal measurement direction and probability of determining the state correctly. This is the focus of this research.

These properties along with entanglement are what give qubits their power. Using qubits, we can perform algorithms such as Shor's algorithm [Sho97], super dense coding [BW92], quantum teleportation [BBC$^+$93], and quantum key distribution (qkd) [BB87]. Shor's algorithm finds prime factors in polynomial time, super dense coding sends two bits of data in one qubit, quantum teleportation teleports a quantum state, and qkd creates secure shared keys. These algorithms are possible because of the properties of qubits.

In this research, we consider the optimal distinguishability of two orthogonal two-qubit quantum states. We consider a scenario that consists of three parties: Alice, Bob, and Charlie. Charlie prepares one of two orthogonal states and sends one qubit to Alice and the other to Bob. Their goal is to correctly identify which state Charlie sent. In our research, we restrict the scenario to one-way classical communication from Alice to Bob. Under this setting, we study two figures of merit (i) Alice's optimal probability, P, of identifying the state , and (ii) Alice's optimal probability, $P^{\perp}$, of identifying the state along with helping Bob identify the state perfectly. We show that in general $P \neq P^{\perp}$ and we prove a theorem for when $P = P^{\perp}$. We also found that the maximum of $P - P^{\perp}$ can arbitrarily approach $1/2$.

Before going into the main part of the research, we will do a short introduction of the crucial parts of quantum information theory used in this research. The introduction will cover parts of Dirac notation, quantum states, and quantum measurements.

CHAPTER 1.1
DIRAC NOTATION

The mathematical language used in quantum information theory is linear algebra because it models quantum systems accurately. Dirac notation (or bra-ket notation) is a linear algebra notation that is used for its convenience. In Dirac notation, we represent vectors with kets such as $|\alpha\rangle$, where α is a label. Every ket stands for a vector. For example, the zero and one vectors in two dimensions can be written as

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1}$$

In (1) the $|0\rangle$ and $|1\rangle$ are kets. The inner product between two vectors $|\alpha\rangle$ and $|\beta\rangle$ is written as

$$\langle\alpha|\,|\beta\rangle = \langle\alpha|\beta\rangle \tag{2}$$

where $\langle\alpha|$ is the dual vector of $|\alpha\rangle$ and $\langle\alpha| = |\alpha\rangle^{\dagger}$. The symbol $\dagger$ stands for complex conjugate transpose. Thus, the operation is equivalent to taking the complex conjugate of the elements and then taking the transpose. The term on the right of the equal sign in (2) is just a simplification of the notation, where we eliminated one of the vertical bars. We can also define the outer product as

$$|\alpha\rangle\langle\beta| \tag{3}$$

The result of an outer product is a matrix. Finally, the tensor product between two vectors is written as

$$|\alpha\rangle \otimes |\beta\rangle = |\alpha\beta\rangle \tag{4}$$

where the term on the right of the equal sign in (4) is a simplification of the notation. An important fact to note, is that any calculation done using bra-ket notation can be done with matrices and vectors. Bra-ket is used because it is convenient.

CHAPTER 1.2
QUANTUM STATE

In quantum mechanics, the state of any physical systems can be represented by a unit vector. More specifically,

Postulate 1. *The state of a system is described by a unit vector in a Hilbert space* $\mathcal{H}$

[KLM07].

For finite dimensions a Hilbert space is just a vector space with an inner product. A qubit is a quantum system that exists as a superposition of two states and is an element in a two dimensional Hilbert space. Thus, in two dimensions, we can represent any qubit $|a\rangle$ as

$$|a\rangle = c_0 |0\rangle + c_1 |1\rangle \tag{5}$$

where c_0 and c_1 are complex numbers and the norm $\| |a\rangle \| = 1$. When dealing with more than one qubit, the kets involve tensor products. For example, a two-qubit state can be represented as

$$|a\rangle = c_0 |00\rangle + c_1 |01\rangle + c_2 |10\rangle + c_3 |11\rangle \tag{6}$$

Two classical bits can be in the state 00, 01, 10, or 11. A two-qubit system can exist as a superposition of these four possible states. An equivalent way of representing a quantum system is through a density matrix. For our purposes, the density matrix of a quantum system is represented by the outer product of its state vector. More concretely,

$$\rho = |\Psi\rangle\langle\Psi| \tag{7}$$

where $|\Psi\rangle$ is the state vector of the system.

CHAPTER 1.3

MEASUREMENT

In quantum mechanics, measurement outcomes are probabilistic as shown in (8). Measurements can be mathematically represented with a set of matrices $\{M_i\}$.

$$P(i) = \langle\Psi| M_i^\dagger M_i |\Psi\rangle \tag{8}$$

(8) gives the probability of getting the measurement result i for the state $|\Psi\rangle$. If we get the result i, then that means the measurement operator M_i acted on the system. We know that measurement in quantum mechanics affects the state of the system. Thus, the post measurement state of the system is

$$|\Psi'\rangle = \frac{M_i |\Psi\rangle}{\sqrt{P(i)}} \tag{9}$$

where $\sqrt{P(i)}$ is a normalizing constant. In terms of density matrices, the measurement probability is given by

$$P(i) = Tr(M_i^\dagger M_i \rho) \tag{10}$$

and the post measurement density matrix is

$$\rho' = \frac{M_i \rho M_i^\dagger}{P(i)} \tag{11}$$

Before moving forward, there are two useful properties involving tensor products that we will use extensively. The first is that the † distributes over tensor products. For example,

$$|ab\rangle^\dagger = |a\rangle^\dagger \otimes |b\rangle^\dagger = \langle ab| \text{ or } (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger \tag{12}$$

where A and B are matrices. Second, multiplication involving tensor products occurs as follows

$$(A \otimes B)(C \otimes D) = AC \otimes BD \text{ and } (A \otimes B)|ab\rangle = A|a\rangle \otimes B|b\rangle \tag{13}$$

where A, B, C, and D are matrices. Since we are dealing with state discrimination, it is important to determine what kind of measurement operators are optimal. It turns out that the optimal measurements when measuring on single qubits are projective measurements [BC09]. Projectors are matrices that satisfy the properties that $P = P^\dagger$ and $P^2 = P$. We can prove the optimality of projectors as follows.

First, consider a qubit that is in one of two possible states $|\Psi_0\rangle$ and $|\Psi_1\rangle$. The probability that the qubit has the state $|\Psi_0\rangle$ is $P(|\Psi_0\rangle) = P_0$ and the probability that it has the state $|\Psi_1\rangle$ is $P(|\Psi_1\rangle) = P_1$. Then, we can calculate the error of misidentifying the state as follows

$$P_{error} = P(1|\Psi_0)P(\Psi_0) + P(0|\Psi_1)P(\Psi_1) \tag{14}$$

$$= \langle\Psi_0| A_1 |\Psi_0\rangle P_0 + \langle\Psi_1| A_0 |\Psi_1\rangle P_1 \tag{15}$$

$$= P_0 - Tr[(P_0 |\Psi_0\rangle\langle\Psi_0| - P_1 |\Psi_1\rangle\langle\Psi_1|)A_0] \tag{16}$$

We used the property $A_1 = I - A_0$ and switched to density operators to get the last line. To minimize the error we need to maximize $Tr[(P_0 |\Psi_0\rangle\langle\Psi_0| - P_1 |\Psi_1\rangle\langle\Psi_1|)A_0]$. Since $T = P_0 |\Psi_0\rangle\langle\Psi_0| - P_1 |\Psi_1\rangle\langle\Psi_1|$ is a normal matrix, it has a spectral decomposition. Therefore, to minimize the error, A_0 should project onto the subspaces of T that have positive eigenvalues. Thus, projectors are optimal.

CHAPTER 2

PROBLEM STATEMENT

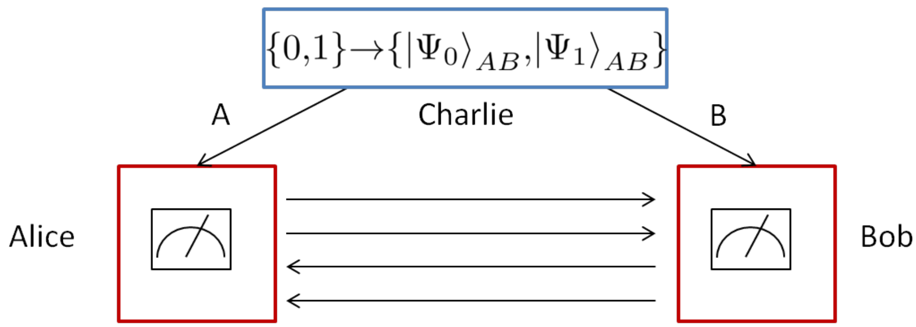

Figure 1: Charlie encodes a classical bit into one of two orthogonal quantum states. To recover the classical bit Alice and Bob have to determine which state their qubit came from. Alice and Bob are limited to local operations and classical communication (LOCC).

This research studies the optimal distinguishing of qubits, which is fundamental in quantum information science. The original motivation for the research can be illustrated by a game between three parties: Alice, Bob, and Charlie as shown in Figure 1. Charlie possesses a classical bit and depending on its value he prepares one of two bipartite globally orthogonal states $|\Psi_0\rangle$ and $|\Psi_1\rangle$. The form of $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are completely known to Alice and Bob. Charlie then sends the first qubit of the prepared state to Alice and the other qubit to Bob.

Charlie chooses each state with equal probability. The task of Alice and Bob is to then guess which state their qubits are from. Alice and Bob succeed if they both guess correctly. If one or both guess incorrectly, they fail. Furthermore, the only tools Alice and Bob have are local operations and a classical communication channel (LOCC) between them. When we say local operations, we mean Alice and Bob are in separate locations so that they are only able to perform measurements on their qubit. Thus, they are limited to performing local measurement operations, but they are able to communicate their measurement results using the classical communication channel between them.

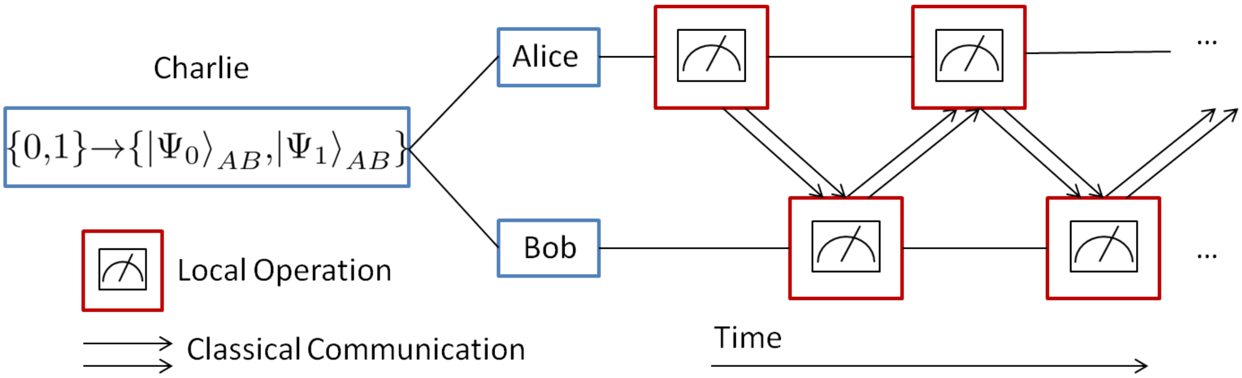

Figure 2: LOCC protocol between Alice to Bob. Alice measures and sends Bob the result. Then after Bob receives the result, Bob measures and sends Alice the result. This process can repeat any number of times.

Essentially, they can perform some sequence of measurements and classical communication as demonstrated in Figure 2.

The discrimination problem for two orthogonal bipartite states using LOCC has been completely solved [WSHV00]. It was proven by Walgate et al. that two orthogonal states of arbitrary dimension can always be distinguished perfectly with LOCC. Walgate et al. proved that the two orthogonal states can always be written as

$$|\Psi_0\rangle = |1\rangle_A |\eta_1\rangle_B + ... + |L\rangle_A |\eta_L\rangle_B \tag{17}$$

$$|\Psi_1\rangle = |1\rangle_A |\eta_1^\perp\rangle_B + ... + |L\rangle_A |\eta_L^\perp\rangle_B \tag{18}$$

where $\{|i\rangle_A$, for i = 1 to L$\}$ is an orthonormal basis. In this basis, $|\eta_i\rangle_B$ and $|\eta_i^\perp\rangle$ are perpendicular. With LOCC, the optimal success strategy is for Alice to measure in this basis and communicate to Bob the measurement results. Bob then perfectly distinguishes between two orthogonal states and communicates the results to Alice. Thus, they both know the state at the end of the protocol. Essentially, Walgate et al. proved that for two orthogonal states LOCC is just as powerful as global measurements.

In this research, our focus is on one way LOCC from Alice to Bob as shown in

Figure 3. The states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are still globally orthogonal bipartite states. Since the classical channel is only from Alice to Bob, Alice is able to assist Bob, but Bob is unable to assist Alice. In this setting, we examine how the original two-way protocol with perfect discrimination changes.

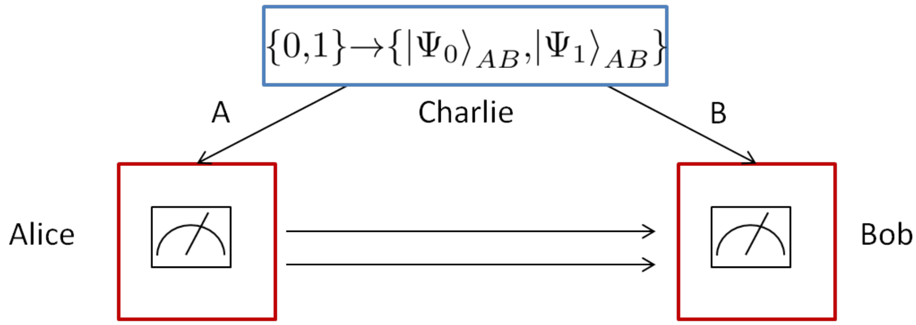

Figure 3: Bipartite state discrimination with one-way LOCC from Alice to Bob. Alice can help Bob distinguish, but Bob can't help Alice.

There are two figures of merit that we are interested in: (i) the optimal probability, P, that Alice determines the state, and (ii) the optimal probability, $P^{\perp}$, that Alice determines the state with the additional constraint that Bob is able to perfectly distinguish. We know from Walgate et al. [WSHV00] that Alice can always help Bob perfectly distinguish by measuring in way that Bob is left to distinguish two orthogonal states. However, we want to know what it costs Alice to accomplish this task. Thus, we are interested in the gap $P - P^{\perp}$, which measures the drop in probability of Alice determining the state correctly.

CHAPTER 3

GENERAL RESULT

With qubits, we know from Walgate et al. [WSHV00] that we can write two orthogonal bipartite states as

$$|\Psi_0\rangle = |1'\rangle_A |\eta_1\rangle_B + |2'\rangle_A |\eta_2\rangle_B$$
$$|\Psi_1\rangle = |1'\rangle_A \left|\eta_1^{\perp}\right\rangle_B + |2'\rangle_A \left|\eta_2^{\perp}\right\rangle_B$$

where $\{|i'\rangle_A$, for i = 1 to 2$\}$ form an orthonormal basis and $|\eta_i\rangle_B$ and $\left|\eta_i^{\perp}\right\rangle_B$ are perpendicular, but not necessarily unit vectors. Using local unitaries, we can rotate $|1'\rangle$ to $|0\rangle$ and $|\eta_1\rangle$ to $\sqrt{p}\,|0\rangle$ to get

$$|\Psi_0\rangle = \sqrt{p}\,|00\rangle + \sqrt{1-p}\left|1\hat{b}\right\rangle \tag{19}$$
$$|\Psi_1\rangle = \sqrt{q}\,|01\rangle + \sqrt{1-q}\,|1\rangle\left|-\hat{b}\right\rangle \tag{20}$$

where $\left|\hat{b}\right\rangle = \cos\frac{\alpha}{2}|0\rangle + \exp(i\phi)\sin\frac{\alpha}{2}|1\rangle$, $\left|-\hat{b}\right\rangle = \exp(i\kappa)(\sin\frac{\alpha}{2}|0\rangle - \exp(i\phi)\cos\frac{\alpha}{2}|1\rangle)$, and p and q are normalizing constants.

We will first determine when the gap is zero and nonzero. To do this, we will find the projectors that optimize P. Then, we will calculate the post measurement states of $|\Psi_0\rangle$ and $|\Psi_1\rangle$ and find the necessary conditions that allow Bob to perfectly distinguish. For Bob to distinguish perfectly, the possible post measurement states of his qubit given Alice's measurement result must be a set of orthogonal vectors.

Theorem 1. $P^{\perp} = P$ *or zero gap occurs iff* (21) *is satisfied.*

$$
\begin{aligned}
&[\sqrt{p}\sqrt{1-q}\left(\sqrt{p-p^2}\exp(-i\kappa)+\sqrt{q-q^2}\exp(i\phi)\right) \\
&+\sqrt{q}\sqrt{1-p}\left(\sqrt{p-p^2}\exp(i\phi)+\sqrt{q-q^2}\exp(-i\kappa)\right)]\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}=0
\end{aligned} \tag{21}
$$

In particular, zero gap occurs whenever (i) $\cos\frac{\alpha}{2}=0$, (ii) $\sin\frac{\alpha}{2}=0$, (iii) $p=q$ and $\kappa+\phi=\pi$, or (iv) $q=1-p$ and $\kappa+\phi=\pi$. Case (ii) corresponds to product states and case (iii) corresponds to Alice having identical reduced density matrices for both states. In the latter situation, $P=1/2$.

Proof. Let us first focus on optimizing P. To do this, we can trace out Bob's half of the states because Bob is unable to help Alice due to the lack of a communication channel going from Bob to Alice.

$$
\begin{aligned}
\rho_0 =& p\left|0\middle\rangle\middle\langle 0\right| \otimes \left|0\middle\rangle\middle\langle 0\right| + \sqrt{p(1-p)}\left|0\middle\rangle\middle\langle 1\right| \otimes \left|0\middle\rangle\middle\langle \hat{b}\right| \\
&+\sqrt{p(1-p)}\left|1\middle\rangle\middle\langle 0\right| \otimes \left|\hat{b}\middle\rangle\middle\langle 0\right| + (1-p)\left|1\middle\rangle\middle\langle 1\right| \otimes \left|\hat{b}\middle\rangle\middle\langle \hat{b}\right|
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
\rho_1 =& q\left|0\middle\rangle\middle\langle 0\right| \otimes \left|1\middle\rangle\middle\langle 1\right| + \sqrt{q(1-q)}\left|0\middle\rangle\middle\langle 1\right| \otimes \left|1\middle\rangle\middle\langle -\hat{b}\right| \\
&+\sqrt{q(1-q)}\left|1\middle\rangle\middle\langle 0\right| \otimes \left|-\hat{b}\middle\rangle\middle\langle 1\right| + (1-q)\left|1\middle\rangle\middle\langle 1\right| \otimes \left|-\hat{b}\middle\rangle\middle\langle -\hat{b}\right|
\end{aligned} \tag{23}
$$

Taking the partial trace of Bob's system of both states we get

$$
\rho_0^A = p\left|0\middle\rangle\middle\langle 0\right| + \sqrt{p(1-p)}\left|0\middle\rangle\middle\langle 1\right|\cos\frac{\alpha}{2} + \sqrt{p(1-p)}\left|1\middle\rangle\middle\langle 0\right|\cos\frac{\alpha}{2} + (1-p)\left|1\middle\rangle\middle\langle 1\right| \tag{24}
$$

$$
\begin{aligned}
\rho_1^A =& q\left|0\middle\rangle\middle\langle 0\right| + \sqrt{q(1-q)}\left|0\middle\rangle\middle\langle 1\right|\left[-\exp(-i\kappa-i\phi)\cos\frac{\alpha}{2}\right] \\
&+\sqrt{q(1-q)}\left|1\middle\rangle\middle\langle 0\right|\left[-\exp(i\kappa+i\phi)\cos\frac{\alpha}{2}\right] + (1-q)\left|1\middle\rangle\middle\langle 1\right|
\end{aligned} \tag{25}
$$

Then, the success rate P is

$$P = \frac{1}{2}\left[\mathrm{Tr}\left(A_0\rho_0^A\right) + \mathrm{Tr}\big(A_1\rho_1^A\big)\right] \tag{26}$$

Since we know that $A_0 + A_1 = I$, we can substitute for A_1 in (26) and get $P = \frac{1}{2}(1 + \mathrm{Tr}\big[A_0(\rho_0^A - \rho_1^A)\big])$. Then by the spectral decomposition, we get

$$P = \frac{1}{2}(1 + \lambda_{max}) \tag{27}$$

where λ_{max} is the largest eigenvalue of the hermitian matrix $M = \rho_0^A - \rho_1^A$. Let

$$M = \begin{bmatrix} p-q & [\sqrt{p-p^2} + \sqrt{q-q^2}\exp(-i\kappa - i\phi)]\cos\frac{\alpha}{2} \\ [\sqrt{p-p^2} + \sqrt{q-q^2}\exp(i\kappa + i\phi)]\cos\frac{\alpha}{2} & q-p \end{bmatrix} \tag{28}$$

From here, notice that if the eigenvector associated with λ_{max} also orthogonalizes Bob's states then we have zero gap. Also, notice that we can do the following relabeling.

$$M = \begin{bmatrix} a & b^* \\ b & -a \end{bmatrix} \tag{29}$$

Solving for the eigenvalues, we get $\lambda = \pm\sqrt{a^2 + |b|^2}$. However, since we only want the maximum value, we get $\lambda_{max} = \lambda = \sqrt{a^2 + |b|^2}$, where the redefinition of λ is done out of convenience. Thus, we have the eigenvector

$$|\lambda\rangle = n\begin{bmatrix} 1 \\ \frac{b}{\sqrt{a^2 + |b|^2} + a} \end{bmatrix} = n\begin{bmatrix} 1 \\ \frac{b}{\lambda + a} \end{bmatrix} \tag{30}$$

where n is the normalizing constant given by

$$n = \frac{1}{\sqrt{1 + \left|\frac{b}{\lambda + a}\right|^2}} \tag{31}$$

Now, we have to see if this projector, $A_0 = |\lambda\rangle\langle\lambda|$, results in orthogonal states for Bob. We also have to check if A_1 results in orthogonal states for Bob. If we show that the possible post measurement states are orthogonal for both measurements then we have zero gap. First, let us rewrite the original states in (19) as

$$|\Psi_0\rangle = |0\rangle_A |\eta_0\rangle_B + |1\rangle_A |\eta_1\rangle_B \tag{32}$$

$$|\Psi_1\rangle = |0\rangle_A |\nu_0\rangle_B + |1\rangle_A |\nu_1\rangle_B \tag{33}$$

where

$$|\eta_0\rangle = \sqrt{p}\,|0\rangle \tag{34}$$

$$|\eta_1\rangle = \sqrt{1-p}\,\left|\hat{b}\right\rangle \tag{35}$$

$$|\nu_0\rangle = \sqrt{q}\,|1\rangle \tag{36}$$

$$|\nu_1\rangle = \sqrt{1-q}\,\left|-\hat{b}\right\rangle \tag{37}$$

and $\left|\hat{b}\right\rangle = \cos\frac{\alpha}{2}|0\rangle + \exp(i\phi)\sin\frac{\alpha}{2}|1\rangle$, $\left|-\hat{b}\right\rangle = \exp(i\kappa)(\sin\frac{\alpha}{2}|0\rangle - \exp(i\phi)\cos\frac{\alpha}{2}|1\rangle)$. Then, the post measurement states from projector A_0 are

$$(|\lambda\rangle\langle\lambda| \otimes I)|\Psi_0\rangle = |\lambda\rangle \otimes (\langle\lambda|0\rangle|\eta_0\rangle + \langle\lambda|1\rangle|\eta_1\rangle) = |\lambda\rangle \otimes |\eta_0'\rangle \tag{38}$$

$$(|\lambda\rangle\langle\lambda| \otimes I)|\Psi_1\rangle = |\lambda\rangle \otimes (\langle\lambda|0\rangle|\nu_0\rangle + \langle\lambda|1\rangle|\nu_1\rangle) = |\lambda\rangle \otimes |\nu_0'\rangle \tag{39}$$

Thus, Bob is left to distinguish the states

$$|\eta_0'\rangle = \langle\lambda|0\rangle\,|\eta_0\rangle + \langle\lambda|1\rangle\,|\eta_1\rangle = n(\sqrt{p}\,|0\rangle + \sqrt{1-p}\frac{b^*}{\lambda+a}\left|\hat{b}\right\rangle) \tag{40}$$

$$|\nu_0'\rangle = \langle\lambda|0\rangle\,|\nu_0\rangle + \langle\lambda|1\rangle\,|\nu_1\rangle = n(\sqrt{q}\,|1\rangle + \sqrt{1-q}\frac{b^*}{\lambda+a}\left|-\hat{b}\right\rangle) \tag{41}$$

Taking the inner product $\langle\nu_0'|\eta_0'\rangle$ and setting it to zero

$$\sqrt{p}\sqrt{1-q}\left(\frac{b}{\lambda+a}\right)\left\langle-\hat{b}\middle|0\right\rangle + \sqrt{q}\sqrt{1-p}\frac{b^*}{\lambda+a}\left\langle 1\middle|\hat{b}\right\rangle = 0 \tag{42}$$

$$\sqrt{p}\sqrt{1-q}\left(\frac{b}{\lambda+a}\right)\exp(-i\kappa)\sin\frac{\alpha}{2} + \sqrt{q}\sqrt{1-p}\frac{b^*}{\lambda+a}\exp(i\phi)\sin\frac{\alpha}{2} = 0$$

and substituting for b and b^* in the numerator we get

$$\sqrt{p}\sqrt{1-q}\left(\frac{\left[\sqrt{p-p^2}\exp(-i\kappa) + \sqrt{q-q^2}\exp(i\phi)\right]\cos\frac{\alpha}{2}}{\lambda+a}\right)\sin\frac{\alpha}{2} \tag{43}$$

$$+\sqrt{q}\sqrt{1-p}\left(\frac{\left[\sqrt{p-p^2}\exp(i\phi) + \sqrt{q-q^2}\exp(-i\kappa)\right]\cos\frac{\alpha}{2}}{\lambda+a}\right)\sin\frac{\alpha}{2} = 0$$

which we can simplify to

$$[\sqrt{p}\sqrt{1-q}\left(\sqrt{p-p^2}\exp(-i\kappa) + \sqrt{q-q^2}\exp(i\phi)\right) \tag{44}$$

$$+\sqrt{q}\sqrt{1-p}\left(\sqrt{p-p^2}\exp(i\phi) + \sqrt{q-q^2}\exp(-i\kappa)\right)]\frac{\cos\frac{\alpha}{2}\sin\frac{\alpha}{2}}{\lambda+a} = 0$$

Then, it is clear that when $\cos\frac{\alpha}{2}$ or $\sin\frac{\alpha}{2}$ is zero the equation is true.

Next, let us study the term inside the square braces in (44) and set it equal to zero.

$$\sqrt{p}\sqrt{1-q}\left(\sqrt{p-p^2}\exp(-i\kappa) + \sqrt{q-q^2}\exp(i\phi)\right) \tag{45}$$

$$+\sqrt{q}\sqrt{1-p}\left(\sqrt{p-p^2}\exp(i\phi) + \sqrt{q-q^2}\exp(-i\kappa)\right) = 0$$

We then solve for the exponentials and take the magnitude of both sides.

$$\exp(i\phi + i\kappa) = \frac{-\left(\sqrt{p(1-q)(p-p^2)} + \sqrt{q(1-p)(q-q^2)}\right)}{\sqrt{p(1-q)(q-q^2)} + \sqrt{q(1-p)(p-p^2)}} \tag{46}$$

$$\sqrt{p(1-q)(q-q^2)} + \sqrt{q(1-p)(p-p^2)} = \sqrt{p(1-q)(p-p^2)} + \sqrt{q(1-p)(q-q^2)}$$

After rearranging, we can factor out common terms.

$$\begin{aligned} \sqrt{p(1-q)}\left(\sqrt{q-q^2} - \sqrt{p-p^2}\right) =& \sqrt{q(1-p)}\left(\sqrt{q-q^2} - \sqrt{p-p^2}\right) \\ \left(\sqrt{p(1-q)} - \sqrt{q(1-p)}\right)&\left(\sqrt{q-q^2} - \sqrt{p-p^2}\right) = 0 \end{aligned} \tag{47}$$

Set the first term in parenthesis to zero and we get

$$\begin{aligned} \sqrt{p(1-q)} - \sqrt{q(1-p)} = 0 \\ p = q \end{aligned} \tag{48}$$

Thus, if $p = q$ and $\phi + \kappa = \pi$ we have zero gap . This is also when $\lambda + a = 0$ and Alice's reduced density matrices are equal.

Now, we need to examine the other term of (47).

$$\begin{aligned} \sqrt{q-q^2} - \sqrt{p-p^2} &= 0 \\ q - q^2 &= p - p^2 \\ q - p &= q^2 - p^2 \\ q - p &= (q+p)(q-p) \\ q &= 1 - p \end{aligned} \tag{49}$$

Thus, if $q = 1 - p$ and $\phi + \kappa = \pi$ we have zero gap. Now, we need to examine the other projector. The post measurement states for projector $A_1 = I - A_0$ is given by

$$|\Psi_0''\rangle = [(I - A_0) \otimes I] |\Psi_0\rangle \tag{50}$$

$$|\Psi_1''\rangle = [(I - A_0) \otimes I] |\Psi_1\rangle \tag{51}$$

Then

$$\begin{aligned} \langle \Psi_1'' | \Psi_0'' \rangle &= \langle \Psi_1 | (I - A_0)(I - A_0) \otimes I |\Psi_0\rangle \\ &= \langle \Psi_1 | (I - 2A_0 + A_0) \otimes I |\Psi_0\rangle \\ &= \langle \Psi_1 | (I - A_0) \otimes I |\Psi_0\rangle \\ &= - \langle \Psi_1 | A_0 \otimes I |\Psi_0\rangle \end{aligned} \tag{52}$$

Thus, all the previous calculations hold because $A_1 = |\lambda^\perp\rangle\langle\lambda^\perp|$, where $\langle\lambda|\lambda^\perp\rangle = 0$. Therefore, A_1 will also result in orthogonal vectors for Bob as long as theorem 1 is satisfied. This ends the proof of the theorem. □

Let us examine the states that satisfy theorem 1 and have no gap in more detail. When $\sin\frac{\alpha}{2} = 0$, we have product states. When $p = q$ and $\phi + \kappa = \pi$, Alice's reduced density matrices are equal. Thus, Alice can always use the projection matrices that will result in distinguishing orthogonal states for Bob. When $\cos\frac{\alpha}{2} = 0$, we have the states

$$\begin{aligned} |\Psi_0\rangle &= \sqrt{p}\,|00\rangle \pm \sqrt{1-p}\exp(i\phi)\,|11\rangle \\ |\Psi_1\rangle &= \sqrt{q}\,|01\rangle \pm \sqrt{1-q}\exp(i\kappa)\,|10\rangle \end{aligned} \tag{53}$$

When $q = 1 - p$ and $\phi + \kappa = \pi$, we have the states

$$\begin{aligned} |\Psi_0\rangle &= \sqrt{p}\,|00\rangle + \sqrt{1-p}\,\left|1\hat{b}\right\rangle \\ |\Psi_1\rangle &= \sqrt{1-p}\,|01\rangle + \sqrt{p}\,|1\rangle\left|-\hat{b}\right\rangle \end{aligned} \tag{54}$$

where $\left|\hat{b}\right\rangle = \cos\frac{\alpha}{2}\left|0\right\rangle + \exp(i\phi)\sin\frac{\alpha}{2}\left|1\right\rangle$ and $\left|-\hat{b}\right\rangle = \exp(i\kappa)(\sin\frac{\alpha}{2}\left|0\right\rangle - \exp(i\phi)\cos\frac{\alpha}{2}\left|1\right\rangle)$. These states are interesting because Alice's reduced density matrices do not have to be equal.

CHAPTER 4

QUANTIFYING THE GAP

The question now is how large can the gap get? We know intuitively that the gap is in the range of zero to $1/2$. First, we showed that the gap can be zero. Second, we know that best probability Alice can ever obtain is one and the worst probability that Alice can obtain is $1/2$ because Alice can always just randomly guess. To figure out the actual maximum gap, we need to calculate P and $P^{\perp}$. From (27), we know that

$$P = \frac{1}{2}(1 + \lambda_{max}) \tag{55}$$

Using the value we calculated for λ we get,

$$P = \frac{1}{2}\left(1 + \sqrt{(p-q)^2 + \cos^2\frac{\alpha}{2}[p - p^2 + q - q^2 + 2\sqrt{p-p^2}\sqrt{q-q^2}\cos(\kappa+\phi)]}\right) \tag{56}$$

Next, we need to calculate $P^{\perp}$. This calculation is a little tricky. For the form of the states that we are using, if Alice measures in the computational basis $|0\rangle$ and $|1\rangle$, Bob's possible post measurement states will be orthogonal vectors. Thus, Bob can perfectly distinguish. However, measuring in the computational basis might not be optimal. There might be another basis that allows Alice to distinguish at a higher probability and Bob is still left with distinguishing orthogonal vectors. Thus, we need to use arbitrary projectors for the calculation of $P^{\perp}$.

CHAPTER 4.1

FORM OF $P^{\perp}$

As we said in (32), we can always write our two initial states as

$$|\Psi_0\rangle = |0\rangle_A |\eta_0\rangle_B + |1\rangle_A |\eta_1\rangle_B$$
$$|\Psi_1\rangle = |0\rangle_A |\nu_0\rangle_B + |1\rangle_A |\nu_1\rangle_B$$

When Alice measures, she will be projecting onto the optimal basis $\{U|0\rangle, U|1\rangle\}$, where U is the unitary transformation for rotating the states. This measurement will leave Bob with the task of distinguishing orthogonal states. With these projectors, we get

$$P^{\perp} = \frac{1}{2}\left[\langle\Psi_0|\left(U|0\rangle\langle 0|U^{\dagger}\otimes P_{B0}\right)|\Psi_0\rangle + \langle\Psi_1|\left(U|1\rangle\langle 1|U^{\dagger}\otimes P_{B1}\right)|\Psi_1\rangle\right] \quad (57)$$

where P_{B0} and P_{B1} are Bob's measurements. We can simplify this equation by applying the unitary matrices to the states. The physical interpretation of this is that Alice rotates the state first and then measures. Thus, we get

$$P^{\perp} = \frac{1}{2}\left[\langle\Psi_0'|\left(|0\rangle\langle 0|\otimes P_{B0}\right)|\Psi_0'\rangle + \langle\Psi_1'|\left(|1\rangle\langle 1|\otimes P_{B1}\right)|\Psi_1'\rangle\right] \quad (58)$$

Notice that

$$|\Psi_0'\rangle = |0\eta_0'\rangle + |1\eta_1'\rangle$$
$$|\Psi_1'\rangle = |0\nu_0'\rangle + |1\nu_1'\rangle$$

These are the states after the unitary rotation by Alice. Also, $|\eta_i'\rangle$ and $|\nu_i'\rangle$ are Bob's possible post measurement states. Furthermore, $|\eta_i'\rangle$ and $|\nu_i'\rangle$ are orthogonal as we require. Thus, we get

$$P^{\perp} = \frac{1}{2}\left(\langle\eta_0'|\,P_{B0}\,|\eta_0'\rangle + \langle\nu_1'|\,P_{B1}\,|\nu_1'\rangle\right) \tag{59}$$

Next, we maximize the success by letting $P_{B0} = \frac{|\eta_0'\rangle\langle\eta_0'|}{\langle\eta_0'|\eta_0'\rangle}$ and $P_{B1} = \frac{|\nu_1'\rangle\langle\nu_1'|}{\langle\nu_1'|\nu_1'\rangle}$, where $\langle\eta_0'|\eta_0'\rangle$ and $\langle\nu_1'|\nu_1'\rangle$ are for normalization. Simplifying eq. 59 further we get

$$P^{\perp} = \frac{1}{2}\left(\langle\eta_0'|\eta_0'\rangle + \langle\nu_1'|\nu_1'\rangle\right) \tag{60}$$

This probability is not always 1 because $|\eta_0'\rangle$ and $|\nu_1'\rangle$ are not necessarily normalized. Notice that we get the same success probability even when we substitute the identity matrix for both P_{B0} and P_{B1} in (59). Thus, the success probability depends entirely on Alice's ability to distinguish the states correctly. This makes sense because Bob can perfectly distinguish between orthogonal states.

CHAPTER 4.2
CALCULATING $|\eta_i'\rangle$ and $|\nu_i'\rangle$

To calculate (60), we have to calculate the post measurement states using Alice's projectors.

$$\left(U\,|0\rangle\langle 0|\,U^{\dagger}\otimes\mathbb{I}\right)|\Psi_0\rangle = U\,|0\rangle\otimes|\eta_0'\rangle \tag{61}$$

$$\left(U\,|1\rangle\langle 1|\,U^{\dagger}\otimes\mathbb{I}\right)|\Psi_1\rangle = U\,|1\rangle\otimes|\nu_1'\rangle \tag{62}$$

This simplifies to

$$U\,|0\rangle\otimes|\eta_0'\rangle = U\,|0\rangle\otimes\left[\left(\langle 0|\,U^{\dagger}\,|0\rangle\right)|\eta_0\rangle + \left(\langle 0|\,U^{\dagger}\,|1\rangle\right)|\eta_1\rangle\right] \tag{63}$$

$$U\,|1\rangle\otimes|\nu_1'\rangle = U\,|1\rangle\otimes\left[\left(\langle 1|\,U^{\dagger}\,|0\rangle\right)|\nu_0\rangle + \left(\langle 1|\,U^{\dagger}\,|1\rangle\right)|\nu_1\rangle\right] \tag{64}$$

Thus,

$$|\eta_0'\rangle = \big(\langle 0| U^\dagger |0\rangle\big) |\eta_0\rangle + \big(\langle 0| U^\dagger |1\rangle\big) |\eta_1\rangle \tag{65}$$

$$|\nu_1'\rangle = \big(\langle 1| U^\dagger |0\rangle\big) |\nu_0\rangle + \big(\langle 1| U^\dagger |1\rangle\big) |\nu_1\rangle \tag{66}$$

Additionally, we use the general unitary given by [NC10]. This is different from the unitary matrix used by Walgate et al. in [WSHV00].

$$U = \exp(i\gamma) \begin{bmatrix} \cos\theta \exp(i(-\delta-\omega)) & -\sin\theta \exp(i(-\delta+\omega)) \\ \sin\theta \exp(i(\delta-\omega)) & \cos\theta \exp(i(\delta+\omega)) \end{bmatrix} \tag{67}$$

Thus, we can simplify (65) to

$$\begin{aligned} |\eta_0'\rangle &= \exp(-i\gamma)(\cos\theta \exp[i(\delta+\omega)] |\eta_0\rangle + \sin\theta \exp[i(-\delta+\omega)] |\eta_1\rangle) \\ |\nu_1'\rangle &= \exp(-i\gamma)(-\sin\theta \exp[i(\delta-\omega)] |\nu_0\rangle + \cos\theta \exp[-i(\delta+\omega)] |\nu_1\rangle) \end{aligned} \tag{68}$$

Now we can calculate $P^{\perp}$. From the original states we have

$$\begin{aligned} |\eta_0\rangle &= \sqrt{p}\, |0\rangle \\ |\eta_1\rangle &= \sqrt{1-p}\, \left|\hat{b}\right\rangle \\ |\nu_0\rangle &= \sqrt{q}\, |1\rangle \\ |\nu_1\rangle &= \sqrt{1-q}\, \left|-\hat{b}\right\rangle \end{aligned} \tag{69}$$

where $\left|\hat{b}\right\rangle = \cos\frac{\alpha}{2} |0\rangle + \exp(i\phi) \sin\frac{\alpha}{2} |1\rangle$, $\left|-\hat{b}\right\rangle = \exp(i\kappa)(\sin\frac{\alpha}{2} |0\rangle - \exp(i\phi) \cos\frac{\alpha}{2} |1\rangle)$.

Substituting (69) into (68), we get

$$
\begin{aligned}
\langle \eta_0' | \eta_0' \rangle &= \cos^2\theta \langle \eta_0 | \eta_0 \rangle + \sin\theta\cos\theta \exp(i2\delta) \langle \eta_1 | \eta_0 \rangle \\
&\quad + \sin\theta\cos\theta \exp(-i2\delta) \langle \eta_0 | \eta_1 \rangle + \sin^2\theta \langle \eta_1 | \eta_1 \rangle \\
&= \sin^2\theta + p\cos 2\theta + \sqrt{p - p^2} \sin 2\theta \cos\frac{\alpha}{2} \cos 2\delta \\
\langle \nu_1' | \nu_1' \rangle &= \sin^2\theta \langle \nu_0 | \nu_0 \rangle - \sin\theta\cos\theta \exp(i2\delta) \langle \nu_1 | \nu_0 \rangle \\
&\quad - \sin\theta\cos\theta \exp(-i2\delta) \langle \nu_0 | \nu_1 \rangle + \cos^2\theta \langle \nu_1 | \nu_1 \rangle \\
&= \cos^2\theta - q\cos 2\theta + \sqrt{q - q^2} \sin 2\theta \cos\frac{\alpha}{2} \cos(2\delta - \kappa - \phi)
\end{aligned} \tag{70}
$$

Thus, we can substitute (70) into (60) to get the final form for $P^{\perp}$.

$$
\begin{aligned}
P^{\perp} = \frac{1}{2}[1 + (p - q)\cos 2\theta + \sqrt{p - p^2} \sin 2\theta \cos\frac{\alpha}{2} \cos 2\delta \\
+ \sqrt{q - q^2} \sin 2\theta \cos\frac{\alpha}{2} \cos(2\delta - \kappa - \phi)]
\end{aligned} \tag{71}
$$

CHAPTER 4.3
RESTRICTIONS ON THE UNITARY

Before we can go any further we need to restrict the unitary. The unitary needs to cause Bob's possible post measurement states to be orthogonal. As Walgate et al. [WSHV00] showed, the unitary matrix acts on a square 2x2 matrix.

$$
U^* M U^T = U^* \begin{bmatrix} x & y \\ z & t \end{bmatrix} U^T \tag{72}
$$

where x, y, z and t are just relabeling done to ease calculation. In our case,

$$\begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} \langle\nu_0|\eta_0\rangle & \langle\nu_1|\eta_0\rangle \\ \langle\nu_0|\eta_1\rangle & \langle\nu_1|\eta_1\rangle \end{bmatrix} \tag{73}$$

This matrix is the result of writing all of Bob's kets in matrix form with respect to the same basis. The trace of this matrix is zero. Thus, the goal of the unitary is to make the terms along the diagonal equal, which forces them to be zero.

The equalizing condition on our unitary matrix is then given by

$$(x-t)\cos 2\theta - \sin 2\theta \left(y \exp(i2\omega) + z \exp(-i2\omega)\right) = 0 \tag{74}$$

The equalizing condition given in [WSHV00] is

$$(x-t)\cos 2\theta + \sin 2\theta (y \exp(-i\omega) + z \exp(i\omega)) = 0 \tag{75}$$

Our equation (74) is different from (75) because we used the most general unitary. We must make sure that we satisfy (74). Directly substituting for x, y, z, t in (74), the diagonalizing condition is

$$-\sin 2\theta \sin\frac{\alpha}{2}\left(\sqrt{p(1-q)}\exp(-i\kappa)\exp(i2\omega) + \sqrt{q(1-p)}\exp(i\phi)\exp(-i2\omega)\right) = 0 \tag{76}$$

We can use this equation to put restrictions that will let us calculate the maximum gap. Recall that zero gap occurs when (i) $\cos\frac{\alpha}{2} = 0$, (ii) $\sin\frac{\alpha}{2} = 0$, (iii) $p = q$ and $\kappa + \phi = \pi$, or (iv) $q = 1 - p$ and $\kappa + \phi = \pi$.

Let us assume that the gap is not zero. Thus, we either have (v) $\cos\frac{\alpha}{2} \neq 0$, $\sin\frac{\alpha}{2} \neq 0$, and $\phi + \kappa \neq \pi$ or (vi) $\cos\frac{\alpha}{2} \neq 0$, $\sin\frac{\alpha}{2} \neq 0$, $p \neq q$, and $q \neq 1 - p$. Looking at (v) first, we can see that this alone will not give us a maximum gap of $1/2$. The term in parenthesis in (76) can always be made zero by ω unless $p = q$. We can see this by setting it to zero and explicitly solving for ω.

$$\exp(i4\omega) = -\frac{\sqrt{q(1-p)}}{\sqrt{p(1-q)}}\exp(i(\phi+\kappa)) \tag{77}$$

By taking the magnitude of both sides we get

$$\sqrt{p(1-q)} = \sqrt{q(1-p)} \tag{78}$$

which is only true if $p = q$. For (vi), with $p \neq q$, we force $\sin 2\theta = 0$ in (76) because $\sin\frac{\alpha}{2} \neq 0$. Therefore, (71) reduces to

$$P^{\perp} = \frac{1}{2}[1 + (p-q)] \tag{79}$$

and we still have the original equation for P given by (56), which we restate here for convenience.

$$P = \frac{1}{2}\left(1 + \sqrt{(p-q)^2 + \cos^2\frac{\alpha}{2}[p - p^2 + q - q^2 + 2\sqrt{p-p^2}\sqrt{q-q^2}\cos(\kappa+\phi)]}\right)$$

Since (79) only reaches its minimum when $p = q$, this means that the maximum gap can never reach $1/2$. However, we can arbitrarily approach $1/2$ by making p and q approach $1/2$ from opposite directions and α approach zero. More concretely, we can write $p = 1/2 + \epsilon$, $q = 1/2 - \epsilon$, $\alpha/2 = \epsilon$, and $\kappa + \phi = 0$ and substitute into (79) and (56) to get

$$P^{\perp} = \frac{1}{2}[1 + 2\epsilon] \tag{80}$$

$$P = \frac{1}{2}(1 + ((2\epsilon)^2 + \cos^2(\epsilon)[1 - (\frac{1}{2}+\epsilon)^2 - (\frac{1}{2}-\epsilon)^2 + 2\sqrt{(\frac{1}{2}+\epsilon) - (\frac{1}{2}+\epsilon)^2}\sqrt{(\frac{1}{2}-\epsilon) - (\frac{1}{2}-\epsilon)^2}])^{1/2}) \tag{81}$$

Then, when we take the limit of P and $P^{\perp}$ as ϵ goes to zero, we get

$$P^{\perp} = \frac{1}{2} \tag{82}$$

$$P = 1 \tag{83}$$

Thus, the gap can arbitrarily approach $1/2$, but never reach it. This discontinuity is interesting because the gap suddenly vanishes as it's about to reach the maximum.

CHAPTER 5
CONCLUSIONS

We examined bipartite quantum state discrimination in the setting of one-way LOCC from Alice to Bob. We looked at two figures of merit (i) the optimal probability P that Alice determines the state, and (ii) the optimal probability $P^{\perp}$ that Alice determines the state with the additional constraint that Bob is able to perfectly distinguish. In general, the two quantities are not equal. Alice has to sacrifice some of her probability of distinguishing correctly when she helps Bob distinguish perfectly.

In addition, we were able to prove the minimum and maximum probability trade-off. In the best case scenario, Alice can just use the projectors that will provide the optimal probability of her distinguishing the state correctly. These same measurements will help Bob distinguish perfectly. In the worst case scenario, Alice's ability to distinguish the state decreases by an amount that can arbitrarily approach $1/2$. The gap can never reach $1/2$ because the gap does a discontinuous jump to zero.

For future work, we can look to extend the results to multidimensions and multipartite systems. Also, we can analyze why the gap is discontinuous. It is not immediately clear why the gap suddenly drops to zero. Another extension of this work could be to look at the amount of entanglement preserved when trying to determine the state.

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