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CLASSIFICATION OF EIGENVALUES OF OCTONIONIC HERMITIAN MATRICES

Kalpa Madhawa Thudewaththage
Southern Illinois University Carbondale, kmthude@gmail.com

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by

Kalpa Madhawa Thudewaththage

B.S., University of Peradeniya, Sri Lanka, 2013

M.S., Southern Illinois University Carbondale, 2017

A Dissertation
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DISSERTATION APPROVAL

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Approved by:

Dr. Jerzy Kocik, Ph.D., Chair

Dr. John McSorley, Ph.D.

Dr. Jianhong Xu, Ph.D.

Dr. Bhaskar Batacharya, Ph.D.

Dr. Thushari Jayasekara, Ph.D.

Graduate School
Southern Illinois University Carbondale
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TITLE: CLASSIFICATION OF EIGENVALUES OF OCTONIONIC HERMITIAN MATRICES

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There are four normed division algebras over $\mathbb{R}$, namely real numbers, complex numbers, quaternions, and octonions. Lack of commutativity and associativity make it difficult to investigate algebraic and geometric properties of octonions. Eigenvalue problem of octonionic Hermitian matrices is one of the interesting studies where we can see this difficulty of extending the basic properties from complex Hermitian matrices to octonionic Hermitian matrices. This includes the notion of orthogonality and decomposition of a Hermitian matrix using its eigenvalues and eigenvectors.

Liping Huang and Wasin So derived explicit formulas for computing the roots of quaternionic quadratic equations. We extend their work to octonionic case and solve octonionic left quadratic equation $x^2 + bx + c = 0$, where a, b are octonions. We represent left spectrum of 2×2 octonionic Hermitian matrix using the solutions to corresponding octonionic left quadratic equation and identify the family of matrices which admit non-real left eigenvalues. For 3×3 case we review previous work by Tevian Dray and Corinne Manogue of real eigenvalue problem and study characteristic equations which admit non-real roots correspond to non-real left eigenvalues. Finally, we discuss the right spectrum using the associator method, and provide examples using "pyoctonion" python library. Interesting applications and open problems for future studies in this literature are also included.

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Finally, I would like to thank my family and friends for all their love and encouragement and my parents who raised me with a love and supported me in all my pursuits.

DEDICATION

To my loving family and friends...

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OVERVIEW

Finding the eigenvalue and eigenvectors of a given matrix is one of the basic techniques in linear algebra. The eigenvalue problem is usually formulated over a field, typically either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. Over the years, Many mathematician work on the generalization of eigenvalue problem to the other normed division algebras, namely quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$.

Right eigenvalues of quaternionic matrices are well studied in literature [11],[16],[17], [20],[23], [24], [29] and [32]. In 1985, R. Wood [33] used a topological method to proved that $n \times n$ quaternionic matrix has at least one left eigenvlaue. However, Wood's proof is existential and provided no algorithm for computing left eigenvalues. At the end of his paper, Wood suggested an algebraic approach to find left eigenvalues. He demonstrated that the left eigenvalues of a 2×2 quaternionic matrix can be found by solving a quaternionic quadratic equation, but he did not actually solve the quadratic equation. Later, L. Huang and W. So [23] explained how to solve quadratic formulas for quaternions and used it to fully classify left eigenvalues of a 2×2 quaternionic matrix [24]. Also reduced the existence of a left eigenvalue of a 3×3 quaternionic matrix to the existence of a solution of a generalized quaternionic polynomial of degree 3, which is guaranteed by the work of S. Eilenberg and I. Niven [17], The Fundamental theorem of algebra for quaternions.

Real eigenvalue problem for octonionic Hermitian matrices were first discussed by H. Goldstine and L. P. Horwitz [13]. In their paper, They showed 3×3 octonionic Hermitian matrices admit 24 real eigenvalues, corresponding to eigenvectors which are independent over $\mathbb{R}$. Later, O. V. Ogievetsky [27] explained about characteristic equation for 3×3 matrices over the octonions. He explained octonionic eigenvalues do not satisfy the general characteristic equation which satisfied by the complex eigenvalues. He also proved that real eigenvalues of 3×3 octonionic matrix generically come in 6 sets of multiplicity 4 rather than the expected 3 sets of multiplicity 8.

Tevian Dray and Corinne Manogue discussed eigenvalue problem for 2×2 and 3×3 octonionic Hermitian matrices in [5],[2] and [6]. They gave the general solution for real eigenvalue problem and show there are also solutions with non-real eigenvalues. In 2×2 and 3×3 cases, they discussed real, left and right eigenvalue problems and generalized the notion of orthogonality of eigenvectors corresponding to real eigenvalues. Furthermore, they explained about decomposition of octonionic Hermitian for above cases and also showed that in the 3×3 case Gram-Schmidt orthogonalization procedure no longer works nevertheless able to find orthonormal eigenvectors with repeated eigenvalues. They also indroduced Mathematica package for finding octonionic eigenvectors [4]. Susumu Okubo discussed eigenvalue problem for symmetric 3×3 octonionic matrix [35] and orthonormal eigenbases [3] which also motivated our studies in this literature.

Even though right eigenvalue problem for quaternionic Hermitian matrices always gives real eigenvalues. There are quaternionic Hermitian matrices which admits left eigenvalues non-real. L. Huang and W. So's work [24] can be used to figure out such a family of quaternionic Hermitian matrices. We study their work and thought about how to extend their work to find a family of 2×2 octonionic Hermitian matrices which admit non-real eigenvalues. We finally able to do it by representing the left spectrum of 2×2 octonionic Hermitian matrix using the solutions to corresponding octonionic left quadratic equation. This allow us to give full classification of eigenvalues of 2×2 octonionic Hermitian matrices. We also discuss interesting properties of eigenvectors corresponding to non-real eigenvalues. For solve octonionic left quadratice equations, we introduce python library [26] called "pyoctonion", which is very usefull to do calculations in octonions.

For 3×3 octonionic Hermitian matrices, Tevian Dray and Corinne Manogue[6] explained the method of finding real eigenvalues by solving special characteristic equation. However, there is no known method for classify non-real eigenvalues. Study's determinant can be used to obtain a characteristic equation which gives the left eigenvalues of a octonionic Hermitian matrix. Due to difficulty of solve some version of these characteristic equations, it is not possible in general to determine all the non-real eigenvalues of a given Hermitian matrix. However, our pyoctonion library can be used to solve selected characteristic equations computationally and find the non-real eigenvalues. This library also can use to provide counter examples that decomposition of 3×3 octonionic Herimtian matrix is not possible with its non-real eigenvalues. It is still remain as a open question that how to prove it analytically.

INTRODUCTION

0.1 BACKGROUND AND HISTORY

0.1.1 Algebra of quaternions

Quaternions arose historically from Sir William Rowan Hamilton's attempts in the midnineteenth century[1] to generalize complex numbers in some way that would applicable to three-dimensional space [18],[19],[9] and [6]. The quaternions are denoted by $\mathbb{H}$; the H is for "Hamilton", they are spanned by the identity element 1 and three imaginary units i, j and k. Quaternion q can be represented as four real numbers (q_1, q_2, q_3, q_4), usually written

$$q = q_1 + q_2 i + q_3 j + q_4 k \tag{0.1.1}$$

where $i^2 = j^2 = k^2 = ijk = -1$ and the multiplication table is cyclic,

$$ij = k = -ji,$$

$$jk = i = -kj,$$

$$ki = j = -ik$$

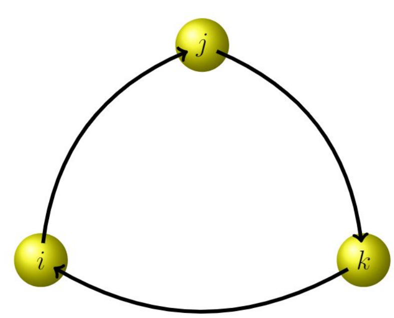

Figure 0.1.1. The quaternionic multiplication table

[1]After struggling unsuccessfully to construct an algebra in three dimensions. On 16 October 1843, as Hamilton was walking along a canal in Dublin, He realized how to construct an algebra in four dimensions.

notice that these units anticommute. Figure 0.1.1 shows multiplying two of these quaternionic units in the direction of the arrow yields the third; going against the arrow contributes an additional minus sign. Since equation 0.1.1 can be written in the form

$$q = (q_1 + q_2 i) + (q_3 + q_4 i)j \tag{0.1.2}$$

we see that a quaternion can be viewed as a pair of complex numbers, we can write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ in direct analogy to the construction of $\mathbb{C}$ from $\mathbb{R}$. The quaternionic conjugate $\bar{q}$ of a quaternion q is defined via the(real) linear map which reverses the sign of each imaginary unit, so that

$$\bar{q} = q_1 - q_2 i - q_3 j - q_4 k \tag{0.1.3}$$

if q is given by 0.1.1 conjugation leads directly to the norm of quaternion $|q|$, defined by

$$|q|^2 = q\bar{q} = q_1^2 + q_2^2 + q_3^2 + q_4^2. \tag{0.1.4}$$

the only quaternion with norm zero is zero and every nonzero quaternion has a unique inverse, namely

$$q^{-1} = \frac{\bar{q}}{|q|^2} \tag{0.1.5}$$

quaternionic conjugation satisfies the identity $\overline{pq} = \bar{q}\bar{p}$ from which it follows that the norm satisfies $|pq| = |p||q|$. Squaring both sides and expanding the result in terms of components yields the $4-$squares identity,

$$\begin{aligned}
&(p_1q_1 - p_2q_2 - p_3q_3 - p_4q_4)^2 + (p_2q_1 + p_1q_2 - p_4q_3 + p_3q_4)^2 \\
&+ (p_3q_1 + p_4q_2 + p_1q_3 - p_2q_4)^2 + (p_4q_1 - p_3q_2 + p_2q_3 + p_1q_4)^2 \\
&= (p_1^2 + p_2^2 + p_3^2 + p_4^4)(q_1^2 + q_2^2 + q_3^2 + q_4^2)
\end{aligned} \tag{0.1.6}$$

this identity can be used to see that the quaternions form a normed division algebra, that is, not only are there inverses, but there are no zero divisors which mean if a product is zero, one of the factors must be zero. It is important to realize that $\pm i, \pm j$, and $\pm k$ are not the only quaternionic square roots of -1, any imaginary quaternion squares to a negative number, so it is only necessary to choose its norm to be one in order to get a square root of -1. The imaginary quaternions of norm one form a $2-$dimensional sphere($q_1 = 0$); in the above notation, this is the set of points

$$q_2^2 + q_3^2 + q_4^2 = 1 \tag{0.1.7}$$

any such unit imaginary quaternion u can be used to construct a complex subalgebra of $\mathbb{H}$, which we will also denote by $\mathbb{C}$, namely

$$\mathbb{C} = a + bu$$

with $a, b \in \mathbb{R}$. Furthermore, we can use the Eular's identity to write

$$e^{u\theta} = \cos\theta + u\sin\theta$$

this means that any quaternion can be written in the form $q = re^{u\theta}$ where $r = |q|$ and u denotes the direction of the imaginary part of q.

0.1.2 Algebra of octonions

The day after Hamilton's discovery of quaternions, he sent a letter describing quaternions to his friend John T. Graves. Graves wrote back describing the octonions [21], which he called octaves[2]. In analogy to the previous construction of $\mathbb{C}$ and $\mathbb{H}$, an octonion x can

[2]However, Graves did not publish this work until 1845. Arthur Cayley published of his own discovery of octonions. For this reason octonions are also known as Cayley numbers.

be thought of as a pair of quaternions, so that

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\, l$$

we will denote i times l simply as il, and similarly with j and k. It is easy to see that l,il, jl, and kl all square to -1; there are now seven independent imaginary units, and we could write

$$x = x_0 + x_1 i + x_2 j + x_3 k + x_4 l + x_5 il + x_6 jl + x_7 kl \quad \text{where } x_i \in \mathbb{R},\ i = 1, 2, ..., 7. \quad (0.1.8)$$

which can be thought of as a point or vector in $\mathbb{R}^8$. The real part of x is just x_0; the imaginary part of x is everything else. We sometime use $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ to denote imaginary units $\{i, j, k, l, il, jl, kl\}$.
octonionic conjugation $\bar{x}$ of an octonion x is the (real) linear map which reverses the sign of the each imaginary basis units

$$\bar{x} = x_0 - \sum_{i=1}^{7} x_i e_i$$

Algebraically, we could define real part of x,

$$\Re(x) = \frac{(x + \bar{x})}{2} \quad (0.1.9)$$

imaginary part of x,

$$\Im(x) = \frac{(x - \bar{x})}{2} \quad (0.1.10)$$

the imaginary part is differs slightly from the standard usage of these terms for complex numbers, where $\Im(z)$ normally refers to a real number, the coefficient of i. This convention is not possible here, since the imaginary part has seven degrees of freedom, and can be thought of as a vector in $\mathbb{R}^7$. The full multiplication table is summarized in Figure 0.1.2 by means of the 7−point projective plane. Each point corresponds to an imaginary unit.

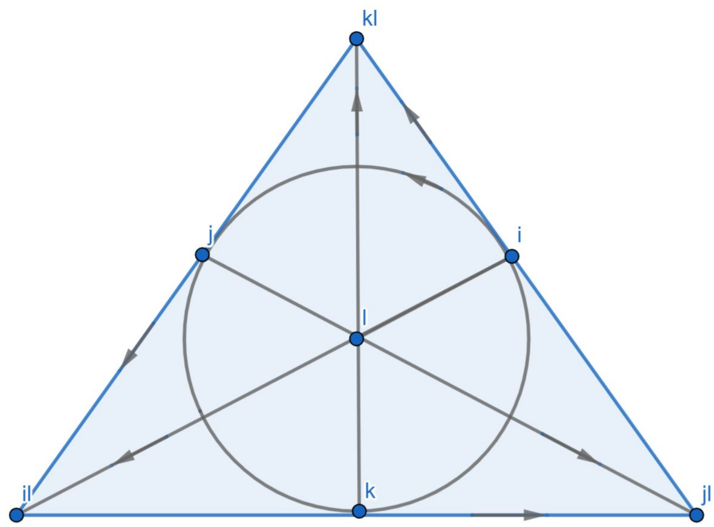

Figure 0.1.2. The octonionic multiplication table

Each line corresponds to a quaternionic triple, much like $\{i, j, k\}$, with the arrow giving the orientation [22]. For example,

$$\begin{aligned} l(kl) &= k \\ (kl)k &= l \\ (jl)l &= -j \end{aligned} \tag{0.1.11}$$

and each of these products anticommutes, reversing the order contributes a minus sign. if x is given by 0.1.8. Direct computation shows that

$$\overline{xy} = \bar{y}\bar{x} \tag{0.1.12}$$

The norm of an octonion $|x|$ is defined by

$$|x|^2 = x\bar{x} = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 \tag{0.1.13}$$

the only octonion with norm zero is zero, and every nonzero octonion has a unique inverse, namely

$$x^{-1} = \frac{\bar{x}}{|x|^2} \tag{0.1.14}$$

as with the other division algebras, the norm satisfies the identity

$$|xy| = |x||y| \tag{0.1.15}$$

writing out this expression in terms of components yields the 8−squares identity. We can see the octonions therefore also form a normed division algebra.

A remarkable property of the octonions is that they are not associative. For example,

$$\begin{aligned} (ij)(l) &= +(k)(l) = +kl \\ (i)(jl) &= (i)(jl) = -kl \end{aligned} \tag{0.1.16}$$

Now let

$$a = a_0 + a_1 i + a_2 j + a_3 k + a_4 l + a_5 il + a_6 jl + a_7 kl\,,$$

and

$$b = b_0 + b_1 i + b_2 j + b_3 k + b_4 l + b_5 il + b_6 jl + a_7 kl\,,$$

The inner product on $\mathbb{O}$ take similar to the usual dot product of vectors in $\mathbb{R}^8$, namely

$$\langle a, b\rangle = \sum_i a_i b_i$$

which can be rewriten as

$$\langle a, b\rangle = \frac{1}{2}(a\bar{b} + b\bar{a}) = \frac{1}{2}(\bar{b}a + \bar{a}b) \tag{0.1.17}$$

which satisfies the identities

$$\langle a, xb\rangle = \langle b, \bar{x}a\rangle \tag{0.1.18}$$

$$\langle ax, bx\rangle = \langle |x|^2 a, b\rangle \tag{0.1.19}$$

for any $a, b, x \in \mathbb{O}$.

The associator of three octonions is

$$[a, b, c] = (ab)c - a(bc) \tag{0.1.20}$$

which is totally antisymmetric in its arguments, has no real part, and changes sign if any one of its arguments is replaced by its octonionic conjugate. Although the associator does not vanish in general, the octonions do satisfy a weak from of associative known as alternativity, namely

$$[b, a, a] = [b, a, \bar{a}] = 0 \tag{0.1.21}$$

the underlying reason for alternativity is Artin's Theorem [31], which states that any two octonions lie in a quaternionic subalgebra of $\mathbb{O}$, so that any product containing two octonionic directions is associative. Also consider the associator identity

$$[a, b, c]d + a[b, c, d] = [ab, c, d] - [a, bc, d] + [a, b, cd] \tag{0.1.22}$$

for any $a, b, c, d \in \mathbb{O}$.

Another octonionic product having two properties, totally antisymmetric, and change sign if any one of their arguments is replaced by its conjugate is given by

$$\Phi(a,b,c) = \frac{1}{2}\Re\big([a,\bar{b}]c\big) \tag{0.1.23}$$

where $\Phi(a,b,c)$ is the associative 3−form [13],[14] and [15].

$$\Phi(a,b,c) = \Re(a \times b \times c) = \frac{1}{2}\big(a(\bar{b}c) - c(\bar{b}a)\big) \tag{0.1.24}$$

which reduces to the vector triple product when a, b, c are imaginary quaternions. Note that $a \times b \times c := \frac{1}{2}\Re\big(a(\bar{b}c) - c(\bar{b}a)\big)$ is the triple cross product, not the iterated cross product.

A consequence of alternativity is that the Moufang identities,

$$\begin{aligned} (aba)c &= a(b(ac)) \\ c(aba) &= ((ca)b)a \\ (ab)(ca) &= a(bc)a \end{aligned} \tag{0.1.25}$$

0.1.3 Eigenvalues and eigenvectors

Eigenvalues are often introduced in the context of linear algebra or matrix theory. Historically, however, they arose in the study of quadratic forms and differential equations. In the 18th century, Leonhard Euler studied the rotational motion of a rigid body, and discovered the importance of the principal axes. Joseph-Louis Lagrange realized that the principal axes are the eigenvectors of the inertia matrix.

Definition 0.1.1. Let V and W be vector space over the same field $\mathbb{F}$.

A function $T : V \longrightarrow W$ is said to be a linear transformation if for any two vectors $v, u \in V$

and any scalar $c \in \mathbb{F}$, the following two conditions are satisfied:

$$\begin{aligned} T(u+v) &= T(u) + T(v) \quad \text{(Additivity)} \\ T(cu) &= cT(u) \quad \text{(Homogeneity)} \end{aligned} \tag{0.1.26}$$

Definition 0.1.2. An eigenvector v of a linear transformation T is a nonzero vector that, when T is applied to it, does not change the direction. Applying T to the eigenvector only scales the eigenvector by the scalar value λ, called an eigenvalue. This condition can be written as the equation

$$T(v) = \lambda v \tag{0.1.27}$$

referred to as the eigenvalue equation or eigenequation. In general, λ may be any scalar.

Finding the eigenvalue and eigenvectors of a given matrix is one of the basic technique in linear algebra. The simple case is that of complex Hermitian matrices, generalizing the familiar case of real symmetric matrices. This simple case is nevertheless very important, for instance in quantum mechanics, where the fact that such matrices have real eigenvalues allows them to represent physical observable quantities. The eigenvalue problem is usually formulated over a field, typically either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. We consider here the generalization to the other normed division algebras, namely quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. We find that most of the basic properties are retained, provided they are reinterpreted to take into account the lack of commutativity of $\mathbb{H}$ and $\mathbb{O}$ and the lack of associativity of $\mathbb{O}$.
Due to non commutativity, there are two eigenvalue problems we must consider in $\mathbb{H}$ and $\mathbb{O}$. Following are the definitions for right and left eigenvalues for general $n \times n$ octonionic Hermitian matrix.

Definition 0.1.3. Given $A \in M_{n\times n}(\mathbb{O}), \lambda \in \mathbb{O}$ is called a right eigenvalues of A if

$$Ax = x\lambda$$

for some nonzero vector $x \in M_{n\times 1}(\mathbb{O})$. The set of distinct right eigenvalues is called the right spectrum of A, denoted $\sigma_r(A)$.

Definition 0.1.4. Given $A \in M_{n\times n}(\mathbb{O}), \lambda \in \mathbb{O}$ is called a left eigenvalues of A if

$$Ax = \lambda x$$

for some nonzero vector $x \in M_{n\times 1}(\mathbb{O})$. The set of distinct left eigenvalues is called the left spectrum of A, denoted $\sigma_l(A)$.

0.2 STATEMENT AND SUMMARY OF MAIN RESULTS

We will state here some of the main results of this study and the previous works which motivate our study of classification eigenvalues of octonionic Hermitian matrices.

Let us start previous studies of quaternionic eigenvalue problem. It is well known that right spectrum is always nonempty and well studied in literature [20],[29],[17],[23] and [11]. on the other hand, left spectrum are less known. R. Wood [33] used a topological method to confirm that the left eigenvalues always exists but provided no algorithm for computing left eigenvalues. Huang and So [24] solve quaternionic quadratic equation to find left eigenvalues of general 2×2 quaternionic matrix.

Even though right eigenvalue problem for quaternionic Hermitian matrices always gives real eigenvalues there are quaternionic Hermitian matrices which admits left eigenvalues non-real. We can use [24] to figure out such a family of quaternionic Hermitian matrices. We extend above work and study classification of left eigenvalues of octonionic Hermitian matrices. This is a generalization of quaternion case and also we give proof to solving special type of octonionic quadratic equation $ax^2 + bx + c = 0$ which we called as octonionic left quadratic equation [3]. We express left spectrum of 2×2 octonionic Hermitian matrix using the solutions to corresponding octonionic left quadratic equation and identify the family of matrices which admit non-real left eigenvalues

For 3×3 octonionic Hermitian matrices, Tevian Dray and Corinne Manogue[6] explain the method of finding real eigenvalues by solving special characteristic equation. However, there is no known method for classify non-real eigenvalues. We use Study's determinant and characteristic function to obtain left eigenvalues of octonionic Hermitian matrices. The roots of characteristic functions give left eigenvalues of such matrices. We discuss two different cases, the first one is called polynomial case when there exists some zero entry outside the diagonal of the octonionic Hermitian matrix and the most generalize case called rational case. Finally, discuss right eigenvalues of octonionic Hermitian matrices using associator identity.

For calculaion in octonions we use python library called "pyoctonion" which we introduce for this study. The pyoctonion library is very helpfull for solving octonionic quadratic left equation and finding eigenvalues which satisfy characteristic function. It also help us to provide ample examples of non-real eigenvalues of 3×3 case.

Now we will see some of the main results of this study.

Result 1: Left eigenvalues of a 2×2 octonionic Hermitian matrix using the roots of octonionic left quadratic equation:

Let

$$A = \begin{bmatrix} p & \bar{a} \\ a & m \end{bmatrix}$$

where $a \in \mathbb{O}$ and $p, m \in \mathbb{R}$.

[3]We called this form of quadratic equation as left octonion quadratic equation because we can consider $x^2 + xb + c = 0$ as a different case due to non-commutativity of octonions.

(i) if $a\bar{a} = |a|^2 = 0$,i.e $a = 0$, then $\sigma_l(A) = \{p, m\}$

(ii) if $a \neq 0$, then $\sigma_l(A) = \{p + \bar{a}t : t^2 + (\bar{a})^{-1}(p - m)t - (\bar{a})^{-1}a = 0\}$ i.e,

$$\sigma_l(A) = \{p + \bar{a}t : t^2 + \frac{a(p-m)t}{|a|^2} - \frac{a^2}{|a|^2} = 0\}$$

Result 2: We explicitly solve above octonionic left quadratic equation of the form $x^2 + bx + c = 0$ where $b, c \in \mathbb{O}$ by considering four different cases. Also we code a python program using pyoctonion library to get roots of any octonionic left quadratic equation. This result helps us to classify real and non-real left spectrum of 2×2 octonionic Hermitian matrices.

Result 3: Tevian Dray and Corrine Manogue [6] explain real eigenvalue problem for 3×3 octonionic Hermitian matrices. However, there is no well known method for obtaining eigenvalues which are non-real. We discuss characteristic functions having their roots as left eigenvalues for 3×3 octonionic Hermitian matrices. Finally, we study right eigenvalues problem using associator method and using "pyoctonion" library we provide some interesting examples of 3×3 octonionic Hermitian matrices with non-real eigenvalues.

0.3 ORGANIZATION

The organization of this dissertation will be as follows:

In Chapter 1, we recall the standard eigenvalue problem. First we start mentioning properties of eigenvalues and eigenvectors of complex Hermitian matrices such as eigenvalues are always real, eigenvectors corresponding to different eigenvalues are orthogonal and decompose of complex Hermitian matrix using their eigenvalues and orthonormal eigenbasis. Then we move on to the quaternionic eigenvalue problem. Due to lack of commutativity of quaternions, we must consider two eigenvalue problems called right and left eigenvalue problems. It turns out that right eigenvalues of quaternionic Hermitian matrices recover the properties enjoy by complex Hermitian matrices. However, some quaternion Hermitian matrices admits left eigenvalues are non-real. In octonionic case, both right and left eigenvalue problems may give non-real eigenvalues and hence it is interesting to classify eigenvalues of octonionic Hermitian matrices mainly in 2×2 and 3×3 cases.

Chapter 2 consists of discussion on method to solve octonionic left quadratic equation of the form $x^2 + bx + c = 0$ where $b, c \in \mathbb{O}$, which help to give full classification of left spectrum of 2×2 octonionic Hermitian matrices. We identify the family of octonionic Hermitian matrices which gives non-real left eigenvalues.

Chapter 3 will be primary devoted to recall Jordan product, Jordan matrices, and previous work of Tevian Dray and Corinne Manogue [6] of real eigenvalue problem of 3×3 octonionic Hermitian matrices. We use our "pyoctonionic" library to find associate $3-$form $\Phi(a, b, c)$ and other octonionic calculations which we need to obtain real eigenvalues. At the end of the chapter we discuss eigenvectors and decomposition.

We will discuss right spectrum and left spectrum of 3×3 octonionic Hermitian matrices using associator and characteristic function method respectively in chapter 4. Provide examples for certain cases and mention some applications and open problems for future studies in this literature.

CHAPTER 1

PRELIMINARIES

1.1 THE STANDARD EIGENVALUE PROBLEM

The eigenvalue problem as usually stated is to find solutions λ, v to the equation

$$Av = \lambda v \tag{1.1.1}$$

for a given square matrix A. The basic properties of the eigenvalue problem for $n \times n$ complex Hermitian matrices are well-understood.

Lemma 1.1.1. An $n \times n$ complex Hermitian matrix A has n real eigenvalue(counting multiplicity)

Proof. Let A, v, λ satisfy equation 1.1.1, with $A^\dagger = A$. Then

$$(Av)^\dagger v = v^\dagger A^\dagger v = v^\dagger Av = \lambda v^\dagger v \tag{1.1.2}$$

$$(Av)^\dagger v = (\lambda v)^\dagger v = \bar{\lambda} v^\dagger v \tag{1.1.3}$$

from equations 1.1.2 and 1.1.3, if $v \neq 0$ we have $v^\dagger v \neq 0$, which implies $\bar{\lambda} = \lambda$. Thus, A has real eigenvalues. □

Lemma 1.1.2. Eigenvectors of an $n \times n$ complex Hermitian matrix A corresponding to different eigenvalues are orthogonal.

Proof. For $m = 1, 2$, let v_m be an eigenvector of $A = A^\dagger$ with eigenvalue λ_m. By the previous lemma, $\lambda_m \in \mathbb{R}$. Then

$$(Av_1)^\dagger v_2 = v_1^\dagger A^\dagger v_2 = v_1^\dagger Av_2 = \lambda_2 v_1^\dagger v_2 \tag{1.1.4}$$

$$(Av_1)^\dagger v_2 = \bar{\lambda}_1 v_1^\dagger v_2 = \lambda_1 v_1^\dagger v_2 \tag{1.1.5}$$

i.e, if $\lambda_1 \neq \lambda_2$ then $v_1^\dagger v_2 = 0$. Thus, eigenvectors corresponding to different eigenvalues are orthogonal. □

Lemma 1.1.3. For any $n \times n$ complex Hermitian matrix A, there exists an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A.

Proof. If all eigenvalues have multiplicity one, the result follows from the lemma 1.1.2. The Gram-Schmidt orthogonalization process can be used on any eigenspace corresponding to an eigenvalue with multiplicity greater than one. □

These lemmas are equivalent to the standard result that a complex Hermitian matrix can always be diagonalized by a unitary transformation. It is important for what follows to realize that the form of the proofs given above relies on both the commutativity and the associativity of $\mathbb{C}$.

Next we will prove that any complex Hermitian matrix A admits a decomposition in terms of an orthonormal basis of eigenvectors.

Theorem 1.1.4. Let A be an $n \times n$ complex Hermitian matrix. Then A can be expanded as

$$A = \sum_{m=1}^{n} \lambda_m v_m v_m^\dagger \tag{1.1.6}$$

where $v_m; m = 1, ..., n$ is an orthonomal basis of eigenvectors corresponding to eigenvalues λ_m.

Proof. By lemma 1.1.3, there exist an orthonormal basis v_m of eigenvectors. Consider

$$\sum_{m=1}^{n} \lambda_m v_m v_m^\dagger v_k = \lambda_k v_k = A v_k \tag{1.1.7}$$

thus,

$$A = \sum_{m=1}^{n} \lambda_m v_m v_m^\dagger$$

□

1.2 THE QUATERNIONIC EIGENVALUE PROBLEM

1.2.1 The left eigenvalue problem

The eigenvalue problem 1.1.1 for Hermitian matrices A over $\mathbb{H}$ immediately yields that first unexpected result: The eigenvalues need not be real. An example is given by [5]

Example 1.2.1.

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 1 \\ k \end{bmatrix} = \begin{bmatrix} 1-j \\ k-i \end{bmatrix} = (1-j) \begin{bmatrix} 1 \\ k \end{bmatrix} \tag{1.2.1}$$

Furthermore, because of the lack of commutativity, multiples of eigenvectors are not necessarily eigenvectors. For instance, the vector

$$v = \begin{bmatrix} \sqrt{2} \\ 1-i \end{bmatrix} \tag{1.2.2}$$

is an eigenvector of the matrix

$$A = \begin{bmatrix} 0 & 1+i \\ 1-i & 0 \end{bmatrix} \tag{1.2.3}$$

with eigenvalue $\sqrt{2}$, but jv is not an eigenvector of A. This example illustrates an important point: We must distinguish between right and left multiplication. Since

$$A(vq) = (Av)q \tag{1.2.4}$$

by associativity, right multiples of eigenvectors are indeed eigenvectors. For example, vj is an eigenvector of the matrix A above, with the same eigenvalue $\sqrt{2}$. Hence, we must carefully distinguish between the left eigenvalues problem 1.1.1 and the right eigenvalue problem.

1.2.2 The right eigenvalue problem

$$Av = v\lambda \tag{1.2.5}$$

It turns out that the right eigenvalue problem satisfy the properties enjoyed by complex Hermitian matrices which we discussed above.

Lemma 1.2.2. The right eigenvalues of an $n \times n$ quaternionic Hermitian matrix ($A = A^\dagger$) are real.

Proof.

$$(Av)^\dagger v = (v^\dagger A^\dagger)v = v^\dagger(Av) = (v^\dagger v)\lambda \tag{1.2.6}$$

$$(Av)^\dagger v = (v\lambda)^\dagger v = (\bar{\lambda} v^\dagger)v = \bar{\lambda}(v^\dagger v) \tag{1.2.7}$$

since $v^\dagger v \in \mathbb{R}$ above two equations yields $\lambda = \bar{\lambda}$. □

Lemma 1.2.3. Right eigenvectors of an $n \times n$ quaternionic Hermitian matrix A corresponding to different eigenvalues are orthogonal.

Proof. For $m = 1, 2$, let v_m be an eigenvector of $A = A^\dagger$ with eigenvalue λ_m. By the previous lemma, $\lambda_m \in \mathbb{R}$. Then

$$(Av_1)^\dagger v_2 = (v_1^\dagger A^\dagger)v_2 = v_1^\dagger(Av_2) = v_1^\dagger(v_2\lambda_2) = \lambda_2(v_1^\dagger v_2) \tag{1.2.8}$$

$$(Av_1)^\dagger v_2 = (v_1\lambda_1)^\dagger v_2 = (\lambda_1 v_1^\dagger)v_2 = \lambda_1(v_1^\dagger v_2) \tag{1.2.9}$$

since $\lambda_m \in \mathbb{R}$, either $\lambda_1 = \lambda_2$ or $v_1^\dagger v_2 = 0$
Thus, eigenvectors corresponding to different eigenvalues are orthogonal. □

The right eigenvalue problem over $\mathbb{H}$ is therefore just a straightforward extension of the complex eigenvalue problem.

1.3 THE OCTONIONIC EIGENVALUE PROBLEM

The use of associativity in Lemma 1.2.2 leads one to suspect that even the right eigenvalues of octonionic Hermitian matrices admit right eigenvalues which are not real.

Example 1.3.1.

$$\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} j \\ l \end{bmatrix} = \begin{bmatrix} j \\ l \end{bmatrix} (1 - kl) \tag{1.3.1}$$

This leads both left and right eigenvalue problem of octonionic Hermitian matrices do not follow standard properties enjoyed by complex Hermitian matrices.

In his papers Tevian Dray [5] discuss about eigenvalue problem of 2×2 octonionic Hermitian matrices. We provide different approach in chapter 2 for identify the left spectrum of 2×2 octonionic Hermitian matrix and conclude that eigenvalues can be finite or infinite. In Finite case it has at most two distinct real eigenvalues. Finally, we classify the left spectrum of 2×2 octonionic Hermitian matrix and discuss interesting properties of matrices with non-real left spectrum.

CHAPTER 2

CLASSIFICATION OF LEFT EIGENVALUES OF 2×2 OCTONIONIC HERMITIAN MATRICES

2.1 OCTONIONIC LEFT QUADRATIC EQUATION

Liping Huang and Wasin So [23] derive explicit formulas for computing the roots of quaternionic quadratic equations. In this chapter we extend their work to octonionic case and solve octonionic left quadratic equation of the form $x^2 + bx + c = 0$, where a, b are octonions. Finally, we represent the left spectrum of 2×2 octonionic matrix using solutions to the corresponding octonionic left quadratic equation. To begin, consider following two lemmas.

Lemma 2.1.1. Let B, E, and D be real numbers such that

1. $D \neq 0$, and
2. $B < 0$ implies $B^2 < 4E$

Then the cubic equation

$$y^3 + 2By^2 + (B^2 - 4E)y - D^2 = 0$$

has exactly one positive solution.

Proof. Let

$$f(y) = y^3 + 2By^2 + (B^2 - 4E)y - D^2 = 0$$

Note that $f(0) = -D^2 < 0$ and $\lim_{y \to +\infty} f(y) = +\infty$. According to the intermediate value theorem: if f is a continuous function over an interval $[a, b]$, then f takes all values between $f(a)$ and $f(b)$. Since above cubic polynomial is a continuous function, its graph

must intersect the x-axis at some finite points greater than zero. Therefore the equation has at least one positive root. Now let us prove that f has only one positive root.
Suppose that f has three real roots, r_1, r_2 and r_3. Take $r_1 > 0$ be the positive root we found above. We must show that $r_2, r_3 < 0$. Then we have the result.
We know,

$$r_1.r_2.r_3 = D^2 > 0$$

this implies the product of r_2 and r_3 is positive. Therefore, r_2 and r_3 should be in same sign(both positive or negative). Let's assuming $r_2, r_3 > 0$,
if $B < 0$ implies $r_1 + r_2 + r_3 = -2B > 0$ but

$$r_1.r_2 + r_1.r_3 + r_2.r_3 = B^2 - 4E \tag{2.1.1}$$

2.1.1 should be positive. This is a contradiction due to the condition 2 of Lemma 2.1.1. Thus, we have only one positive solution to $f(y) = 0$. $\square$

Lemma 2.1.2. Let B, E, and D be real numbers such that

1. $E \geq 0$, and

2. $B < 0$ implies $B^2 < 4E$

Then the real system

$$N^2 - (B + T^2)N + E = 0, \tag{2.1.2}$$

$$T^3 + (B - 2N)T + D = 0, \tag{2.1.3}$$

has at most two solutions (T, N) satisfying $T \in \mathbb{R}$ and $N \geq 0$ as follows.

1. $T = 0, N = \frac{(B \pm \sqrt{B^2 - 4E})}{2}$ provided that $D = 0, B^2 \geq 4E$.

2. $T = \pm\sqrt{2\sqrt{E} - B}, N = \sqrt{E}$ provided that $D = 0, B^2 < 4E$.

3. $T = \pm\sqrt{z}, N = \frac{(T^3+BT+D)}{2T}$ provided that $D \neq 0$ and z is the unique positive root of the real polynomial $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2$.

Proof. (a) If $D = 0$ by 2.1.3 then we have $T = 0$ and 2.1.2 gives $N = \frac{(B\pm\sqrt{B^2-4E})}{2}$
(b) If $D = 0$ and $B^2 < 4E$, by 2.1.3 we have

$$T(T^2 - (2N - B)) = 0 \text{ Note that } 2N - B > 0$$

therefore,

$$T = \pm\sqrt{2N - B}$$

by 2.1.2 we have,

$$N^2 - (B + 2N - B)N + E = 0$$

$$N^2 = E \text{ implies } N = \sqrt{E}$$

Hence,

$$T = \pm\sqrt{2\sqrt{E} - B} \text{ and } N = \sqrt{E}$$

(c) If $D \neq 0$ from 2.1.3, $N = \frac{(T^3+BT+D)}{2T}$ plug this N value back to 2.1.2 we have

$$\Big(\frac{T^3 + BT + D}{2T}\Big)^2 - (B + T^2)\Big(\frac{T^3 + BT + D}{2T}\Big) + E = 0,$$

$$\Big(T^3 + BT + D\Big)^2 - 2T(B + T^2)\Big(T^3 + BT + D\Big) + 4T^2E = 0,$$

$$\Big(T^3 + BT + D\Big)^2 - 2T(B + T^2)\Big(T^3 + BT + D\Big) + 4T^2E = 0,$$

$$T^6 + 2BT^4 + (B^2 - 4E)T^2 - D^2 = 0,$$

Let $T = \pm\sqrt{z}$ then, by Lemma 2.1.1 we have z as unique positive root satisfies cubic equation $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2$. □

Theorem 2.1.3. The solutions of octonionic left quadratic equation $x^2 + bx + c = 0$ can be obtained by formulas according to the following cases.

Case 1: If $b, c \in \mathbb{R}$ and $b^2 \geq 4c$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Case 2: If $b, c \in \mathbb{R}$ and $b^2 < 4c$, then

$$x = \frac{1}{2}(-b + \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} + \delta\mathbf{l} + p\mathbf{il} + q\mathbf{jl} + r\mathbf{kl})$$

where

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + p^2 + q^2 + r^2 = 4c - b^2 \quad \text{and} \quad \alpha, \beta, \gamma, \delta, p, q, r \in \mathbb{R}$$

Case 3: If $b \in \mathbb{R}$ and $c \notin \mathbb{R}$, then

$$x = \frac{-b}{2} \pm \frac{\rho}{2} \mp \frac{c_1}{\rho}\mathbf{i} \mp \frac{c_2}{\rho}\mathbf{j} \mp \frac{c_3}{\rho}\mathbf{k} \mp \frac{c_4}{\rho}\mathbf{l} \mp \frac{c_5}{\rho}\mathbf{il} \mp \frac{c_6}{\rho}\mathbf{jl} \mp \frac{c_7}{\rho}\mathbf{kl}$$

where $$c = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} + c_4\mathbf{l} + c_5\mathbf{il} + c_6\mathbf{jl} + c_7\mathbf{kl}$$

and $$\rho = \sqrt{\frac{\left(b^2 - 4c_0 + \sqrt{(b^2 - 4c_0)^2 + 16\sum_{i=1}^{7} c_i^2}\right)}{2}}$$

Case 4: If $b \notin \mathbb{R}$, then

$$x = \frac{-\Re(b)}{2} - (b' + T)^{-1}(c' - N),$$

where

$$b' = b - \Re(b) = \Im(b), \;\; c' = c - \left(\frac{\Re(b)}{2}\right)\left(b - \frac{\Re(b)}{2}\right), \text{ and } (T, N) \text{ is chosen as follows.}$$

1. $T = 0, N = \frac{(B \pm \sqrt{B^2 - 4E})}{2}$ provided that $D = 0,\ B^2 \geq 4E$.

2. $T = \pm\sqrt{2\sqrt{E} - B}, N = \sqrt{E}$ provided that $D = 0, B^2 < 4E$.

3. $T = \pm\sqrt{z}, N = \frac{(T^3 + BT + D)}{2T}$ provided that $D \neq 0$ and z is the unique positive root of the real polynomial $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2$,

   where $B = b'\bar{b}' + c' + \bar{c}' = |b'|^2 + 2\Re(c'),\ E = c'\bar{c}' = |c'|^2,\ D = \bar{b}'c' + \bar{c}'b' = 2\Re(\bar{b}'c')$, are real numbers.

Proof. Case 1: $b, c \in \mathbb{R}$ and $b^2 \geq 4c$. There are at most two solutions, both are real

$$\frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Case 2: $b, c \in \mathbb{R}$ and $b^2 < 4c$. Note that x is a solution if and only if $q^{-1}xq$ is also a solution for octonion $q \neq 0$. To check that, consider

$$t = q^{-1}xq \tag{2.1.4}$$

plug t to the quadratic function $x^2 + bx + c$,

$$t^2 + bt + c = (q^{-1}xq)(q^{-1}xq) + b(q^{-1}xq) + c = q^{-1}(-bx - c)q + b(q^{-1}xq) + c = 0 \tag{2.1.5}$$

thus, $t = q^{-1}xq$ is also a solution to $x^2 + bx + c = 0$, and there are at least two complex solutions

$$\frac{-b \pm \sqrt{4c - b^2}\mathbf{i}}{2}$$

hence, the solution set is

$$\left\{q^{-1}\Big(\frac{-b + \sqrt{4c - b^2}\mathbf{i}}{2}\Big)q : q \neq 0\right\}$$

Let $R^2 = 4c - b^2 > 0$ and $q \in \mathbb{O}$

$$\left\{ q^{-1}\Big(\frac{-b+R\mathbf{i}}{2}\Big)q : q \neq 0 \right\}$$

$$\left\{ \frac{-b+q^{-1}R\mathbf{i}q}{2} : q \neq 0 \right\}$$

$$\left\{ \frac{-b+Rq^{-1}\mathbf{i}q}{2} : q \neq 0 \right\}$$

Note that $Rq^{-1}iq$ is pure octonion. To see that consider

$$(Rq^{-1}iq)(Rq^{-1}iq) = R^2 i^2 = -R^2$$

thus, we can write $Rq^{-1}iq$ as,

$$Rq^{-1}iq = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} + \delta\mathbf{l} + p\mathbf{il} + q\mathbf{jl} + r\mathbf{kl} \in \Im(\mathbb{O})$$

and

$$\overline{Rq^{-1}iq} = -Rq^{-1}iq$$

therefore,

$$|Rq^{-1}iq| = (Rq^{-1}iq)\overline{(Rq^{-1}iq)} = -(Rq^{-1}iq)(Rq^{-1}iq) = R^2 = 4c - b^2$$

implies,

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + p^2 + q^2 + r^2 = 4c - b^2$$

hence solution set is,

$$= \left\{ \frac{1}{2}(-b + \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} + \delta\mathbf{l} + p\mathbf{il} + q\mathbf{jl} + r\mathbf{kl}) : \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + p^2 + q^2 + r^2 = 4c - b^2 \right\}$$

How we will describe having infinitely many solutions?

For Geometric view,

Consider

$$|x|^2 = \frac{(-b)^2}{4} + \frac{\alpha^2}{4} + \frac{\beta^2}{4} + \frac{\gamma^2}{4} + \frac{\delta^2}{4} + \frac{p^2}{4} + \frac{q^2}{4} + \frac{r^2}{4} = \frac{b^2 + 4c - b^2}{4} = c$$

Conclusion: Solutions are set of all points on S^7 (7 - sphere) in 8-dimension with norm equal to $\sqrt{c}$ (Note that $c \in \mathbb{R}$).

Case 3: $b \in \mathbb{R}$ and $c \notin \mathbb{R}$. Let

$$x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} + x_4\mathbf{l} + x_5\mathbf{il} + x_6\mathbf{jl} + x_7\mathbf{kl} \text{ and } c = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} + c_4\mathbf{l} + c_5\mathbf{il} + c_6\mathbf{jl} + c_7\mathbf{kl}$$

then $x^2 + bx + c = 0$ becomes the real system,

$$(x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2) + bx_0 + c_0 = 0$$

$$\left(x_0 + \frac{b}{2}\right)^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2 = \frac{b^2}{4} - c_0 \tag{2.1.6}$$

$$(2x_0 + b)x_i = -c_i \text{ where } \ i = 1, 2, 3, 4, 5, 6, 7 \tag{2.1.7}$$

since c is non-real, $(2x_0 + b)$ is non-zero. From equation 2.1.6

$$(2x_0 + b)^2 - 4x_1^2 - 4x_2^2 - 4x_3^2 - 4x_4^2 - 4x_5^2 - 4x_6^2 - 4x_7^2 = (b^2 - 4c_0)$$

by equation 2.1.7

$$(2x_0 + b)^4 - 4c_1^2 - 4c_2^2 - 4c_3^2 - 4c_4^2 - 4c_5^2 - 4c_6^2 - 4c_7^2 = (2x_0 + b)^2(b^2 - 4c_0)$$

$$(2x_0 + b)^4 - (b^2 - 4c_0)(2x_0 + b)^2 - 4(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2 + c_7^2) = 0$$

$$(2x_0 + b)^2 = \frac{(b^2 - 4c_0) \pm \sqrt{(b^2 - 4c_0)^2 + 16(c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 + c_6^2 + c_7^2)}}{2}$$

$$(2x_0 + b) = \pm\sqrt{\frac{\left(b^2 - 4c_0 + \sqrt{(b^2 - 4c_0)^2 + 16\sum_{i=1}^{7} c_i^2}\right)}{2}}$$

$$x_0 = \frac{(-b \pm \rho)}{2} \quad \text{where } \rho \neq 0$$

$$x = \frac{-b}{2} \pm \frac{\rho}{2} \mp \frac{c_1}{\rho}\mathbf{i} \mp \frac{c_2}{\rho}\mathbf{j} \mp \frac{c_3}{\rho}\mathbf{k} \mp \frac{c_4}{\rho}\mathbf{l} \mp \frac{c_5}{\rho}\mathbf{il} \mp \frac{c_6}{\rho}\mathbf{jl} \mp \frac{c_7}{\rho}\mathbf{kl}$$

Case 4: $b \notin \mathbb{R}$. Let $y = x + \frac{\Re(b)}{2}$ and $b' = b - \Re(b) = \Im(b)$

then

$$y^2 = x^2 + (\Re(b))x + (\frac{\Re(b)}{2})^2 \tag{2.1.8}$$

$$b'y = (\Im(b))x + \frac{\Re(b)}{2}(b - \Re(b)) \tag{2.1.9}$$

by adding equations 2.1.8 and 2.1.9 we have,

$$y^2 + b'y = x^2 + bx + \frac{\Re(b)}{2}(b - \frac{\Re(b)}{2}) = -c + \frac{\Re(b)}{2}(b - \frac{\Re(b)}{2}) \tag{2.1.10}$$

take

$$c' = c - \frac{\Re(b)}{2}(b - \frac{\Re(b)}{2})$$

then 2.1.10 becomes,

$$y^2 + b'y + c' = 0 \tag{2.1.11}$$

by the idea of I. Niven[16], the solution of the equation 2.1.11 also satisfies

$$y^2 - Ty + N = 0, \tag{2.1.12}$$

where $N = \bar{y}y \geq 0$ and $T = y + \bar{y} \in \mathbb{R}$. Hence, by 2.1.11 and 2.1.12 we have,

$$(b' + T)y + (c' - N) = 0,$$

and so

$$y = -(b' + T)^{-1}(c' - N)$$

because $T \in \mathbb{R}$ and $b' \notin \mathbb{R}$ implies that $b' + T \neq 0$. To solve for T and N, we substitute y back into the definitions $N = \bar{y}y$ and $T = y + \bar{y}$, and simplify to obtain the real system

$$N = \left(\frac{N - \bar{c'}}{\bar{b'} + T}\right)\left(\frac{N - c'}{b' + T}\right)$$

$$N = \frac{N^2 - (c' + \bar{c'})N + \bar{c'}c'}{|b|^2 + (\bar{b'} + b')T + T^2}$$

$$N^2 - (|\bar{b}|^2 + 2\Re(c') + T^2)N + |c'|^2 = 0 \tag{2.1.13}$$

take $B = |b'|^2 + 2\Re(c')$ and $E = |c'|^2$. Note that both B and E in $\mathbb{R}$. Now we can write 2.1.13 as follow,

$$N^2 - (B + T^2)N + E = 0, \tag{2.1.14}$$

similarly,

$$T = y + \bar{y} = \frac{N - c'}{b' + T} + \frac{N - \bar{c'}}{\bar{b'} + T}$$

$$T = \frac{\bar{b'}N - \bar{b'}c' + TN - Tc' + b'N - b'c' + TN - T\bar{c'}}{|b'|^2 + (b' + \bar{b'})T + T^2}$$

$$T^3 + (|b'|^2 + 2\Re(c') - 2N)T + 2\Re(\bar{b'}c') = 0$$

$$T^3 + (B - 2N)T + D = 0 \tag{2.1.15}$$

where $D = \bar{b'}c' + \bar{c'}b' = 2\Re(\bar{b'}c')$ is also a real number. Note that $E = |c'|^2 \geq 0$. If $B < 0$, then $c' + \bar{c'} < 0$ and $B^2 - 4E = |b'|^2B + |b'|^2(c' + \bar{c'}) + (c' - \bar{c'})^2 \leq 0$ because of the fact that $(c' - \bar{c'})^2 \leq 0$. It follows that $B^2 - 4E < 0$, otherwise $B^2 - 4E = 0$ and so $|b'|^2B = |b'|^2(c' + \bar{c'}) = (c' - \bar{c'}) = 0$, i.e., $b' = 0 \in \mathbb{R}$, a contradiction. Hence, by lemma 2.1.2 such system can be solved explicitly as claimed. consequently,

$$x = -\frac{\Re(b)}{2} - (b' + T)^{-1}(c' - T).$$

□

2.2 EXAMPLES

Now we discuss some of the examples of solving octonionic left quadratic equaions. Theorem 2.1.3 is being used for this and we can use the code [37] from ”pyoctonionic” library [26] for solving any octonionic left quadratic equations.

Example 2.2.1. Consider the equation $x^2 + 5x + 6 = 0$, Find solutions $x \in \mathbb{O}$

Solution:

$b = 5$ and $c = 6$, therefore, $b^2 > 4c$. From theorem 2.1.3, case 1.

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 6}}{2} = \frac{-5 \pm 1}{2} = -3 \text{ or} -2$$

Example 2.2.2. Consider the equation $x^2 + 1 = 0$, Find solutions $x \in \mathbb{O}$

Solution:

$b = 0$ and $c = 1$, therefore, $b^2 < 4c$. theorem 2.1.3, case 2.

$$x = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} + \delta\mathbf{l} + p\mathbf{il} + q\mathbf{jl} + r\mathbf{kl}$$

where,

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + p^2 + q^2 + r^2 = 1 \text{ and } \alpha, \beta, \gamma, \delta, p, q, r \in \mathbb{R}$$

there are infinitely many solutions to the equation $x^2 + 1 = 0$ in $\mathbb{O}$.

Example 2.2.3. Consider the equation $x^2 - x + \mathbf{l} = 0$, Find solutions $x \in \mathbb{O}$

Solution:

$b = -1$ and $c = \mathbf{l}$, from theorem 2.1.3, case 3.

$$x = \frac{1}{2} \pm \frac{\rho}{2} \mp \frac{\mathbf{l}}{\rho}$$

where

$$\rho = \sqrt{\frac{1 + \sqrt{17}}{2}}$$

Example 2.2.4. Consider the equation $x^2 + \mathbf{l}x + \mathbf{j} = 0$, Find solutions $x \in \mathbb{O}$

Solution:

$b = \mathbf{l} \notin \mathbb{R}$ and $c = \mathbf{j}$, from theorem 2.1.3, case 4

$$x = -\frac{\Re(b)}{2} - (b' + T)^{-1}(c' - N)$$

since $\Re(b) = 0$ we have $x = -(b' + T)^{-1}(c' - N)$ where note that $B^2 < 4E$ therefore, $T = \pm\sqrt{2\sqrt{E} - B} = \pm 1$ and $N = \sqrt{E} = 1$

thus, $x = -(\mathbf{l}+1)^{-1}(\mathbf{j}-1)$ or $x = -(\mathbf{l}-1)^{-1}(\mathbf{j}-1)$

finally,

$$x = \frac{(\mathbf{l}-1)(1-\mathbf{j})}{-2} = \frac{\mathbf{l}+\mathbf{jl}-1+\mathbf{j}}{-2} = \frac{1-\mathbf{j}-\mathbf{l}-\mathbf{jl}}{2}$$

or

$$x = \frac{(\mathbf{l}+1)(1-\mathbf{j})}{-2} = \frac{\mathbf{l}+\mathbf{jl}+1-\mathbf{j}}{-2} = \frac{-1+\mathbf{j}-\mathbf{l}-\mathbf{jl}}{2}$$

2.3 LEFT SPECTRUM OF 2×2 OCTONIONIC HERMITIAN MATRICES

We can find left spectrum of a 2×2 octonionic Hermitian matrix by solving corresponding octonionic left quadratic equation. Let's begin with following lemma :

Lemma 2.3.1. Let

$$A = \begin{bmatrix} p & \bar{a} \\ a & m \end{bmatrix}$$

For q and $r \in \mathbb{O}$, left spectrum of $(qI + rA)$ is $\sigma_l(qI + rA) = \{q + rt : t \in \sigma_l(A)\}$, where I is the 2×2 identity matrix.

Proof. Let t be a left eigenvalue of A with eigenvector v,

$$Av = tv$$

consider

$$(qI + rA)v = qIv + rAv = qv + rtv = (q + rt)v$$

therefore, $q + rt$ where $t \in \sigma_l(A)$ represent the set of left spectrum of $(qI + rA)$. □

Theorem 2.3.2. Let

$$A = \begin{bmatrix} p & \bar{a} \\ a & m \end{bmatrix}$$

where $a \in \mathbb{O}$ and $p, m \in \mathbb{R}$.

(i) if $a\bar{a} = |a|^2 = 0$, i.e $a = 0$, then $\sigma_l(A) = \{p, m\}$

(ii) if $a \neq 0$, then $\sigma_l(A) = \{p + \bar{a}t : t^2 + (\bar{a})^{-1}(p-m)t - (\bar{a})^{-1}a = 0\}$ i.e,

$$\sigma_l(A) = \{p + \bar{a}t : t^2 + \frac{a(p-m)t}{|a|^2} - \frac{a^2}{|a|^2} = 0\}$$

Proof. (i) A is a triangular matrix, then results follows.

(ii) Using Lemma 2.3.1, we have

$$\sigma_l(A) = \sigma_l \left[pI + \bar{a} \begin{bmatrix} 0 & 1 \\ (\bar{a})^{-1}a & (\bar{a})^{-1}(m-p) \end{bmatrix} \right]$$

Let t be any left eigenvalue of $\begin{bmatrix} 0 & 1 \\ (\bar{a})^{-1}a & (\bar{a})^{-1}(m-p) \end{bmatrix}$, then there exists non-zero vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$, where $x, y \in \mathbb{O}$ such that

$$\begin{bmatrix} 0 & 1 \\ (\bar{a})^{-1}a & (\bar{a})^{-1}(m-p) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} x \\ y \end{bmatrix}$$

$$y = tx, \tag{2.3.1}$$

$$(\bar{a})^{-1}ax + (\bar{a})^{-1}(m-p)y = ty \tag{2.3.2}$$

From equations 2.3.1 and 2.3.2, we have

$$t^2 + (\bar{a})^{-1}(m-p)t - (\bar{a})^{-1}a = 0. \tag{2.3.3}$$

$$t^2 + \frac{a(m-p)}{|a|^2}t - \frac{a^2}{|a|^2} = 0. \tag{2.3.4}$$

We can use Theorem 2.1.3 to find roots of 2.3.4 and explicitly find left spectrum of given 2×2 octonionic Hermitian matrix A. $\square$

2.4 CLASSIFICATION OF LEFT SPECTRUM

Consider the 2×2 octonionic Hermitian matrix A. From theorem 2.3.2, the set of left eigenvalues of A can be written as,

$$\sigma_l(A) = \left\{ p + \bar{a}x : x^2 + \frac{a(p-m)x}{|a|^2} - \frac{a^2}{|a|^2} = 0 \right\}$$

We will consider different cases for solving this octonionic left quadratic equation

$x^2 + bx + c = 0$, where $b = \frac{(p-m)a}{|a|^2}$, $c = -\frac{a^2}{|a|^2}$ and classify the left spectrum of the 2×2 octonionic Hermitian matrix A $= \begin{bmatrix} p & \bar{a} \\ a & m \end{bmatrix}$.

Case 1: If $a \in \mathbb{R}$, then we have,

$$b^2 = \left(\frac{(p-m)a}{|a|^2} \right)^2 = (p-m)^2 > 4c = -4(\frac{a^2}{|a|^2}) = -4$$

therefore, we always get two distinct real solutions for the octonionic quadratic equation. When $a \in \mathbb{R}$, we have conclusion that 2×2 octonionic Hermitian matrix always have two distinct real eigenvalues.(This is not surprise as our matrix reduce to real symmetric matrix when $a \in \mathbb{R}$).

Example 2.4.1. Let

$$A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$$

From above case 1,

$$\sigma_l(A) = \Big\{4 + x : x^2 + x - 1 = 0\Big\}$$

solutions to $x^2 + x - 1 = 0$ are

$$x = \frac{-1 - \sqrt{5}}{2} \quad \text{or} \quad x = \frac{-1 + \sqrt{5}}{2}$$

therefore,

$$\sigma_l(A) = \Big\{\frac{7 - \sqrt{5}}{2}, \frac{7 + \sqrt{5}}{2}\Big\}$$

Case 2: If $a \in \Im(\mathbb{O})$, then $b \notin \mathbb{R}$ and $c = 1$, In this case our quadratic equation becomes,

$$x^2 + \Big(\frac{(p - m)a}{|a|^2}\Big)x + 1 = 0 \tag{2.4.1}$$

We will solve above equation using following subcases.

Subcase 1: If $p = m$, then above equation become

$$x^2 + 1 = 0 \tag{2.4.2}$$

from theorem 2.1.3 case 2, equation 2.4.2 has infinitely many solutions in $\mathbb{O}$, recall the example 2.2.2 we discussed above.

$$x = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} + \delta\mathbf{l} + p\mathbf{il} + q\mathbf{jl} + r\mathbf{kl}$$

where

$$\alpha^2+\beta^2+\gamma^2+\delta^2+p^2+q^2+r^2=1 \text{ and } \alpha,\beta,\gamma,\delta,p,q,r\in\mathbb{R}$$

therefore, set of left eigenvalues become

$$\sigma_l(A)=\Big\{p+\bar{a}x : x \text{ is unit imaginary octonion}\Big\}$$

We have two possible set of left eigenvalues for A

First case when :

$x=\pm\frac{a}{|a|}$ in this case A is always has two real eigenvalues.

Second case when :

$x\neq\pm\frac{a}{|a|}$ in this case A is always has infinitely many non-real left eigenvalues.

Example 2.4.2. Let

$$A=\begin{bmatrix} 2 & \mathbf{k}+\mathbf{l}+\mathbf{il} \\ -\mathbf{k}-\mathbf{l}-\mathbf{il} & 2 \end{bmatrix}$$

From above case 2: subcase 1:

$$\sigma_l(A)=\Big\{2+(\mathbf{k}+\mathbf{l}+\mathbf{il})x : x^2+1=0\Big\}$$

where x is an unit imaginary octonion,

$$x=\alpha\mathbf{i}+\beta\mathbf{j}+\gamma\mathbf{k}+\delta\mathbf{l}+p\mathbf{il}+q\mathbf{jl}+r\mathbf{kl}$$

where,

$$\alpha^2+\beta^2+\gamma^2+\delta^2+p^2+q^2+r^2=1 \text{ and } \alpha,\beta,\gamma,\delta,p,q,r\in\mathbb{R}$$

Here we have two possibilities,

If $x = \pm\frac{(\mathbf{k+l+il})}{\sqrt{3}}$ then $\sigma_l(A) = \left\{2 \pm \sqrt{3}\right\}$. That is only two real eigenvalues exists.

If $x \neq \pm\frac{(\mathbf{k+l+il})}{\sqrt{3}}$ then $\sigma_l(A)$ has infinitely many non-real left eigenvalues.

Subcase 2: $p \neq m$

Then $b = \frac{(p-m)a}{|a|^2} \notin \mathbb{R}$ and $c = 1$. From theorem 2.1.3 case 4,

$$x = -\frac{\Re(b)}{2} - (b' + T)^{-1}(c' - N)$$

where $b' = \frac{(p-m)a}{|a|^2}$ and $c' = 1$

Note that $\Re(b) = 0$ since $b \in \Im(\mathbb{O})$

Also we have,

$B = \frac{(p-m)^2}{|a|^2} + 2$, $E = 1$ and $D = 0$

$$T = 0 \quad \text{and} \quad N = \frac{B \pm \sqrt{B^2 - 4E}}{2}$$

therefore,

$$x = -\Big(\frac{(p-m)a}{|a|^2}\Big)^{-1}\Big(1 - \frac{B \mp \sqrt{B^2 - 4E}}{2}\Big)$$

Let

$$L = \Big(1 - \frac{B \mp \sqrt{B^2 - 4E}}{2}\Big) \quad \text{which is real number.}$$

thus,

$$x = -\frac{L|a|^2\bar{a}}{(p-m)a\bar{a}} = -\frac{L\bar{a}}{(p-m)} = \frac{La}{(p-m)}$$

Therefore set of left eigenvalues are,

$$\sigma_l(A) = \Big\{p + \frac{\bar{a}La}{(p-m)}\Big\} = \Big\{p + \frac{|a|^2L}{(p-m)}\Big\}$$

Note that the set of left eigenvalues in this case is always real.

Example 2.4.3. Let

$$A = \begin{bmatrix} 4 & \mathbf{i}+\mathbf{k}+\mathbf{l} \\ -\mathbf{i}-\mathbf{k}-\mathbf{l} & 2 \end{bmatrix}$$

From above case 2: subcase 2:

$$\sigma_l(A) = \Big\{p + \frac{|a|^2 L}{(p-m)}\Big\}$$

where

$p = 4 \;\; |a|^2 = 3 \;\; (p-m) = 2$ and $B = \frac{10}{3} \;\; E = 1.$

$$L = \Big(1 - \frac{B \mp \sqrt{B^2 - 4E}}{2}\Big)$$
$$L = \Big(1 - \frac{\frac{10}{3} \mp \sqrt{(\frac{10}{3})^2 - 4}}{2}\Big)$$
$$L = -2 \text{ or } L = \frac{2}{3}$$

hence,

$$\sigma_l(A) = \Big\{1, 5\Big\}$$

Case 3:

If $a \in \mathbb{O}$

Subcase 1: If $p = m$

In this case our quadratic equation becomes,

$$x^2 - \frac{a^2}{|a|^2} = 0$$
$$x = \pm\frac{a}{|a|}$$

This mean x is unit octonion and therefore, the set of left eigenvalues become,

$$\sigma_l(A) = \left\{p \pm \frac{\bar{a}a}{|a|}\right\} = \left\{p \pm |a|\right\}$$

This conclude that, if $p = m$ and $a \in \mathbb{O}$, then we always get real eigenvalues.

Example 2.4.4. Let

$$A = \begin{bmatrix} 5 & 2 + \mathbf{k} + \mathbf{l} \\ 2 - \mathbf{k} - \mathbf{l} & 5 \end{bmatrix}$$

From above case 3: subcase 1:

$$\sigma_l(A) = \left\{p \pm |a|\right\} = \left\{5 \pm \sqrt{6}\right\}$$

Subcase 2: If $p \neq m$

In this case our quadratic equation becomes,

$$x^2 + \frac{a(p-m)x}{|a|^2} - \frac{a^2}{|a|^2} = 0$$

since

$$b = \frac{a(p-m)}{|a|^2} \notin \mathbb{R}$$

solution to above quadratic equation will be given by theorem 2.1.3:(Case 4).

thus, set of left eigenvalues become,

$$\sigma_l(A) = \left\{p + \bar{a}x\right\}$$

where

$$x = -\frac{\Re(b)}{2} - (b' + T)^{-1}(c' - T).$$

and

$$b' = b - \Re(b) = \Im(b),\ c' = c - \Big(\frac{\Re(b)}{2}\Big)\Big(b - \frac{\Re(b)}{2}\Big),\ \text{and } (T,N) \text{ is chosen as follows.}$$

1. $T = 0, N = \frac{(B \pm \sqrt{B^2-4E})}{2}$ provided that $D = 0,\ B^2 \geq 4E$.
2. $T = \pm\sqrt{2\sqrt{E} - B}, N = \sqrt{E}$ provided that $D = 0, B^2 < 4E$.
3. $T = \pm\sqrt{z}, N = \frac{(T^3+BT+D)}{2T}$ provided that $D \neq 0$ and z is the unique positive root of the real polynomial $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2$.

Example 2.4.5. Let

$$A = \begin{bmatrix} 5 & 2+\mathbf{k}+\mathbf{l} \\ 2-\mathbf{k}-\mathbf{l} & 3 \end{bmatrix}$$

From above case 3: subcase 2:

$$\sigma_l(A) = \Big\{p + \bar{a}x : x^2 + \frac{(2-\mathbf{k}-\mathbf{l})x}{3} - \frac{(1-2\mathbf{k}-2\mathbf{l})}{3} = 0\Big\} \tag{2.4.3}$$

$$x = \frac{-\Re(b)}{2} - (b'+T)^{-1}(c'-N),$$

where,

$$b' = b - \Re(b) = \Im(b) = \frac{(-\mathbf{k}-\mathbf{l})}{3}, c' = c - \Big(\frac{\Re(b)}{2}\Big)\Big(b - \frac{\Re(b)}{2}\Big) = \frac{(-4+7\mathbf{k}+7\mathbf{l})}{9}$$

$$B = |b'|^2 + 2\Re c' = -\frac{2}{3},\quad E = |c'|^2 = \frac{38}{27},\quad D = 2\Re(\bar{b}'c') = -\frac{28}{27}$$

thus,

(T,N) chosen as follows,

1. $T = \pm\sqrt{z}, N = \frac{(T^3+BT+D)}{2T}$ provided that $D \neq 0$ and z is the unique positive root of

the real polynomial $z^3 + 2Bz^2 + (B^2 - 4E)z - D^2$

Now using the python code [37] we can solve octonionic left quadratic equation

$$x^2 + \frac{(2 - \mathbf{k} - \mathbf{l})x}{3} - \frac{(1 - 2\mathbf{k} - 2\mathbf{l})}{3} = 0$$

Two solutions are:

$$\begin{aligned} x_1 &= 0.5486 - 0.2743\mathbf{k} - 0.2743\mathbf{l} \\ x_2 &= -1.2153 + 0.6076\mathbf{k} + 0.6076\mathbf{l} \end{aligned} \tag{2.4.4}$$

Lets consider the equation 2.4.3 to find the left spectrum of A

$$\sigma_l(A) = \Big\{5 + \bar{a}x_1, 5 + \bar{a}x_2\Big\} = \Big\{6.6458, 1.3542\Big\}$$

Note that in this case gives two real eigenvalues.

Now we will show that in above subcase 2, left spectrum can always be obtained by using theorem 2.1.3 case 4.

Theorem 2.4.6. Consider the octonionic Hermitian matrix

$$A = \begin{bmatrix} p & \bar{a} \\ a & m \end{bmatrix}$$

If $p \neq m$ and $a \in \mathbb{O}$ then left spectrum of A can be obtained using theorem 2.1.3 case 4..

Proof. If we take $a = r_1 + \mathrm{I}$ where $r_1 \in \mathbb{R}, r_1 \neq 0$ and I is pure imaginary octonion. From theorem 2.3.2 the set of left eigenvalues become,

$$\sigma_l(A) = \Big\{p + (r_1 - \mathrm{I})x : x^2 + \frac{(r_1 + \mathrm{I})(p - m)x}{|r_1 + \mathrm{I}|^2} - \frac{(r_1 + \mathrm{I})^2}{|r_1 + \mathrm{I}|^2} = 0\Big\}$$

take $\mathrm{I}^2 = -r$ where r is a positive real number.

$$b = \frac{r_1(p - m)}{|r_1 + \mathrm{I}|^2} + \frac{(p - m)\mathrm{I}}{|r_1 + \mathrm{I}|^2} \notin \mathbb{R}$$
$$b' = b - \Re(b) = \frac{(p - m)\mathrm{I}}{|r_1 + \mathrm{I}|^2}$$
$$c = -\frac{(r_1 + \mathrm{I})^2}{|r_1 + \mathrm{I}|^2}$$

now let $L = \frac{(p-m)}{|r_1+\mathrm{I}|^2} \in \mathbb{R}$ and $L \neq 0$ since $p \neq m$

then,

$$c' = c - \Big(\frac{\Re(b)}{2}\Big)\Big(b - \frac{\Re(b)}{2}\Big)$$
$$c' = -\frac{(r_1 + \mathrm{I})^2}{|r_1 + \mathrm{I}|^2} - \frac{r_1 L}{2}\Big(\frac{r_1 L}{2} + L\mathrm{I}\Big)$$
$$c' = \Big(\frac{(r - r_1^2)}{|r_1 + \mathrm{I}|^2} - \frac{r_1^2 L^2}{4}\Big) + \Big(\frac{-2r_1}{|r_1 + \mathrm{I}|^2} - \frac{r_1 L^2}{2}\Big)\mathrm{I} = P - Q\mathrm{I}$$

where

$$P = \Big(\frac{(r - r_1^2)}{|r_1 + \mathrm{I}|^2} - \frac{r_1^2 L^2}{4}\Big) \in \mathbb{R}$$

and

$$Q = \Big(\frac{2r_1}{|r_1 + \mathrm{I}|^2} + \frac{r_1 L^2}{2}\Big) \in \mathbb{R}$$

to find, $D = 2\Re(\bar{b}'c')$

consider

$$\bar{b}'c' = -L\mathrm{I}(P - Q\mathrm{I}) = -\mathrm{LPI} - \mathrm{LQr}$$

thus,

$$\Re(\bar{b}'c') = -LQr$$

Note that $r, L \neq 0$. therefore $\Re(\bar{b}'c') = 0$ if and only if $Q = 0$.

Now consider

$$Q = \Big(\frac{2r_1}{|r_1 + \mathrm{I}|^2} + \frac{r_1 L^2}{2}\Big)$$

since $L = \frac{(p-m)}{|r_1+\mathrm{I}|^2}$

$$Q = \Big(\frac{2r_1}{|r_1 + \mathrm{I}|^2} + \frac{r_1(p-m)^2}{2|r_1 + \mathrm{I}|^4}\Big)$$

$$Q = \Big(\frac{r_1\big(4|r_1 + \mathrm{I}|^2 + (\mathrm{p} - \mathrm{m})^2\big)}{2|r_1 + \mathrm{I}|^4}\Big)$$

since $r_1 \neq 0$, Q is always non-zero. Therefore we have $D = 2\mathrm{Re}\ \bar{b}'c' \neq 0$. Thus in this subcase we have to use theorem 2.1.3, case 4: with

$$T = \pm\sqrt{z},\ N = \frac{(T^3 + BT + D)}{2T}$$

provided that $D \neq 0$ and z is the unique positive root of the real polynomial

$$z^3 + 2Bz^2 + (B^2 - 4E)z - D^2,$$

where $B = |b'|^2 + 2\Re(c')$, $E = |c'|^2$, and $D = 2\Re(\bar{b}'c')$. □

Following diagram summarize the above discussion of classification of left spectrum.

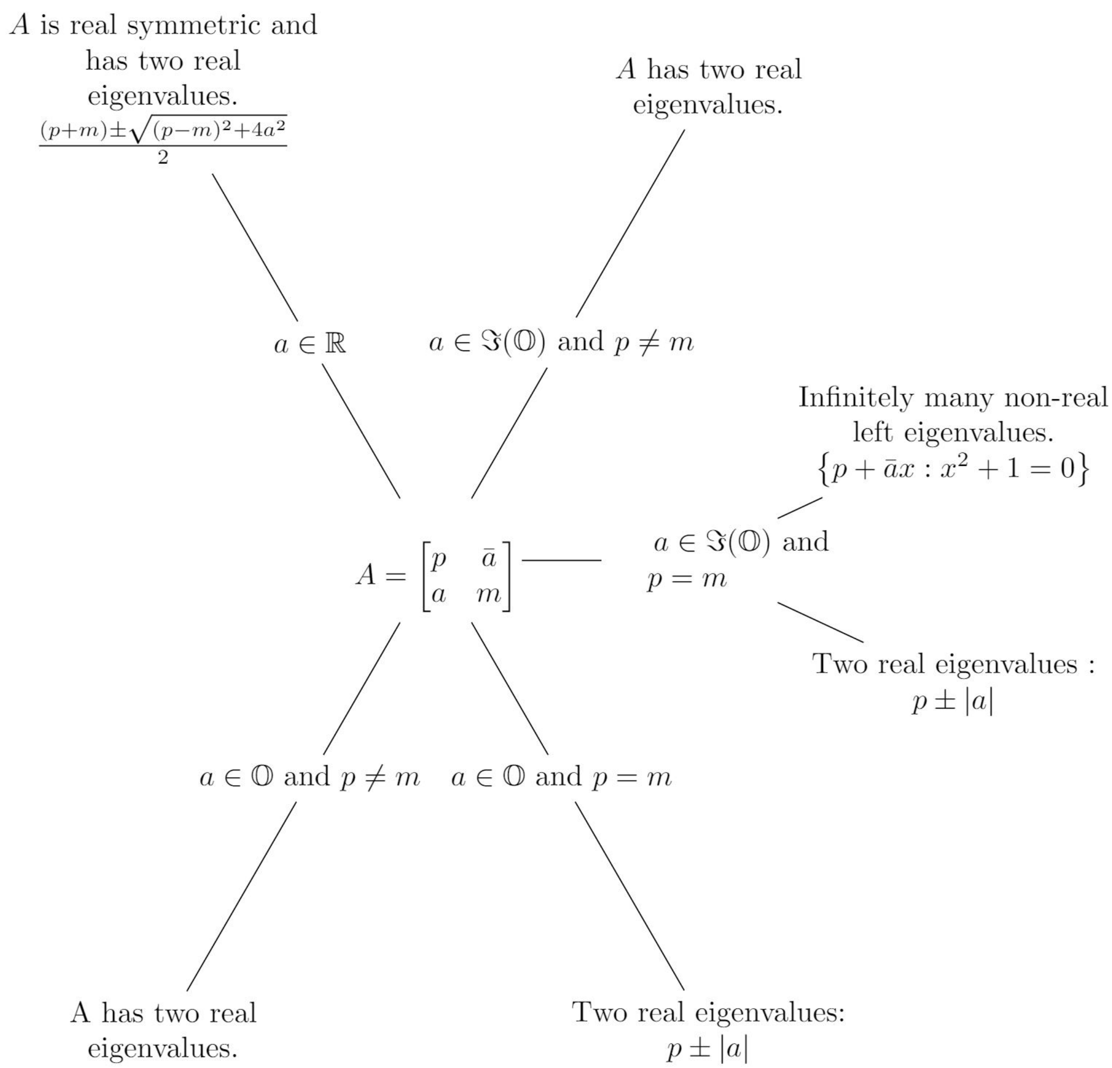

Figure 2.4.1. Classification of left eigenvalues of 2×2 octonionic Hermitian matrices

2.5 EIGENVECTORS FOR NON-REAL LEFT EIGENVALUES

In this section we discuss eigenvectors for the family of 2×2 octonionic Herimitian matrices which admit left eigenvalues are not real [2], [3] and [7].

Let

$$J(\hat{r}) = \begin{bmatrix} 0 & -\hat{r} \\ \hat{r} & 0 \end{bmatrix} \tag{2.5.1}$$

for any pure imaginary unit octonion $\hat{r}$, and noting that this latter condition can be written as $\hat{r}^2 = -1$

Lemma 2.5.1. The set of 2×2 Hermitian matrices A for which left eigenvalues exist which are not real is

$$\mathbb{A} := \left\{ A : A = pI + qJ(\hat{r}); p, q \in \mathbb{R}, p \neq 0, q \neq 0, \hat{r}^2 = -1 \right\} \tag{2.5.2}$$

The set $\mathbb{A}$ has some remarkable properties, which will be further discussed below. without loss of generality, we can take $\hat{r} = i$, so that A takes the form

$$A = \begin{bmatrix} p & -iq \\ iq & p \end{bmatrix} \tag{2.5.3}$$

Let us find the general solution of the left eigenvalues problem for these matrices. Taking A as in 2.5.3 and take $v = \begin{bmatrix} x \\ y \end{bmatrix}$. Thus, we can write $Av = \lambda v$ as

$$\begin{aligned} \frac{\lambda - p}{q} x &= -iy \\ \frac{\lambda - p}{q} y &= ix \end{aligned} \tag{2.5.4}$$

Taking the norm of both sides immediately yields

$$|x|^2 = |y|^2 \tag{2.5.5}$$

and we can normalize both of these to 1 without loss of generality. We thus obtain

$$\begin{aligned} \frac{\lambda - p}{q} &= -(iy)\bar{x} = (ix)\bar{y} \\ &= -[i, y, \bar{x}] - i(y\bar{x}) = [i, x, \bar{y}] + i(x\bar{y}) \end{aligned} \tag{2.5.6}$$

but since

$$[z, y, \bar{x}] = -[z, y, x] = [z, x, y] = -[z, x, \bar{y}] \tag{2.5.7}$$

for any z, the two associators cancel, and we are left with

$$\langle x, y \rangle = 0 \tag{2.5.8}$$

Thus, x and y correspond to orthonormal vectors in $\mathbb{O}$ thought of as $\mathbb{R}^8$. This argument is fully reversible; any suitably normalized x and y which are orthogonal yield an eigenvector of A. We have therefore shown that all matrices in $\mathbb{A}$ have the same left eigenvectors:

Lemma 2.5.2. The set of left eigenvectors for any matrix $A \in \mathbb{A}$ is

$$\mathbb{V} = \left\{ \begin{bmatrix} x \\ \\ y \end{bmatrix} : |x|^2 = |y|^2; \langle x, y \rangle = 0 \right\} \tag{2.5.9}$$

The left eigenvalue is given in each case by 2.5.4. Furthermore, left multiplication by an arbitrary octonion preserves the set $\mathbb{V}$, so that matrices in $\mathbb{A}$ have the property that left multiplication of left eigenvectors yields another left eigenvector. It follows from 2.5.4 and 2.5.5 that

$$|\lambda - p| = q \tag{2.5.10}$$

Inserting this into either of 2.5.4, multiplying both sides by i, and using the identities 0.1.18

and 0.1.19 then shows that 2.5.8 forces

$$\langle \lambda, i \rangle = 0 \tag{2.5.11}$$

However, these are the only restrictions on λ, since 2.5.4 can be used to construct eigenvectors having any eigenvalues satisfying these two conditions.

Example 2.5.3. Let $A = \begin{bmatrix} p & -l \\ l & p \end{bmatrix}$
choose $x = i$ and $y = j$ which satisfies equation 2.5.5 then,

$$\begin{bmatrix} p & -l \\ l & p \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix} = \begin{bmatrix} pi + jl \\ -il + pj \end{bmatrix} = (p - kl) \begin{bmatrix} i \\ j \end{bmatrix}$$

notice that $\lambda = p - kl$ satisfies the two conditions given in equations 2.5.10 and 2.5.11.

CHAPTER 3

REAL EIGENVALUE PROBLEM OF 3×3 OCTONIONIC HERMITIAN MATRICES.

It is not immediately obvious that 3×3 octonionic Hermitian matrices have a well defined determinant, let alone a charasteristic equation. In this chapter, we review properties of these matrices and the real eigenvalues of 3×3 octonionic Hermitian matrices.

3.1 JORDAN PRODUCT AND JORDAN MATRICES

Let A and B Hermitian matrices, that is $A^\dagger = A$, and $B^\dagger = B$. But then their product $C = AB$ may not necessarily be Hermitian since $C^\dagger = (AB)^\dagger = B^\dagger A^\dagger = BA$, which is not the same as $C = AB$, unless A and B commute. In order to avoid this difficulty, Jordan[1] proposed [36] to use the Jordan product.

$$C = A \circ B := \frac{1}{2}(AB + BA) \tag{3.1.1}$$

which is Hermitian, $C^\dagger = C$ if $A^\dagger = A$ and $B^\dagger = B$. Therefore, the correct product is $A \circ B$ and not the ordinary product.
Jordan product also satisfies following properties:

$$A^2 \equiv A \circ A \tag{3.1.2}$$

$$A^3 := A^2 \circ A = A \circ A^2 \tag{3.1.3}$$

[1]Jordan algebras were first introduced by Pascual Jordan (1933) to formalize the notion of an algebra of observables in quantum mechanics.

Theorem 3.1.1. (Jordan-Von Neuman-Wigner) If X is a finite-dimensional simple Jordan algebra, then X can be obtained in only on of the following two ways:

Case 1: The Jordan algebra can be obtained from some associative algebra as in equation 3.1.1. Such an algebra is known as a special Jordan algebra.

Case 2: Otherwise, the Jordan algebra is a 27− dimensional algebra which can be constructed as follows.

Let A be a 3×3 matrix whose entries are octonionic numbers of the form

$$A = \begin{bmatrix} p & a & \bar{b} \\ \bar{a} & m & c \\ b & \bar{c} & n \end{bmatrix} \tag{3.1.4}$$

where $p, m, n \in \mathbb{R}$ and $a, b, c \in \mathbb{O}$ with their conjugates $\bar{a}, \bar{b}, \bar{c}$. Define the product of two such matrices by equation 3.1.1. Then this defines the 27− dimensional exceptional Jordan algebra(it is exceptional because the octonions are not associative). Note that AB is a matrix product but it is not associative in general, since its octonionic elements are not associative.

The 3×3 octonionice Hermitian matrices are called as Jordan matrices, form the exceptional Jordan algebra under the Jordan product. Remarkably, with these definitions, Jordan matrices satisfy the characteristic equation.

$$A^3 - (\text{tr}A)A^2 + \sigma(A)A - \det(A)I = 0 \tag{3.1.5}$$

where $\sigma(A)$ is defined by

$$\sigma(A) := \frac{1}{2}\big((\text{tr}A)^2 - \text{tr}(A^2)\big) \tag{3.1.6}$$

and where the determinant $\det A$ of A is defined abstractly in terms of the Freudenthal product [12],

$$A * B = A \circ B - \frac{1}{2}\Big(A\text{tr}(B) + B\text{tr}(A)\Big) + \frac{1}{2}\Big(\text{tr}(A)\text{tr}(B) - \text{tr}(A \circ B)\Big) \tag{3.1.7}$$

The determinant can then be defined as

$$\det A = \frac{1}{3}\text{tr}\Big((A * A) \circ A\Big) \tag{3.1.8}$$

Therefore, if

$$A = \begin{bmatrix} p & a & \bar{b} \\ \bar{a} & m & c \\ b & \bar{c} & n \end{bmatrix} \tag{3.1.9}$$

with $p, m, n \in \mathbb{R}$ and $a, b, c \in \mathbb{O}$ then,

$$\text{tr}A = p + m + n \tag{3.1.10}$$

$$\sigma(A) = pm + pn + mn - |a|^2 - |b|^2 - |c|^2 \tag{3.1.11}$$

$$\det A = pmn + b(ac) + \overline{b(ac)} - n|a|^2 - m|b|^2 - p|c|^2 \tag{3.1.12}$$

3.2 THE REAL EIGENVALUE PROBLEM

$n \times n$ Hermitian matrices over any of the normed division algebras can be rewritten as symmetric $kn \times kn$ real matrices, where k denotes the dimension of the underlying division algebra, it is clear that a 3×3 octonionic Hermitian matrix must have $8 \times 3 = 24$ real eigenvalues [5], However, as we will see, instead of having (a maximum of) 3 distict real eigenvalues, each with multiplicity 8, there are (a maximum of) 6 distinct real eigenvalues, each with multiplicity 4.

The reason for this is that, somewhat surprisingly, a real eigenvalue λ of a Jordan matrix A does not in general satisfy the characteristic equation 3.1.5.

To see this, consider the eigenvalue equation $Av = \lambda v$, with A as in 3.1.9, $\lambda \in \mathbb{R}$, and

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{3.2.1}$$

where, $x, y, z \in \mathbb{O}$. By solving $Av = \lambda v$ explicitly, we have

$$(\lambda - p)x = ay + \bar{b}z \tag{3.2.2}$$

$$(\lambda - m)y = cz + \bar{a}x \tag{3.2.3}$$

$$(\lambda - n)z = bx + \bar{c}y \tag{3.2.4}$$

so that

$$(\lambda - p)(\lambda - m)y = (\lambda - p)(cz + \bar{a}x) = (\lambda - p)cz + \bar{a}(ay + \bar{b}z) \tag{3.2.5}$$

which implies

$$\left[(\lambda - p)(\lambda - m) - |a|^2\right]y = \bar{a}(\bar{b}z) + (\lambda - p)cz \tag{3.2.6}$$

Assume first that $\lambda \neq p$. Using 3.2.2 and 3.2.6 in 3.2.4 leads to

$$\begin{aligned}
&\big[\lambda - p)(\lambda - m) - |a|^2\big](\lambda - p)(\lambda - n)z \\
&\qquad = \big[(\lambda - p)(\lambda - m) - |a|^2\big](\lambda - p)(bx + \bar{c}y) \\
&\qquad = \big[(\lambda - p)(\lambda - m) - |a|^2\big]b(ay + \bar{b}z) + (\lambda - p)\bar{c}\big[\bar{a}(\bar{b}z) + bx + (\lambda - p)cz\big] \\
&\qquad = b\big[a(\bar{a}(\bar{b}z) + (\lambda - p)cz)\big] + \big[(\lambda - p)(\lambda - m) - |a|^2\big]b(\bar{b}z) + (\lambda - p)\bar{c}\big[\bar{a}(\bar{b}z) + (\lambda - p)cz\big] \\
&\qquad = (\lambda - p)\Bigg[(\lambda - m)|b|^2 z + (\lambda - p)|c|^2 z + b(a(cz)) + \bar{c}(\bar{a}(\bar{b}z))\Bigg]
\end{aligned}$$

Expanding this out and comparing with 3.1.10-3.1.12, we have

$$\begin{aligned}
[\det(\lambda I - A)]z &= [\lambda^3 - (\mathrm{tr}A)\lambda^2 + \sigma(A)\lambda - \det A]z \\
&= b(a(cz)) + \bar{c}(\bar{a}(\bar{b}z)) - [b(ac) + (\bar{c}\bar{a})\bar{b}]z
\end{aligned} \tag{3.2.7}$$

Consider the case $\lambda = p$, we still have 3.2.6, which here takes the form

$$-|a|^2 y = \bar{a}(\bar{b}z) \tag{3.2.8}$$

Inserting this into 3.2.3, we can solve for x, obtaining

$$-|a|^2 x = a(cz) + (p - m)\bar{b}z \tag{3.2.9}$$

Finally, inserting 3.2.8 and 3.2.9 in 3.2.4 yield

$$-\big(|a|^2(p - n) + |b|^2(p - m)\big)z = b\big(a(cz)) + \bar{c}(\bar{a}(\bar{b}z)\big) \tag{3.2.10}$$

Comparing with 3.1.10 to 3.1.12 and using $\lambda = p$, we see that 3.2.7 still holds, and thus holds in general.

If a, b, c, and z associate, the RHS of 3.2.7 vanishes, and λ does indeed satisfy the characteristic equation 3.1.5; this will not happen in general. However, since the LHS of 3.2.7 is a real multiple of z, this must also be true of the RHS, so that

$$b\big(a(cz)\big) + \bar{c}\big(\bar{a}(\bar{b}z)\big) - \big[b(ac) + (\bar{c}\bar{a})\bar{b}\big]z = rz \tag{3.2.11}$$

which can be solved to yield a quadratic equation for r as well as constraints on z.

Lemma 3.2.1. The real eigenvalues of the 3×3 octonionic Hermitian matrix A satisfy the modified characteristic equation

$$\det(\lambda I - A) = \lambda^3 - (\mathrm{tr}A)\lambda^2 + \sigma(A)\lambda - \det A = r \tag{3.2.12}$$

where r is either of the two roots of

$$r^2 + 4\Phi(a, b, c)r - \big|[a, b, c]\big|^2 = 0 \tag{3.2.13}$$

with a, b, c as defined by 3.1.9 and where Φ was defined in 0.1.23.

The solutions of 3.2.12 are real, since the corresponding 24×24 real symmetric matrix has 24 real eigenvalues. We will refer to 3 real solutions of 3.2.12 corresponding to a single value of r as a family of eigenvalues of A. There are thus 2 families of real eigenvalues, each corresponding to 4 independent(over $\mathbb{R}$) eigenvectors.
This result is a generalize case of complex and quaternion. If A is complex then the only solution of 3.2.13 is $r = 0$, and we recover the characteristic equation with a unique set of 3 real eigenvalues. If A is quaternionic, then one solution of 3.2.12 is $r = 0$, leading to

the standard set of 3 real eigenvalues and their corresponding quaternionic eigenvectors. However, unless a, b, c involve only two independent imaginary quaternionic directions (in which case $\Phi(a, b, c) = 0 = [a, b, c]$), there will also be a nonzero solution for r, leading to a second set of 3 real eigenvalues.

Example 3.2.2. Let

$$A = \begin{bmatrix} 1 & 1+i & l+jl \\ 1-i & 2 & l \\ -l-jl & -l & 3 \end{bmatrix}$$

then,

$$\mathrm{tr}A = 1 + 2 + 3 = 6 \tag{3.2.14}$$

from equation 3.1.11 and 3.1.12

$$\sigma(A) = 2 + 3 + 6 - 2 - 2 - 1 = 6 \tag{3.2.15}$$

$$\det A = 6 + 2 - 6 - 4 - 1 = -3 \tag{3.2.16}$$

from lemma 3.2.1, real eigenvalues of A satisfy the modified characteristic equation,

$$\begin{aligned} \lambda^3 - (\mathrm{tr}A)\lambda^2 + \sigma(A)\lambda - \det A = r \\ \lambda^3 - 6\lambda^2 + 6\lambda + 3 = r \end{aligned} \tag{3.2.17}$$

where r is either of the two roots of

$$r^2 + 4\Phi(a, b, c)r - \left|[a, b, c]\right|^2 = 0 \tag{3.2.18}$$

where, $a = 1 + i, b = -l - jl, c = l$. thus, equation 3.2.18 becomes,

$$r^2 - 4 = 0$$
$$r = \pm 2 \tag{3.2.19}$$

therefore from equation 3.2.17 we have two cubic equations for each r

Case 1: $r = 2$

$$\lambda^3 - 6\lambda^2 + 6\lambda + 1 = 0 \tag{3.2.20}$$

Case 2: $r = -2$.

$$\lambda^3 - 6\lambda^2 + 6\lambda + 5 = 0 \tag{3.2.21}$$

All real eigenvalues can be found using the python code [38].

For Case 1: real eigenvalues are:

$$4.6691, 1.4760, -0.1451$$

Each has multiplicity 4.

For Case 2: real eigenvalues are:

$$4.3615, 2.1674, -0.5289$$

Each has multiplicity 4.

3.3 ORTHOGONALITY AND DECOMPOSITION

In [5] already noted since 2×2 octonionic Hermitian matrix A lies in a complex subspace of $\mathbb{O}$, it admits a complete set of complex eigenvectors with real eigenvalues, which can be used to obtain the decomposition

$$A = \sum_{m=1}^{n} \lambda_m v_m v_m^\dagger \tag{3.3.1}$$

Also recall that for any $v \in \mathbb{O}^2$ satisfies the following result

$$(vv^\dagger)(vv^\dagger) = (v^\dagger v)(vv^\dagger) \tag{3.3.2}$$

and $(vv^\dagger)(ww^\dagger) = 0$ if v and w are eigenvectors of 2×2 octonionic Hermitian matrix with distict real eigenvalues, since each term in parantheses lies in $\mathbb{C}$.
How about this concept extended to 3×3 octonionic Hermitian with real eigenvalues.
From [5] we know that new notion of orthogonality,

Definition 3.3.1. Let v and w be two octonionic vectors. We will say that w is orthogonal to v if

$$(vv^\dagger)w = 0 \tag{3.3.3}$$

The vectors v, w are orthonormal if in addition $v^\dagger v = w^\dagger w = 1$.

Let's review some work of [6] and discuss decomposition of 3×3 octonionic Hermitian matrices.

Lemma 3.3.2. Let A be a 2×2 complex Hermitian matrix. Then w is an octonionic eigenvectors of A with real eigenvalue λ if and only if $w = v\xi$, where $\xi \in \mathbb{O}$ is arbitrary and where v is a complex eigenvector of A with the same eigenvalue.

Lemma 3.3.3. If v and w are eigenvectors of the 2×2 octonionic Hermitian matrix A corresponding to different real eigenvalues, then v and w are mutually orthogonal in the sense of equation 3.3.3

Proof. From above lemma 3.3.2, we can write $v = v_1\alpha$ and $w = v_2\beta$ where $v_1, v_2 \in \mathbb{C}; \alpha, \beta \in \mathbb{O}$. Note that $\mathbb{C} \subset \mathbb{O}$ is the complex subspace containing the elements of A.
Consider,

$$(vv^\dagger)w = |\alpha|^2(v_1v_1^\dagger)(v_2\beta) = |\alpha|^2(v_1)(v_1^\dagger v_2)(\beta) = 0$$

□

Proposition 3.3.4. For any octonionic vector $v \in \mathbb{O}^n$,

$$(vv^\dagger)v = v(v^\dagger v) \tag{3.3.4}$$

This proposition shows in particular that any normalized vector v is an eigenvector of the matrix $vv^\dagger$ with eigenvalue 1.

Lemma 3.3.5. If v and w are eigenvectors of the 3×3 octonionic Hermitian matrix A corresponding to different real eigenvalues in the same family(same r value), then v and w are mutually orthogonal in the sense of equation 3.3.3.

Thus, for Jordan matrices, we can obtain two decompositions of the form in equation 3.3.1, coresponding to the two sets of real eigenvalues.

Theorem 3.3.6. Let A be a 3×3 octonionic Hermitian matrix. Then A can be expanded as in 3.3.1, where v_1, v_2, v_3 are orthonormal(in the sense of equation 3.3.3) eigenvectors of A corresponding to the real eigenvalues λ_m, which belong to the same family(same r value).

In the 2×2 case, 3.3.2 tells us that, for normalized $v, vv^\dagger$ squares to itself, and hence is idempotent. However, 3.3.2 fails in the 3×3 case, so that the decomposition in theorem

3.3.6 is therefore not an idempotent decomposition.

It is nevetheless straightforward to show that if u, v and w are orthonormal in the sense of equation 3.3.3, then

$$uu^\dagger + vv^\dagger + ww^\dagger = I \tag{3.3.5}$$

since the left-hand side has eigenvlaue 1 with multiplicity 3. This permits us to view u, v, w as basis of $\mathbb{O}^3$ in the following sense

Lemma 3.3.7. Let $u, v, w \in \mathbb{O}^3$ be orthonormal in the sense of equation 3.3.3, and let g be any vector in $\mathbb{O}^3$. Then

$$g = (uu^\dagger)g + (vv^\dagger)g + (ww^\dagger)g \tag{3.3.6}$$

However, another consequence of the failure of eqution 3.3.2 in the 3×3 case is that the Gram-Schmidt orhthogonalization procedure no longer works. It appears to be fortuitous that we are nevertheless able to find orthonomal eigenvectors in the 3×3 case with repeated eigenvalues.

We can relate our notion of orthonormality to the usual one by noting that n vectors in $\mathbb{O}^n$ which are orthonomal in the sense of equation 3.3.3 satisfy

$$vv^\dagger + ... + ww^\dagger = I \tag{3.3.7}$$

If we define a matrix U whose columns are just $v, ..., w$, then this statement is equivalent to

$$UU^\dagger = I \tag{3.3.8}$$

over the quaternions, left matrix inverses are the same as right matrix inverses, and we would also have

$$U^\dagger U = I \tag{3.3.9}$$

or equivalently

$$v^\dagger v = ... = w^\dagger w = 1, \quad v^\dagger w = 0... \tag{3.3.10}$$

which is just the standard notion of orthogonality. These two notions of orthogonality fail to be equivalent over the octonions; we have been led to view the former as more fundamental.

We can now rewrite the eigenvalue equation $Av = \lambda v$ in the form

$$AU = DU \tag{3.3.11}$$

where D is a diagonal matrix whose entries are the real eigenvalues. Multiplying equation 3.3.11 on the left by $U^\dagger$ yields

$$U^\dagger(AU) = U^\dagger(DU) = (U^\dagger U)D \tag{3.3.12}$$

note that D is real. However, this does not lead to a diagonalization of A since, as noted above, $U^\dagger U$ is not in general equal to the identity matrix. However, theorem 3.3.6 can be rewritten as

$$A = UDU^\dagger \tag{3.3.13}$$

so that in this sense A is diagonolizable. Furthermore, multiplication of equation 3.3.11 on the right by $U^\dagger$ shows that

$$(AU)U^\dagger = (DU)U^\dagger = A = A(UU^\dagger) \tag{3.3.14}$$

and this assertion of associativity can be taken as a restatement of theorem 3.3.6. In the

2×2 case, this associativity holds for a single vector v that is,

$$(Av)v^\dagger = A(vv^\dagger) \tag{3.3.15}$$

which leads to the elegant one-line derivation

$$A = A\Big(\sum_{m=1}^{2} v_m v_m^\dagger\Big) = \sum_{m=1}^{2} A(v_m v_m^\dagger) = \sum_{m=1}^{2} (Av_m)v_m^\dagger = \sum_{m=1}^{2} \lambda v_m v_m^\dagger. \tag{3.3.16}$$

CHAPTER 4

CLASSIFICATION OF NON REAL LEFT AND RIGHT EIGENVALUES OF 3×3 OCTONIONIC HERMITIAN MATRICES

In this chapter we will prove that 3×3 octonionic Hermitian matrix which admits characteristic function, whose roots are left eigenvalues. The characteristic map μ_A will be in most cases a rational fuction instead of a polynomial. Later we discuss right spectrum and provide examples of finding eigenvalues and eigenvectors.

4.1 LEFT SPECTRUM

4.1.1 Basic definitions

Definition 4.1.1. Let the octonionic matrix $A \in M_{n\times n}(\mathbb{O})$ be decomposed as $A = X + Yl$ with $X, Y \in M_{n\times n}(\mathbb{H})$. We call Study's determinat of A the non-negative real number

$$\text{Sdet}(A) := (\det c(A))^{1/2},$$

where $c(A)$ is the matrix $\begin{bmatrix} X & -\bar{Y} \\ Y & \bar{X} \end{bmatrix} \in M_{2n\times 2n}(\mathbb{H})$

Now consider 2×2 octonionic Hermitian matrix $A = \begin{bmatrix} p & \bar{a} \\ a & m \end{bmatrix}$.

Define Study's determinant(Sdet) [25] as follows.

Definition 4.1.2. if $\bar{a} = 0$ then $\text{Sdet}(A) = |pm|$ and if $\bar{a} \neq 0$ then

$$\text{Sdet}(A) = \text{Sdet}\begin{bmatrix} p & \bar{a} \\ a & m \end{bmatrix} = \text{Sdet}\begin{bmatrix} 0 & \bar{a} \\ a - m(\bar{a})^{-1}p & m \end{bmatrix} = \left|\det\begin{bmatrix} 0 & \bar{a} \\ a - \frac{pma}{|a|^2} & m \end{bmatrix}\right|$$

So we have,

$$\text{Sdet}(A) = |\bar{a}||a - \frac{pma}{|a|^2}|$$

Definition 4.1.3. A fuction μ is called characteristic function of the matrix A, if

$$|\mu(\lambda)| = \text{Sdet}(A - \lambda I) \ \forall \ \lambda \in \mathbb{O}$$

Definition 4.1.4. An octonion λ be a left eigenvalue of the matrix A if $Av = \lambda v$ for some nonzero vector v in octonion. Equivalently, the matrix $(A - \lambda I)$ is not invertible, that is $\text{Sdet}(A - \lambda I) = 0$.

Proposition 4.1.5. Sdet(A) satisfies following properties:

1. Sdet(AB)=Sdet(A).Sdet(B);
2. if A is a complex matrix then $\text{Sdet} = |\det(A)|$.

Corollary 4.1.6.

1. $\text{Sdet}(A) > 0$
2. let A and $B = PAP^{-1}$ be similar matrices, then $\text{Sdet}(A) = \text{Sdet}(B)$;
3. Sdet(A) does not change when a (right) multiple of one column is added to another column;
4. Sdet(A) does not change when a (left) multiple of one row is added to another row;
5. Sdet(A) does not change when two columns(or two rows) are permuted.

Definition 4.1.7. We can now define, characteristic function for matrix A
if $\bar{a} \neq 0$ then

$$\text{Sdet}(A - \lambda I) = \text{Sdet}\begin{bmatrix} p - \lambda & \bar{a} \\ a & m - \lambda \end{bmatrix} = \det\begin{bmatrix} 0 & \bar{a} \\ a - (m - \lambda)(\bar{a})^{-1}(p - \lambda) & m - \lambda \end{bmatrix}$$

hence,

$$\text{Sdet}(A-\lambda I) = |\bar{a}||a-(m-\lambda)(\bar{a})^{-1}(p-\lambda)m-\lambda|$$

$$\mu(\lambda) = a-(m-\lambda)(\bar{a})^{-1}(p-\lambda)$$

roots of characteristic function $\mu(\lambda)$ will give left eigenvalues of A.

Theorem 4.1.8. The product $(m-\lambda)(\bar{a})^{-1}(p-\lambda)$ is associative.

Proof. let $m-\lambda = r_1 + I$ and $p-\lambda = r_2 + I$

$$\begin{aligned} ((m-\lambda)(\bar{a})^{-1})(p-\lambda) &= ((r_1+I)(\bar{a})^{-1})(r_2+I) \\ &= (r_1(\bar{a})^{-1} + I(\bar{a})^{-1})(r_2+I) \\ &= r_1r_2(\bar{a})^{-1} + r_1(\bar{a})^{-1}I + r_2I(\bar{a})^{-1} + I(\bar{a})^{-1}I \end{aligned} \tag{4.1.1}$$

and

$$\begin{aligned} (m-\lambda)((\bar{a})^{-1})(p-\lambda)) &= (r_1+I)((\bar{a})^{-1}(r_2+I)) \\ &= (r_1+I)(r_2(\bar{a})^{-1} + (\bar{a})^{-1}I) \\ &= r_1r_2(\bar{a})^{-1} + r_1(\bar{a})^{-1}I + r_2I(\bar{a})^{-1} + I(\bar{a})^{-1}I \end{aligned} \tag{4.1.2}$$

result follows by equations 4.1.1 and 4.1.2. □

4.1.2 Characteristic functions

Let

$$A = \begin{bmatrix} p & a & \bar{b} \\ \bar{a} & m & c \\ b & \bar{c} & n \end{bmatrix}$$

4.1.3 Polynomial case

We begin with the situation, when there exists some zero entry outside the diagonal. There are three possibile cases:

Case 1: If a, b and $c = 0$ then we have triagular matrix and we can define characteristic function as

$$\mu(\lambda) = (p - \lambda)(m - \lambda)(n - \lambda) \tag{4.1.3}$$

Case 2: If $a, b = 0, c \neq 0$

$$A = \begin{bmatrix} p & 0 & 0 \\ 0 & m & c \\ 0 & \bar{c} & n \end{bmatrix}$$

then,

$$\mu(\lambda) = \Big(\bar{c} - (n - \lambda)c^{-1}(m - \lambda)\Big)\big(p - \lambda\big)$$

Proof. Consider

$$\operatorname{Sdet}(A-\lambda I) = \operatorname{Sdet}\begin{bmatrix} p - \lambda & 0 & 0 \\ 0 & m - \lambda & c \\ 0 & \bar{c} & n - \lambda \end{bmatrix} = \operatorname{Sdet}\begin{bmatrix} p - \lambda & 0 & 0 \\ 0 & 0 & c \\ 0 & (\bar{c} - (n - \lambda)c^{-1}(m - \lambda)) & n - \lambda \end{bmatrix}$$

$$\operatorname{Sdet}(A - \lambda I) = |c|\big|\Big(\bar{c} - (n - \lambda)c^{-1}(m - \lambda)\Big)\big(p - \lambda\big)|$$

therefore,

$$\mu(\lambda) = \Big(\bar{c} - (n-\lambda)c^{-1}(m-\lambda)\Big)\big(p-\lambda\big) \tag{4.1.4}$$

□

Case 3: $b = 0, a \neq 0$

$$A = \begin{bmatrix} p & a & 0 \\ \bar{a} & m & c \\ 0 & \bar{c} & n \end{bmatrix}$$

Proof. Consider

$$\text{Sdet}(A - \lambda I) = \text{Sdet}\begin{bmatrix} p-\lambda & a & 0 \\ \bar{a} & m-\lambda & c \\ 0 & \bar{c} & n-\lambda \end{bmatrix} = \text{Sdet}\begin{bmatrix} p-\lambda & 0 & a \\ \bar{a} & c & m-\lambda \\ 0 & n-\lambda & \bar{c} \end{bmatrix}$$

$$= \text{Sdet}\begin{bmatrix} 0 & 0 & a \\ \bar{a} - (m-\lambda)a^{-1}(p-\lambda) & c & m-\lambda \\ -\bar{c}a^{-1}(p-\lambda) & n-\lambda & \bar{c} \end{bmatrix} = |a|\text{Sdet}\begin{bmatrix} \bar{a} - (m-\lambda)a^{-1}(p-\lambda) & c \\ -\bar{c}a^{-1}(p-\lambda) & n-\lambda \end{bmatrix}$$

If $c = 0$

$$\mu(\lambda) = (n-\lambda)(\bar{a} - (m-\lambda)a^{-1}(p-\lambda)) \tag{4.1.5}$$

If $c \neq 0$

$$\mu(\lambda) = -\bar{c}\big(a^{-1}(p-\lambda)\big) - (n-\lambda)\Big(c^{-1}\big(\bar{a} - (m-\lambda)a^{-1}(p-\lambda)\big)\Big) \tag{4.1.6}$$

□

4.1.4 Rational case

Definition 4.1.9. We shall call pole of the matrix $A \in H_{3\times 3}(\mathbb{O})$ the point

$$\lambda_0 = m - (cb^{-1})a$$

Theorem 4.1.10. Let A be 3×3 a octonionic Hermitian matrix such that $b \neq 0$. A characteristic map can be defined as follows:

1. if $\lambda_0 = m - (cb^{-1})a$ is the pole of A, then

$$\mu(\lambda_0) = (\bar{c} - ((n - \lambda_0)b^{-1})a)(\bar{a} - (cb^{-1})(p - \lambda_0))$$

2. for $\lambda \neq \lambda_0$ we define

$$\mu(\lambda) = (\lambda_0 - \lambda)((\bar{b} - (n-\lambda)b^{-1}(p-\lambda)) - ((\bar{c} - (n-\lambda)b^{-1}a)(\lambda_0 - \lambda)(\bar{a} - cb^{-1}(p-\lambda))))$$

Proof. This is the more general situation, when $b \neq 0$, we can compute the Study's determinant of the matrix A as follows. Consider

$$\begin{aligned}\operatorname{Sdet}(A - \lambda I) &= \operatorname{Sdet}\begin{bmatrix} p-\lambda & a & b \\ \bar{a} & m-\lambda & c \\ \bar{b} & \bar{c} & n-\lambda \end{bmatrix} \\ &= \operatorname{Sdet}\begin{bmatrix} 0 & 0 & b \\ \bar{a} - (cb^{-1})(p-\lambda) & (m-\lambda) - (cb^{-1})a & c \\ \bar{b} - (n-\lambda)b^{-1}(p-\lambda) & \bar{c} - ((n-\lambda)b^{-1})a & n-\lambda \end{bmatrix}\end{aligned} \tag{4.1.7}$$

Now let $\lambda_0 = m - (cb^{-1})a$

when $\lambda = \lambda_0$

$$\text{Sdet}(A - \lambda_0 I) = \text{Sdet}\begin{bmatrix} 0 & 0 & b \\ \bar{a} - (cb^{-1})(p - \lambda_0) & 0 & c \\ \bar{b} - (n - \lambda_0)b^{-1}(p - \lambda_0) & \bar{c} - ((n - \lambda_0)b^{-1})a & n - \lambda_0 \end{bmatrix} \tag{4.1.8}$$

$$= |b||(\bar{c} - ((n - \lambda_0)b^{-1})a)(\bar{a} - (cb^{-1})(p - \lambda_0))|$$

Thus,

$$\mu(\lambda_0) = (\bar{c} - ((n - \lambda_0)b^{-1})a)(\bar{a} - (cb^{-1})(p - \lambda_0)) \tag{4.1.9}$$

Next when $\lambda \neq \lambda_0$

$$\text{Sdet}(A - \lambda I) = \text{Sdet}\begin{bmatrix} 0 & 0 & b \\ \bar{a} - (cb^{-1})(p - \lambda) & \lambda_0 - \lambda & c \\ \bar{b} - (n - \lambda)b^{-1}(p - \lambda) & \bar{c} - ((n - \lambda)b^{-1})a & n - \lambda \end{bmatrix}$$

$$= \text{Sdet}\begin{bmatrix} 0 & 0 & b \\ 0 & \lambda_0 - \lambda & c \\ (\bar{b} - (n - \lambda)b^{-1}(p - \lambda)) - \big(((\bar{c} - \big(((n - \lambda)b^{-1})a)(\lambda_0 - \lambda)\big)(\bar{a} - (cb^{-1})(p - \lambda))) & 0 & n - \lambda \end{bmatrix}$$

Thus,

$$\mu(\lambda) = (\lambda_0 - \lambda)((\bar{b} - (n - \lambda)b^{-1}(p - \lambda)) - ((\bar{c} - (n - \lambda)b^{-1}a)(\lambda_0 - \lambda)(\bar{a} - cb^{-1}(p - \lambda)))) \tag{4.1.10}$$

□

4.2 RIGHT SPECTRUM

Let $v, \lambda \in \mathbb{O}$ s.t $Av = v\lambda$ and consider associator identity,

$$[v^\dagger, v, \lambda] := (v^\dagger v)\lambda - v^\dagger(v\lambda) \equiv 0 \tag{4.2.1}$$

which follows for any octonionic vector v and $\lambda \in \mathbb{O}$ by alternativity.
If v is a normalized right eigenvector of A with eigenvalue λ, then

$$v^\dagger(Av) = v^\dagger(v\lambda) = (v^\dagger v)\lambda = \lambda \tag{4.2.2}$$

which yeilds and equation for λ in terms of A and the components of v. A similar construction using the associator

$$[v^\dagger, A, v] := (v^\dagger A)v - v^\dagger(Av) = (Av)^\dagger v - v^\dagger(Av) = (v^\dagger(Av))^\dagger - v^\dagger(Av) \tag{4.2.3}$$

leads for normalized eigenvectors to

$$[v^\dagger, A, v] = (v^\dagger(v\lambda))^\dagger - v^\dagger(v\lambda) = ((v^\dagger v)\lambda))^\dagger - (v^\dagger v)\lambda = \bar{\lambda} - \lambda = -2\Im(\lambda) \tag{4.2.4}$$

Inserting $Av = v\lambda$ into 4.2.2 and simplifiy will give,

$$\lambda = \frac{p|x|^2 + m|y|^2 - n|z|^2 + 2x.(ay)}{|x|^2 + |y|^2 - |z|^2} + \frac{[x, a, y] + [z, b, x] + [y, c, z]}{|x|^2 + |y|^2 + |z|^2} \tag{4.2.5}$$

which gives explicit expressions for the real and imaginary parts of λ. The first term can be rewirtten using cyclic permutation of x, y, z (and a, c, b), and the resulting expressions set equal to obtain

$$\Re(\lambda) = \frac{x.(ay) + z.(bx) + p|x|^2}{|x|^2} \tag{4.2.6}$$

and similar expressions obtained by cyclic permutation. Finally, if v is normalized, the imaginary part of 4.2.5 reduces to

$$\Im(\lambda) = [x, a, y] + [z, b, x] + [y, c, z] \tag{4.2.7}$$

Example 4.2.1. Let

$$A = \begin{bmatrix} p & qi & \frac{q}{2}ks \\ -qi & p & \frac{q}{2}j \\ -\frac{q}{2}ks & -\frac{q}{2}j & p \end{bmatrix} \tag{4.2.8}$$

and

$$s = \frac{\sqrt{5}}{3} - \frac{2}{3}kl$$

then the matrix A becomes,

$$A = \begin{bmatrix} p & qi & \frac{q(\sqrt{5}k+2l)}{6} \\ -qi & p & \frac{q}{2}j \\ -\frac{q(\sqrt{5}k+2l)}{6} & -\frac{q}{2}j & p \end{bmatrix} \tag{4.2.9}$$

eigenvalues which are not real

$$
\begin{aligned}
\lambda_{u_1} &= (p + \frac{\sqrt{5}q}{2}) - \frac{qkl}{2} : \quad u_1 = \begin{bmatrix} 3k \\ \sqrt{5}j - 2il \\ 1 + \sqrt{5}kl \end{bmatrix} \\
\lambda_{u_2} &= (p + \frac{\sqrt{5}q}{2}) + \frac{qkl}{2} : \quad u_2 = \begin{bmatrix} \sqrt{5}k + 2l \\ 3j \\ \sqrt{5} - kl \end{bmatrix} \\
\lambda_{v_1} &= (p - \frac{\sqrt{5}q}{3}) + \frac{2qkl}{3} : \quad v_1 = \begin{bmatrix} \sqrt{5}j - 2il \\ 3k \\ 0 \end{bmatrix} \\
\lambda_{v_2} &= (p - \frac{\sqrt{5}q}{3}) - \frac{2qkl}{3} : \quad v_2 = \begin{bmatrix} 3j \\ \sqrt{5}k + 2l \\ 0 \end{bmatrix} \\
\lambda_{w_1} &= (p - \frac{\sqrt{5}q}{6}) - \frac{qkl}{6} : \quad w_1 = \begin{bmatrix} 3k \\ \sqrt{5}j - 2il \\ -7 - \sqrt{5}kl \end{bmatrix} \\
\lambda_{w_2} &= (p - \frac{\sqrt{5}q)}{2} + \frac{qkl}{2} : \quad w_2 = \begin{bmatrix} \sqrt{5}k + 2l \\ 3j \\ -3\sqrt{5}j - 3kl \end{bmatrix}
\end{aligned}
\tag{4.2.10}
$$

CHAPTER 5

CONCLUSION

We reperest left eigenvalues of a 2×2 octonionic Hermitian matrix using the roots of octonionic left quadratic equation:

Let

$$A = \begin{bmatrix} p & \bar{a} \\ a & m \end{bmatrix}$$

where $a \in \mathbb{O}$ and $p, m \in \mathbb{R}$.

(i) if $a\bar{a} = |a|^2 = 0$,i.e $a = 0$, then $\sigma_l(A) = \{p, m\}$

(ii) if $a \neq 0$, then $\sigma_l(A) = \{p + \bar{a}t : t^2 + (\bar{a})^{-1}(p - m)t - (\bar{a})^{-1}a = 0\}$ i.e,

$$\sigma_l(A) = \{p + \bar{a}t : t^2 + \frac{a(p-m)t}{|a|^2} - \frac{a^2}{|a|^2} = 0\} \tag{5.0.1}$$

By explicitly solving octonionic left quadratic equation in equation 5.0.1, we able to give fully classify of left eigenvalues of 2×2 octonionic Hermitian matrices. It turns out that left eigenvalues of A can be real or non-real. Non-real eigenvalues occurs only when $p = m$ and $a \in \Im(\mathbb{O})$. The family of matrices having non-real left eigenvalues can be written as,

$$\mathbb{A} := \left\{A : A = pI + qJ(\hat{r}); p, q \in \mathbb{R}, p \neq 0, q \neq 0, \hat{r}^2 = -1\right\} \tag{5.0.2}$$

where

$$J(\hat{r}) = \begin{bmatrix} 0 & -\hat{r} \\ \hat{r} & 0 \end{bmatrix}$$

Eigenvectors corresponding to non-real eigenvalues can be write as,

$$\mathbb{V} = \left\{ \begin{bmatrix} x \\ \\ y \end{bmatrix} : |x|^2 = |y|^2; \langle x, y \rangle = 0 \right\} \tag{5.0.3}$$

We used "pyoctonion" python library to solve octonionic left quadratic equation and it also help us to identify the properties of eigenvalues and eigenvectors.

Real eigenvalue problem of a 3×3 octonionic Hermitian matrix were discussed by Tevian Dray and Corinne Mangoue. Our python library can be use to find out all real eigenvalues and verify that generalized notion of orthogonality should be used for matrix decomposition. For non-real case, without being able to solve some version of the characteristic equation, we can not find all the non-real eigenvalues of a given octonionic Hermitian matrix. However, we able to provide non-real eigenvalues for selected 3×3 octonionic Hermitian matrices.

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VITA

Graduate School
Southern Illinois University

Kalpa Madhawa Thudewaththage

kmthude@gmail.com

University of Peradeniya, Sri Lanka
Bachelor of Science, Mathematics, January 2013

Southern Illinois University Carbondale
Master of Science, Mathematics, August 2017

Special Honors and Awards:

John M.H. Olmsted Outstanding Ph.D. Teaching Assistant Award for 2019-2020 at Southern Illinois University Carbondale.

Dissertation Paper Title:
Classification of Eigenvalues of Octonionic Hermitian Matrices.

Major Professor: Dr. Jerzy Kocik