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LOCALLY PRIMITIVELY UNIVERSAL FORMS AND THE PRIMITIVE COUNTERPART TO THE FIFTEEN THEOREM

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by

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TITLE: LOCALLY PRIMITIVELY UNIVERSAL FORMS AND THE PRIMITIVE COUNTERPART TO THE FIFTEEN THEOREM

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An n-dimensional integral quadratic form over $\mathbb{Z}$ is a polynomial of the form $f = f(x_1, \cdots, x_n) = \sum_{1\leqslant\, i,\, j\, \leqslant n} a_{ij}x_ix_j$ where $a_{ij} = a_{ji} \in \mathbb{Z}$. An integral quadratic form is called positive definite if $f(\alpha_1, \cdots, \alpha_n) > 0$ whenever $(0, \ldots, 0) \neq (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$. A positive definite integral quadratic form is said to be almost (primitively) universal if it (primitively) represents all but at most finitely many positive integers. In general, almost primitive universality is a stronger property than almost universality. Main results of this study are: *every primitively universal form nontrivially represents zero over every ring* $\mathbb{Z}_p$ *of* p*-adic integers*, and *every almost universal form in five or more variables is almost primitively universal.* With use of these results and improving a result of G. Pall from 1946, we then provide criteria to determine whether a given integral quadratic lattice over a ring $\mathbb{Z}_p$ of p-adic integers is $\mathbb{Z}_p$-universal or primitively $\mathbb{Z}_p$-universal. The criteria are stated explicitly in terms of a Jordan splitting of the lattice. As an application of the local criteria, we complete the determination of the universal positive definite classically integral quaternary quadratic forms that are almost primitively universal, which was initiated in work of N. Budarina in 2010. Finally, with the use of these local results, we identify 28 positive definite classically integral primitively universal quaternary quadratic forms which were not known previously, introducing a conjecture obtained by a numerical approach, which could possibly be the primitive counterpart to the Fifteen Theorem proved by J.H.

Conway and W.A. Schneeberger in 1993.

DEDICATION

To my loving parents...

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INTRODUCTION

0.1 BACKGROUND AND HISTORY

A positive definite integral quadratic form f is said to be *almost universal* if it represents all sufficiently large positive integers; that is, if the excluded set of positive integers not represented by f is finite. The systematic study of forms with this property was initiated by Ramanujan over a century ago. In a groundbreaking 1917 paper [16], he determined all diagonal quaternary forms of the special type $ax^2 + ay^2 + az^2 + dt^2$ which have this property. Among them is the form $x^2 + y^2 + z^2 + 9t^2$ which represents all positive integers with the single exception of the number 7. Halmos [11] subsequently determined all diagonal quaternary forms for which the excluded set consists of a single integer. There are 88 such forms $ax^2 + by^2 + cz^2 + dt^2$ with $a \leq b \leq c \leq d$. It has now been proved that there are exactly 73 pairs of positive integers that can occur as the excluded set of some positive definite classically integral quadratic form [1].

More recently, Bochnak and Oh [3] completed the determination of effective criteria whereby it can be decided whether or not a general (not necessarily diagonal) positive definite quaternary integral quadratic form f is almost universal. It can be seen that a necessary condition for f to be almost universal is that it be everywhere locally universal; that is, that for every prime p, the equation $f(x_1, \ldots, x_4) = a$ is solvable over the ring $\mathbb{Z}_p$ of p-adic integers for every $a \in \mathbb{Z}_p$. Indeed, if this local condition does not hold for some prime p, then f fails to represent an entire arithmetic progression of positive integers. However, for quaternary forms f, these local conditions are not sufficient to guarantee that f is almost universal. For example, the form $f = x^2 + y^2 + 5^2z^2 + 5^2t^2$ is everywhere locally universal but fails to represent the integers of the type $3 \cdot 2^{2k}$, where k is any nonnegative integer. Note that this f fails to represent zero nontrivially 2-adically; that is, it is anisotropic over $\mathbb{Z}_2$. Bochnak and Oh refer to almost universal forms that are anisotropic for some prime p as *exceptional* and prove that this terminology is indeed appropriate, in the sense that there are only finitely many equivalence classes of such forms. The 144 diagonal quaternary forms of this type were essentially determined by Kloosterman [13].

In light of classical theorems of Tartakowsky [19] and Ross and Pall [17] (see [5, Theorem 1.6, page 204]), the stronger condition that f primitively represents all integers over $\mathbb{Z}_p$ for all primes p does imply that f is almost universal; in fact, such a form primitively represents all sufficiently large integers. A positive definite integral quadratic form f is said to be *almost primitively universal* if for all sufficiently large positive integers a, there exist $x_1, \ldots, x_n \in \mathbb{Z}$ such that $f(x_1, \ldots, x_n) = a$ and g.c.d. $(x_1, \ldots, x_n) = 1$. While it is clear from the definitions that almost primitive universality implies almost universality, the converse is not true. For example, the form $f = x^2 + y^2 + z^2 + 9t^2$ mentioned in the first paragraph does not primitively represent the number 8 over $\mathbb{Z}_2$, and consequently does not primitively represent any member of the arithmetic progression $8+64k$ over $\mathbb{Z}$. So this form f is almost universal, but not almost primitively universal.

As noted above, the identification of almost primitively universal forms reduces completely to a local problem; that is, a positive definite integral quadratic form is almost primitively universal if and only if it is everywhere locally primitively universal. In the paper [4], Budarina began a detailed investigation of the local problem of determining when a form over $\mathbb{Z}_p$ is primitively universal, and used these results to derive some criteria for a form of odd discriminant to be almost primitively universal. For example, in the spirit of the celebrated Fifteen Theorem of Conway and Schneeberger (see [6], [2]), Budarina proved: *a classically integral quadratic form in at least four variables with odd squarefree discriminant is almost primitively universal if and only if it primitively represents the integers* 1, 4 *and* 8. In this dissertation, we will extend this line of inquiry, revisiting and generalizing the local investigations of Budarina. In the process, we produce new, more elementary proofs of the main results in the paper [4], and extend the results there to the case of forms of even discriminant. In contrast to Budarina's arguments, which draw heavily on Zhuravlev's extensive general work on minimal indecomposable representations, as described in [20] and the references therein, our methods make use of nothing more advanced than standard local theory as presented, for example, in the foundational books of O'Meara [14] and Gerstein [10].

Some results which appear in Chapter 3 overlap with those obtained by Pall [15] and Budarina [4], and in a recent paper of Xu and Zhang [20]. In such situations we have chosen to include those results here for the sake of completeness and uniformity. Criteria for $\mathbb{Z}_p$-universality, stated in the classical language of integral quadratic forms, can be found in Lemma 1 of [15]. However that paper does not contain proofs for the assertions which appear in that lemma, and results are stated explicitly only for those 2-adic forms in four variables that admit a diagonal decomposition. Here we generalize Pall's result to include all 2-adic forms and forms of arbitrary rank, and provide complete proofs to cover all cases. The corresponding problem of determining criteria for primitive $\mathbb{Z}_p$-universality was taken up by Budarina in [4], but only partial results were obtained there for the 2-adic case. Completing that work is the main goal of Chapter 3.

As an application of these local results, we complete work initiated by Budarina [4] to determine which universal classically integral quaternary quadratic forms are almost primitively universal. In this process we identify three primitively universal quaternary forms which were not known previously.

0.2 STATEMENT AND SUMMARY OF MAIN RESULTS

We will state here some of the main results of this study in the traditional language of quadratic forms, although the proofs will subsequently by presented from the more modern geometric perspective of quadratic lattices. The first two theorems hold for arbitrary integral quadratic forms $f = \sum_{1\leq i\leq j\leq n} a_{ij}X_iX_j$ with $a_{ij} \in \mathbb{Z}$, with no additional assumption on the cross-term coefficients a_{ij}, $i \neq j$.

Theorem 0.2.1. *If a positive definite integral quadratic form is almost primitively universal, then it non-trivially represents zero over $\mathbb{Z}_p$ for all primes p.*

Thus, no almost universal forms of the exceptional type of Bochnak and Oh can be almost primitively universal. The next result shows that the distinction between almost universality and almost primitive universality is no longer present when the number of variables exceeds four.

Theorem 0.2.2. *If a positive definite integral quadratic form in five or more variables is almost universal, then it is almost primitively universal.*

As an application of this theorem and our 2-adic computations, we prove the following result, which extends Theorem 6 and Corollary 2 of [4]. For this statement, we restrict to forms that are classically integral, in the sense that the cross-term coefficients are even integers.

Theorem 0.2.3. *Let f be a positive definite classically integral quadratic form in n variables such that f represents an odd integer and the discriminant of f is not divisible by p^{n-2} for any prime p. If $n \geq 5$, or if $n = 4$ and the discriminant of f is even, then f is almost primitively universal.*

Moreover, although the main focus of our study is on positive definite forms, the local computations that yield the proof of Theorem 0.2.2 can be applied to arbitrary integral quadratic forms in the indefinite case as well, to obtain the following result:

Theorem 0.2.4. *Let f be an indefinite integral quadratic form in five or more variables. If every integer is represented by the genus of f, then every nonzero integer is primitively represented by f.*

The proofs of Theorem 0.2.1 through 0.2.4 will be presented in Section 3.3.

The primary goal of Chapter 3 will be to give criteria for local universality and local primitive universality. These will be presented in the language of quadratic lattices. A quadratic $\mathbb{Z}_p$-lattice L will be said to be (primitively) $\mathbb{Z}_p$-universal if it (primitively) represents every (nonzero) element of $\mathbb{Z}_p$. Complete criteria, in terms of a Jordan splitting of the lattice, will be presented for both $\mathbb{Z}_p$-universality and primitive $\mathbb{Z}_p$-universality. For odd primes p, the criteria for $\mathbb{Z}_p$-universality appear in Proposition 3.1.5 (for lattices of rank at most 3) and 3.1.6 (for rank at least 4). The corresponding criteria for $\mathbb{Z}_2$-universality appear in Proposition 3.2.13 and Theorem 3.2.14, respectively. The criteria for primitive $\mathbb{Z}_p$-universality for odd primes p are given in Proposition 3.1.8, and those for primitive $\mathbb{Z}_2$-universality are given in Proposition 3.2.20 (for rank at most 3) and Theorem 3.2.21 (for rank at least 4).

In Chapter 4, the local criteria developed in Chapter 3 will be applied to complete the determination of which of the universal positive definite classically integral quaternary quadratic forms are almost primitively universal. The results are summarized in Propositions 4.2.5 and 4.3.6.

Finally, Chapter 5 contains a detailed analysis of the integers that are primitively represented by each of the universal positive definite classically integral quaternary quadratic forms. Here the results of numerical explorations will be presented, along with proofs of primitive universality for 28 of the forms of this type. The list of these 28 forms appears in Proposition 5.3.1. The numerical data lead us to formulate the following conjecture of a potential counterpart to the Fifteen Theorem:

Conjecture 0.2.5. *Let f be a positive definite classically integral quadratic form. If f primitively represents the set of numbers;* $\{$*1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 24, 25, 32, 48, 49, 64* $\}$ *then f is primitively universal.*

0.3 ORGANIZATION

The organization of this dissertation will be as follows:

In Chapter 1, we introduce the notations and the terminologies we use throughout our study. Also, we review a few facts regarding p-adic integers and orthogonal splittings of lattices over $\mathbb{Z}_p$.

Chapter 2 consists of a discussion on universality and isotropy of $\mathbb{Z}_p$-lattices. In this chapter we establish an improved result of [9, Proposition 3.1] which leads to a proof that all primitively $\mathbb{Z}_p$-universal lattices are isotropic.

Chapter 3 will be primarily devoted to determining local criteria for primitively universal quadratic forms. We begin by developing criteria for $\mathbb{Z}_p$-universal forms of all ranks and then discuss about their primitive $\mathbb{Z}_p$-universality. Here we introduce local criteria for dyadic and the non-dyadic cases separately. The chapter will be concluded by supplying proofs of the theorems stated in Section 0.2.

The application of the local criteria for primitive universality to complete the determination of which among the classically integral positive definite quaternary quadratic forms are almost primitively universal will be presented in Chapter 4.

Finally, in Chapter 5 we discuss the methodology we followed to come up with the new conjecture, present the results of our numerical investigations, and provide proofs of the primitive universality of 28 forms we have identified in this study.

CHAPTER 1

PRELIMINARIES

1.1 NOTATION AND TERMINOLOGY FOR LATTICES

From now on, we will abandon the language of forms, and instead adopt the geometric language of lattices. Unexplained terminology and notation will follow that presented in the books of O'Meara [14] and Gerstein [10]. To set the context, let R be an integral domain with field of quotients F of characteristic not 2. By an *R-lattice* L, we will mean a finitely generated R-submodule of a nondegenerate quadratic space V over F equipped with a quadratic map q and corresponding symmetric bilinear form B for which $q(v) = B(v, v)$ for all $v \in V$. We will say that the R-lattice is *integral* if $q(L) \subseteq R$, where $q(L) = \{q(v) : v \in L\}$. A vector $v \in L$ is *primitive* in L, denoted $v \overset{*}{\in} L$, if $\{\alpha \in F : \alpha v \in L\} = R$. An element $a \in F$ is said to be *represented* (*primitively represented*, resp.), denoted $a \to L$ ($a \overset{*}{\to} L$, resp.), if there exists $v \in L$ ($v \overset{*}{\in} L$, resp.) such that $q(v) = a$. For a set S of elements of F, the notation $S \to L$ ($S \overset{*}{\to} L$, resp.) will be used to indicate that $a \to L$ ($a \overset{*}{\to} L$, resp.) for all $a \in S$. The notation $S \not\to L$ will mean that there exists at least one element $a \in S$ such that $a \not\to L$, and analogously for $S \overset{*}{\not\to} L$.

For our purposes, the ring R will always be either the ring $\mathbb{Z}$ of rational integers or a ring $\mathbb{Z}_p$ of p-adic integers for some prime p. Since these rings are principal ideal domains, all lattices under consideration will be free. If $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis for a lattice L, the matrix $(B(v_i, v_j))$ is the *Gram matrix of L with respect to $\mathcal{B}$*. For a symmetric $n \times n$-matrix M, we will write $L \cong M$ to indicate that there exists a basis for L such that M is the Gram matrix of L with respect to that basis. In particular, $L \cong \langle a_1, \ldots, a_n \rangle$ will mean that L has an orthogonal basis for which the Gram matrix is the diagonal matrix with the indicated diagonal entries. When L is a $\mathbb{Z}$-lattice, all Gram matrices of L have the same determinant, which is called the *discriminant of L* and denoted dL. When L is a $\mathbb{Z}_p$-lattice, the determinants of all Gram matrices of L lie in the same coset of $\dot{\mathbb{Q}}_p/(\mathbb{Z}_p^\times)^2$, where $\mathbb{Z}_p^\times$

denotes the group of units of $\mathbb{Z}_p$. This coset is the *discriminant of* L, again denoted by dL. The notation dL is also used to denote the determinant of a specific Gram matrix of L, with the exact meaning generally clear from the context.

1.2 THE P-ADIC INTEGERS

In this subsection we will review a few facts regarding the p-adic integers that will be used frequently. Further discussion of the topics here and in the next subsection can be found, for example, in the books of O'Meara [14] or Gerstein [10]. For a prime p, $\mathbb{Q}_p$ will denote the field of p-adic numbers (that is, the completion of the rational number field $\mathbb{Q}$ with respect to the p-adic metric $|\cdot|_p$) and $\mathbb{Z}_p$ will denote the ring of integers of $\mathbb{Q}_p$ (so $\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p : |\alpha|_p \leq 1\}$). This ring is a local ring with unique maximal ideal $p\mathbb{Z}_p$, and all fractional ideals of $\mathbb{Z}_p$ in $\mathbb{Q}_p$ are of the form $(p\mathbb{Z}_p)^j$ for some $j \in \mathbb{Z}$; hence the nonzero fractional ideals of $\mathbb{Z}_p$ in $\mathbb{Q}_p$ are linearly ordered by inclusion.

The group of units of $\mathbb{Z}_p$ will be denoted by $\mathbb{Z}_p^\times$; thus, $\mathbb{Z}_p^\times = \{\alpha \in \mathbb{Z}_p : |\alpha|_p = 1\}$. So a typical element $a \in \mathbb{Z}_p$ can be written uniquely as $a = p^{\mathrm{ord}_p a} a_0$ with $\mathrm{ord}_p a \in \mathbb{N} \cup \{0\}$ and $a_0 \in \mathbb{Z}_p^\times$. We will make frequent use of the Local Square Theorem, which in the present context asserts that for $\alpha, \beta \in \mathbb{Z}_p^\times$, if $\alpha \equiv \beta (\mathrm{mod}\ 4p\mathbb{Z}_p)$ then $\alpha \in \beta(\mathbb{Z}_p^\times)^2$ (e.g., see [10, Theorem 3.39]). From this it follows that when p is odd, the group $\mathbb{Z}_p^\times$ consists of two squareclasses; that is, $\mathbb{Z}_p^\times = (\mathbb{Z}_p^\times)^2 \cup \Delta(\mathbb{Z}_p^\times)^2$, where Δ denotes any fixed nonsquare in $\mathbb{Z}_p^\times$. For $p = 2$, the group $\mathbb{Z}_2^\times$ consists of four squareclasses with representatives 1, 3, 5 and 7. If $\alpha, \beta \in \mathbb{Z}_2^\times$, then $\alpha \in (\mathbb{Z}_2^\times)^2$ if and only if $\alpha \equiv 1 (\mathrm{mod}\ 8\mathbb{Z}_2)$, and $\alpha(\mathbb{Z}_2^\times)^2 = \beta(\mathbb{Z}_2^\times)^2$ if and only if $\alpha \equiv \beta (\mathrm{mod}\ 8\mathbb{Z}_2)$. For nonzero $\alpha, \beta \in \mathbb{Z}_2$, we will write $\alpha \cong \beta$ when $\alpha/\beta \in (\mathbb{Z}_2^\times)^2$.

1.3 ORTHOGONAL SPLITTINGS AND INVARIANTS OF LATTICES OVER $\mathbb{Z}_P$

In the case of lattices over $\mathbb{Z}_p$, there are several invariants that will frequently be used in the arguments that follow. For such a lattice L, we let $\mathfrak{s}L$, $\mathfrak{n}L$ and $\mathfrak{v}L$ denote the scale, norm and volume ideals, respectively, associated to the lattice L, as defined in [14, §82E]. When a lattice can be decomposed as an orthogonal sum of sublattices whose norm ideals are distinct, it is sometimes possible to transfer information on representations between the entire lattice and the sublattices. For example, we will make frequent use of the following results:

Lemma 1.3.1. *Let L be an integral $\mathbb{Z}_p$-lattice such that $L \cong M \perp K$ for nonzero sublattices M and K of L.*

i) If $M \cong \langle \varepsilon \rangle$, for some $\varepsilon \in \mathbb{Z}_p^\times$, and $\mathfrak{n}K \subseteq 2p\mathbb{Z}_p$, then $\mathbb{Z}_p^\times \not\to L$.

ii) If $\mathbb{Z}_p^\times \to L$ and $\mathfrak{n}K \subseteq 4p\mathbb{Z}_p$, then $\mathbb{Z}_p^\times \to M$.

iii) If $p\mathbb{Z}_p^\times \to L$ and $\mathfrak{n}K \subseteq 4p^2\mathbb{Z}_p$, then $p\mathbb{Z}_p^\times \to M$.

Proof. i) When p is odd, it follows from the Local Square Theorem that the only units that are represented by L are in $\varepsilon(\mathbb{Z}_p^\times)^2$. For $p = 2$, any unit represented by L is congruent to ε modulo $4\mathbb{Z}_2$. Hence at most two squareclasses of units can be represented by L.

ii) Let μ be a unit represented by $M \perp K$. Then $\mu = q(x) + q(y)$ for some x in M and y in K. Since $\mathfrak{n}K \subseteq 4p\mathbb{Z}_p$, we have μ congruent to $q(x)$ modulo $4p\mathbb{Z}_p$. So there exists $\lambda \in \mathbb{Z}_p^\times$ such that $\mu = \lambda^2 q(x) = q(\lambda x) \in q(M)$.

iii) Similar argument as ii). □

The following two lemmas will be used frequently in the remainder of this paper.

Lemma 1.3.2. *Let L be an integral $\mathbb{Z}_p$-lattice such that $L \cong M \perp K$ for nonzero sublattices M and K of L. If $\mathbb{Z}_p \to M$, then $\alpha \overset{*}{\to} L$ for all $0 \neq \alpha \in \mathbb{Z}_p$.*

Proof. Suppose M is $\mathbb{Z}_p$-universal. For any $\alpha \in \mathbb{Z}_p$ and any $v \overset{*}{\in} K$, we have $\alpha - q(v) \in \mathbb{Z}_p$. So $\alpha - q(v) \to M$; hence $\alpha \overset{*}{\to} M \perp K \cong L$. □

Lemma 1.3.3. *Let K be an integral $\mathbb{Z}_p$-lattice such that $\mathbb{Z}_p^\times \to K$. Then for any $\varepsilon \in \mathbb{Z}_p^\times$, $\alpha \overset{*}{\to} \langle \varepsilon \rangle \perp K$ for all $0 \neq \alpha \in \mathbb{Z}_p$.*

Proof. If $\lambda \in p\mathbb{Z}_p$, then $\lambda - \varepsilon \in \mathbb{Z}_p^\times$. So $\lambda - \varepsilon \to K$ and $\lambda \xrightarrow{*} \langle \varepsilon \rangle \perp K$. $\square$

The norm and scale of a lattice are related by the following containments:

$$2\mathfrak{s}L \subseteq \mathfrak{n}L \subseteq \mathfrak{s}L.$$

In particular, when p is odd it is always the case that $\mathfrak{n}L = \mathfrak{s}L$, and when $p = 2$ there are two possibilities, namely $\mathfrak{n}L = \mathfrak{s}L$ or $\mathfrak{n}L = 2\mathfrak{s}L$. The scale and volume of a lattice are related by the containment $\mathfrak{v}L \subseteq (\mathfrak{s}L)^n$, where n is the rank of L. When equality holds, the lattice is said to be $\mathfrak{s}L$-*modular*. A $\mathbb{Z}_p$-modular lattice is called *unimodular*, and a lattice is referred to simply as *modular* if it is A-modular for some fractional ideal A. Thus, a $\mathbb{Z}_p$-lattice L is $p^a\mathbb{Z}_p$-modular if and only if the scaled lattice $L^{a^{-1}}$ is unimodular (here the notation L^α denotes the scaled lattice, as defined in [14, §82J]). For an odd prime p, a modular lattice over $\mathbb{Z}_p$ can always be written as an orthogonal sum of rank 1 sublattices (that is, such a lattice is diagonalizable); a modular lattice over $\mathbb{Z}_2$ can be written as an orthogonal sum of modular sublattices of rank 1 or 2 [14, 93:15]. The modular lattices L over $\mathbb{Z}_2$ which are diagonalizable are precisely those for which $\mathfrak{n}L = \mathfrak{s}L$; these are referred to as *proper*. In the case of an improper unimodular lattice L over $\mathbb{Z}_2$, one can more precisely say that L has an orthogonal splitting

$$L \cong \mathbb{H} \perp \ldots \perp \mathbb{H} \perp \mathbb{P}, \tag{1.1}$$

where $\mathbb{H}$ denotes a hyperbolic plane with matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathbb{P}$ is a binary lattice isometric to either $\mathbb{H}$ or the lattice $\mathbb{A}$ with matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ (e.g., see [10, Corollary 8.10]).

An arbitrary lattice over $\mathbb{Z}_p$ can always be decomposed as an orthogonal sum of modular sublattices. By grouping the sublattices having the same scale, one obtains the so-called *Jordan splitting* for the lattice. For our purposes, we will be considering only $\mathbb{Z}_p$-lattices L for which $\mathfrak{n}L = \mathbb{Z}_p$. Thus $\mathfrak{s}L$ will be either $\mathbb{Z}_p$ or $\frac{1}{2}\mathbb{Z}_p$ by the fundamental containment noted above. The Jordan splitting of such a lattice can thus be written as

$$L \cong L_{(-1)} \perp L_{(0)} \perp L_{(1)} \perp \ldots \perp L_{(t)}, \tag{1.2}$$

where each Jordan component $L_{(i)}$ is either $p^i\mathbb{Z}_p$-modular or 0.[1] Here $L_{(-1)} = 0$ unless $p = 2$. The existence of Jordan splittings and the extent to which such splittings are unique are discussed in detail in [14, §91C]. Of relevance for our purposes is the fact that the ranks, norms and scales of the Jordan components are invariants of the lattice. The rank of the component $L_{(i)}$ will be denoted by r_i. When L is a $\mathbb{Z}_2$-lattice with $\mathfrak{n}L = \mathbb{Z}_2$, the leading Jordan component of L will be either an improper $\frac{1}{2}\mathbb{Z}_2$-modular lattice $L_{(-1)}$ if $\mathfrak{s}L = \frac{1}{2}\mathbb{Z}_2$, or a proper unimodular lattice $L_{(0)}$ if $\mathfrak{s}L = \mathbb{Z}_2$. In order to describe the improper $\frac{1}{2}\mathbb{Z}_2$-modular lattices, it will be convenient to introduce the notations $\widehat{\mathbb{H}}$ and $\widehat{\mathbb{A}}$ to represent the lattices obtained from $\mathbb{H}$ and $\mathbb{A}$, respectively, by scaling by $\frac{1}{2}$. So

$$\widehat{\mathbb{H}} \cong \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad \widehat{\mathbb{A}} \cong \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

[1]Note that our convention for the indexing of the components differs from that of [14].

CHAPTER 2

UNIVERSALITY AND ISOTROPY OF $\mathbb{Z}_P$-LATTICES

2.1 BASIC DEFINITIONS AND EXAMPLES

In this section we will establish some results regarding the set $q^*(L) = \{q(v) : v \overset{*}{\in} L\}$ when L is an integral $\mathbb{Z}_p$-lattice. The nature of this set depends heavily on whether the lattice L is isotropic (that is, there exists a $0 \neq v \in L$ such that $q(v) = 0$ or, equivalently, $0 \in q^*(L)$) or anisotropic, as the following examples of binary modular lattices illustrate.

Example 2.1.1. For any prime p, $q^*(\widehat{\mathbb{H}}) = \mathbb{Z}_p$ (e.g., see [9, Proposition 3.2]).

Example 2.1.2. For $p = 2$, $q^*(\widehat{\mathbb{A}}) = \mathbb{Z}_p^\times$. To see this, let $\widehat{\mathbb{A}}$ have the Gram matrix given above with respect to the basis $\{v_1, v_2\}$; so $q(a_1v_1 + a_2v_2) = a_1^2 + a_1a_2 + a_2^2$. If one or both of the a_i are in $\mathbb{Z}_2^\times$, then $a_1^2 + a_1a_2 + a_2^2$ is also in $\mathbb{Z}_2^\times$, thus giving the containment $q^*(\widehat{\mathbb{A}}) \subseteq \mathbb{Z}_2^\times$. To see the reverse containment, it is only necessary to check that the expression $a_1^2 + a_1a_2 + a_2^2$ takes on values from the four squareclasses in $\mathbb{Z}_2^\times$. We further note that $q(\widehat{\mathbb{A}}) = \{\alpha \in \mathbb{Z}_p : \mathrm{ord}_p\alpha \text{ is even}\}$.

Example 2.1.3. For any prime p, let $L \cong \langle \varepsilon_1, \varepsilon_2 \rangle$ be anisotropic, where $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}_p^\times$. Then $q^*(L) \cap 4p\mathbb{Z}_p = \emptyset$. To see this, let $\{v_1, v_2\}$ be the basis for which the Gram matrix is $\langle \varepsilon_1, \varepsilon_2 \rangle$. Suppose that $v = a_1v_1 + a_2v_2 \overset{*}{\in} L$. Without loss of generality, suppose that $a_1 \in \mathbb{Z}_p^\times$. If $q(v) \in 4p\mathbb{Z}_p$, then also $a_2 \in \mathbb{Z}_p^\times$. Then $a_1^2\varepsilon_1 + a_2^2\varepsilon_2 \equiv 0 \pmod{4p\mathbb{Z}_p}$, and it follows that $-\varepsilon_1\varepsilon_2^{-1} \equiv (a_1^{-1}a_2)^2 \pmod{4p\mathbb{Z}_p}$. By the Local Square Theorem, there exists $\lambda \in \mathbb{Z}_p^\times$ such that $-\varepsilon_1\varepsilon_2^{-1} = \lambda^2$. But then $q(v_1 + \lambda v_2) = 0$, contrary to the assumption that L is anisotropic.

An integral $\mathbb{Z}_p$-lattice L will be said to be $\mathbb{Z}_p$-*universal* if $q(L) = \mathbb{Z}_p$. Note that L is $\mathbb{Z}_p$-universal if and only if $\mathbb{Z}_p^\times \cup p\mathbb{Z}_p^\times \subseteq q(L)$. Further, L is said to be *primitively* $\mathbb{Z}_p$-*universal* if for each $0 \neq \alpha \in \mathbb{Z}_p$, there exists $v \overset{*}{\in} L$ such that $q(v) = \alpha$. It is clear that primitive $\mathbb{Z}_p$-universality implies $\mathbb{Z}_p$-universality, but the converse is not true, as can be seen from the following examples.

Example 2.1.4. Consider $L \cong \langle 1, 1, 3, 3\rangle$ over $\mathbb{Z}_3$. It is easily seen that L represents $1, 2, 3$ and 6, which are representatives of the four squareclasses in $\mathbb{Z}_3^\times \cup 3\mathbb{Z}_3^\times$; thus, L is $\mathbb{Z}_3$-universal. On the other hand, write $L = M \perp K$, where $M \cong \langle 1, 1\rangle$ and $K \cong \langle 3, 3\rangle$, both of which are anisotropic over $\mathbb{Z}_3$. Let $v \overset{*}{\in} L$. Then $v = x + y$, where $x \overset{*}{\in} M$ or $y \overset{*}{\in} K$. If $x \overset{*}{\in} M$, then $q(x) \in \mathbb{Z}_3^\times$ by Example 2.1.3, and hence $q(v) \in 3\mathbb{Z}_3^\times$. Otherwise, $x \in 3M$ and $y \overset{*}{\in} K$. In that case, $q(x) \in 9\mathbb{Z}_3$ and $q(y) \in 3\mathbb{Z}_3^\times$, again by Example 2.1.3, and hence $q(v) \in 3\mathbb{Z}_3^\times$. So $q^*(L) = \mathbb{Z}_3^\times \cup 3\mathbb{Z}_3^\times$ and L is not primitively $\mathbb{Z}_3$-universal.

Example 2.1.5. Consider $L \cong \widehat{\mathbb{A}} \perp \mathbb{A}$ over $\mathbb{Z}_2$. Then L is $\mathbb{Z}_2$-universal, but not primitively $\mathbb{Z}_2$-universal and $q^*(L) = \mathbb{Z}_2^\times \cup 2\mathbb{Z}_2^\times$. The verifications are as in the previous example.

2.2 PRIMITIVELY $\mathbb{Z}_P$-UNIVERSAL LATTICES ARE ISOTROPIC

We are now ready to state the main result of this chapter.

Proposition 2.2.1. *Let p be a prime and L an anisotropic integral $\mathbb{Z}_p$-lattice. Then there exists $l = l(L, p) \in \mathbb{N}$ such that $q^*(L) \cap p^l\mathbb{Z}_p = \emptyset$.*

Before proceeding to the proof of Proposition 2.2.1, we prove the following lemma.

Lemma 2.2.2. *Let $L \cong M \perp K$ be a $\mathbb{Z}_2$-lattice with $M \cong \mathbb{A}^{2^a}$, $K \cong \langle 2^b\beta\rangle \perp \langle 2^c\gamma\rangle$, where a, b, c are non-negative integers and $\beta, \gamma \in \mathbb{Z}_2^\times$. If $\mathfrak{s}L = \mathbb{Z}_2$ and L is anisotropic, then a, b, c are all even and $\beta + \gamma \equiv 4(\mathrm{mod}\, 8\mathbb{Z}_2)$.*

Proof. If a, b had opposite parity, then, for k such that $2k + b \geq a + 1$, we would have $2^{2k+b}\beta \to K$ and $-2^{2k+b}\beta \to M$, and it would follow that L is isotropic. So a, b have the same parity, and, similarly, a, c have the same parity. Since $\mathfrak{s}L = \mathbb{Z}_2$, at least one of a, b, c equals 0, so all of a, b, c must be even. To prove the second assertion, suppose first that $\beta + \gamma \equiv 0(\mathrm{mod}\ 8\mathbb{Z}_2)$. Then $\beta(\mathbb{Z}_2^\times)^2 = -\gamma(\mathbb{Z}_2^\times)^2$ and it would follow that K is isotropic. If

$\beta+\gamma\equiv 2(\text{mod } 4\mathbb{Z}_2)$, then K would represent an element of odd order and again L would be isotropic. Since $\beta+\gamma\in 2\mathbb{Z}_2$, this leaves $\beta+\gamma\equiv 4(\text{mod}\, 8\mathbb{Z}_2)$ as the only remaining possibility, thus completing the proof. $\square$

Proof of Proposition 2.2.1 Note that the result for binary lattices is covered by Examples 2.1.2 and 2.1.3, and for ternary lattices the proof is given in [9, Proposition 3.1]. Moreover, the assertion is vacuous when $rk\ L\geq 5$ since every $\mathbb{Z}_p$-lattice of rank exceeding 4 is isotropic. So we need only consider the case of lattices L of rank 4. Further, by scaling if necessary, we may assume that $\mathfrak{s}L=\mathbb{Z}_p$. So L has a Jordan splitting

$$L\cong L_{(0)}\perp\ldots\perp L_{(t)},$$

for some non-negative integer t, where $L_{(0)}\neq 0$ and $L_{(t)}\neq 0$. Throughout the proof, we let $\{v_1,\ldots,v_4\}$ be a basis for L that gives rise to the indicated Jordan splitting.

Our goal will be to prove that the conclusion of the proposition holds for $l=t+3$, although in some cases a smaller exponent would suffice. So suppose there exists $0\neq v\overset{*}{\in} L$ such that $q(v)\in p^{t+3}\mathbb{Z}_p$. Write $v=\sum_{i=1}^{4}b_iv_i$, where $b_1,\ldots,b_4\in\mathbb{Z}_p$ and $b_k\in\mathbb{Z}_p^\times$ for at least one index k.

Consider first the case when v_k occurs in the orthogonal basis for a diagonalizable Jordan component of L; say $q(v_k)=p^{e_k}\varepsilon_k$, with $\varepsilon_k\in\mathbb{Z}_p^\times$. Writing $v=b_kv_k+w$, where $w=\sum_{i\neq k}b_iv_i$, we have

$$b_k^2p^{e_k}\varepsilon_k+q(w)=q(v)\equiv 0(\text{mod } p^{t+3}\mathbb{Z}_p).$$

It follows that $\text{ord}_p q(w)=e_k$ and

$$-p^{-e_k}\varepsilon_k^{-1}q(w)\equiv b_k^2(\text{mod } p^{t-e_k+3}\mathbb{Z}_p).$$

Since $t-e_k+3\geq 3$, it then follows from the Local Square Theorem that there exists $\lambda\in\mathbb{Z}_p^\times$ such that

$$-p^{-e_k}\varepsilon_k^{-1}q(w)=\lambda^2.$$

Then

$$q(\lambda v_k + w) = \lambda^2 p^{e_k} \varepsilon_k + q(w) = 0,$$

contrary to the assumption that L is anisotropic.

This completes the proof when p is odd, and when $p = 2$ and L is diagonalizable. So we need only further consider the case that $p = 2$ and L has at least one improper Jordan component. Since L is assumed to be anisotropic, this component must be isometric to $\mathbb{A}^{2^s}$, for some non-negative integer s.

Consider first the case that $L \cong \mathbb{A} \perp \mathbb{A}^{2^t}$. If t is even, then L is isotropic; so it suffices to consider odd t. Let $v \overset{*}{\in} L$; say $v = x + y$ with $x \in \mathbb{A}$, $y \in \mathbb{A}^{2^t}$. If $x \overset{*}{\in} \mathbb{A}$, then $q(v) \in 2\mathbb{Z}_2^{\times}$. Otherwise, $y \overset{*}{\in} \mathbb{A}^{2^t}$ and $q(y) \in 2^{t+1} Z_2^{\times}$. Since $q(x)$ has odd order, $\mathrm{ord}_2 q(x) \neq \mathrm{ord}_2 q(y) = t + 1$. So

$$\mathrm{ord}_2 q(v) = \mathrm{ord}_2(q(x) + q(y)) = \min\{\mathrm{ord}_2 q(x), t + 1\} \leq t + 1.$$

So we conclude that $q(v) \notin 2^{t+3}\mathbb{Z}_2$.

In all other cases, the Jordan splitting of L has the form considered in Lemma 2.2.2 for suitable integers a, b, c and units β, γ. By Lemma 2.2.2, we need only consider the case when a, b, c are all even and $\beta + \gamma \equiv 4 (\mathrm{mod}\ 8\mathbb{Z}_2)$. So $L \cong M \perp K$, with $M \cong \mathbb{A}^{2^a}$, $K \cong \langle 2^b \beta \rangle \perp \langle 2^c \gamma \rangle$ in the basis $\{u, w\}$. Let $v \overset{*}{\in} L$, and write $v = x + \alpha u + \delta w$, $x \in \mathbb{A}^{2^a}$, $\alpha, \delta \in \mathbb{Z}_2$. Moreover, we may assume that $x \overset{*}{\in} \mathbb{A}^{2^a}$, as the other cases are covered in the first part of the proof. So $q(x) \in 2^{a+1}\mathbb{Z}_2^{\times}$. Write $\alpha = 2^l \alpha_0$, $\delta = 2^k \delta_0$, $\alpha_0, \delta_0 \in \mathbb{Z}_2^{\times}$. Then

$$q(\alpha u + \delta w) = 2^{2l+b} \alpha_0^2 \beta + 2^{2k+c} \delta_0^2 \gamma.$$

If $2l + b \neq 2k + c$, then $\mathrm{ord}_2 q(\alpha u + \delta w) = \min\{2l + b, 2k + c\}$. If $2l + b = 2k + c$, then $\mathrm{ord}_2 q(\alpha u + \delta w) = 2l + b + 2$ since $\beta + \gamma \equiv 4 (\mathrm{mod}\ 8\mathbb{Z}_2)$. In either case, $\mathrm{ord}_2 q(\alpha u + \delta w)$ is even; hence $\mathrm{ord}_2 q(\alpha u + \delta w) \neq \mathrm{ord}_2 q(x) = a + 1$, since $a + 1$ is odd. So

$$\mathrm{ord}_2 q(v) = \min\{a + 1, \mathrm{ord}_2 q(\alpha u + \delta w)\} \leq t + 1.$$

Once again we conclude that $q(v) \notin 2^{t+3}\mathbb{Z}_2$, and the proof is complete. $\square$

Corollary 2.2.3. *If L is an anisotropic integral $\mathbb{Z}_p$-lattice, then L is not primitively $\mathbb{Z}_p$-universal.*

2.3 ISOTROPY CRITERIA FOR $\mathbb{Z}_P$-LATTICES

If L is primitively $\mathbb{Z}_p$-universal, then L is isotropic. To see when this is the case, it is important to have criteria to recognize when a local lattice or space is isotropic. When the dimension is at least 5, such spaces are known to always be isotropic. For spaces of dimension 3 and 4 explicit conditions in terms of the discriminant and the Hasse symbol of the space are well-known. Such conditions can be found in [10]. We will restate them here for convenient reference.

Let p be a prime and let V be a nondegenerate ternary quadratic space over $\mathbb{Q}_p$. Then by [10, Proposition 4.21], V is isotropic if and only if

$$S_pV = (-1, -dV)_p. \tag{2.1}$$

Here for $a, b \in \dot{\mathbb{Q}}$, $(a, b)_p$ denotes the p-adic Hilbert symbol and S_pV denotes the Hasse symbol of the space V, as defined in [10, §4.2].

For later use, we will apply this criterion to derive conditions under which a particular type of ternary space over $\mathbb{Q}_p$ is isotropic.

Lemma 2.3.1. *Let $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{Z}_2^\times$ and let V be a quadratic space over $\mathbb{Q}_2$ such that $V \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$. Then V is anisotropic if and only if $\varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon_3 (\text{mod } 4\mathbb{Z}_2)$.*

Proof. As isotropy is preserved under scaling, we may assume without loss of generality that $\varepsilon_1 = 1$. To prove the forward implication, assume that it is not the case $\varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon_3 (\text{mod } 4\mathbb{Z}_2)$; say $\varepsilon_2 \equiv 3 (\text{mod } 4\mathbb{Z}_2)$. Then

$$S_2V = (\varepsilon_2, \varepsilon_3)_2 = (-1, \varepsilon_3)_2 = (-1, -\varepsilon_2)_2(-1, \varepsilon_3)_2 = (-1, -\varepsilon_2\varepsilon_3)_2 = (-1, -dV)_2,$$

and it follows from (2.1) that V is isotropic. To prove the converse assume that $\varepsilon_2 \equiv \varepsilon_3 \equiv 1 (\text{mod } 4\mathbb{Z}_2)$. Then $S_2V = (\varepsilon_2, \varepsilon_3)_2 = 1$ and $(-1, -dV)_2 = (-1, -1)_2 = -1$, and it follows from (2.1) that V is anisotropic. □

Now, let V be a nondegenerate quaternary quadratic space over $\mathbb{Q}_p$. Then by [10, Proposition 4.24] V is anisotropic if and only if

$$dV = 1 \text{ and } S_pV = -(-1,-1)_p. \tag{2.2}$$

The following result will provide a useful tool for showing that certain lattices are not primitively $\mathbb{Z}_p$-universal. Although we will be applying this result only for $p = 2$ case, we will provide a general proof.

Lemma 2.3.2. *Let V be a quadratic space of dimension 3 over $\mathbb{Q}_p$, for some prime p. If V is anisotropic, then $-dV \not\to V$.*

Proof. Consider the quaternary space $W \cong V \perp \langle dV \rangle$. Since $dW = 1$, the value of the Hasse symbol S_pW will determine whether or not W is anisotropic. By the general formula given in [10, Proposition 4.18],

$$S_pW = S_pV \cdot S_p(\langle dV \rangle) \cdot (dV, dV)_p.$$

Here $S_p(\langle dV \rangle) = 1$ by definition, and $S_pV = -(-1,-dV)_p$ since V is anisotropic, by (2.1). So

$$S_pW = -(-1,-dV)_p(dV,dV)_p = -(-1,-1)_p.$$

It follows that W is anisotropic, by (2.2), and hence $-dV \not\to V$ by [10, Proposition 2.27]. $\square$

CHAPTER 3

LOCAL CRITERIA FOR UNIVERSAL, PRIMITIVELY UNIVERSAL QUADRATIC FORMS

3.1 (PRIMITIVELY) UNIVERSAL $\mathbb{Z}_P$-LATTICES - NON-DYADIC CASE

Throughout this section, we assume that p is an odd prime and L is an integral $\mathbb{Z}_p$-lattice. So in particular $\mathfrak{n}L = \mathfrak{s}L \subseteq \mathbb{Z}_p$ and L has a Jordan splitting of the form

$$L \cong L_{(0)} \perp \ldots \perp L_{(t)},$$

where each L_i has an orthogonal basis. In this case, L has an orthogonal basis and we write $L \cong \langle p^{\alpha_1}\varepsilon_1, p^{\alpha_2}\varepsilon_2, \ldots, p^{\alpha_n}\varepsilon_n \rangle$, where $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ are non-negative integers and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \mathbb{Z}_p^{\times}$.

Recall that $(a, b)_p$, for $a, b \in \dot{\mathbb{Q}}_p^2$, denotes the p-adic Hilbert symbol. So, in particular, for $\delta \in \mathbb{Z}_p^{\times}$, we have $\delta \in (\mathbb{Z}_p^{\times})^2$ if and only if $(\delta, p)_p = 1$. Thus, for $\varepsilon_1, \varepsilon_2 \in \mathbb{Z}_p^{\times}$, the lattice $\langle \varepsilon_1, \varepsilon_2 \rangle$ is isotropic if and only if $(-\varepsilon_1\varepsilon_2, p)_p = 1$.

Lemma 3.1.1. *Let L be unimodular.*

i) If $rk\ L = 2$, then $\mathbb{Z}_p^{\times} \to L$.

ii) If $rk\ L \geq 3$ or if $rk\ L = 2$ and L is isotropic, then L is primitively universal.

Proof. $i)$ is simply a restatement of [14, 92:1(b)]. For $ii)$, it suffices to note that either $L \cong \mathbb{H}$ or $L \cong \mathbb{H} \perp \langle 1, \ldots, 1, -d \rangle$, where $dL = d(\mathbb{Z}_p^{\times})^2$, by [14, 92:1], and apply the result of Example 2.1.1. □

Proposition 3.1.2. *Let p be an odd prime and let L be an integral $\mathbb{Z}_p$-lattice with $rk\ L \geq 5$. If L is $\mathbb{Z}_p$-universal, then L is primitively $\mathbb{Z}_p$-universal.*

Proof. Assume that L is $\mathbb{Z}_p$-universal. Then $\mathfrak{n}L = \mathfrak{s}L = \mathbb{Z}_p$ and, by Lemma 1.3.1 *i*), $r_0 \geq 2$. If $L_{(0)}$ is isotropic, then L is primitively $\mathbb{Z}_p$-universal by Lemma 3.1.1 *ii*). So we need only consider further the case when $r_0 = 2$ and $L_{(0)}$ is anisotropic. In this case, $q^*(L_{(0)}) \cap p\mathbb{Z}_p = \emptyset$ by Example 2.1.3. If $r_1 = 0$, it would follow from Lemma 1.3.1 *iii*) that $p\mathbb{Z}_p \not\to L$, contrary to the assumption that L is $\mathbb{Z}_p$-universal. So to complete the proof we consider three possibilities for $r_1 = rkL_{(1)}$:

$r_1 = 1$: Since $L_{(0)}$ is anisotropic, $p\mathbb{Z}_p^\times \to L_{(0)} \perp L_{(1)}$ holds if and only if $p\mathbb{Z}_p^\times \to L_{(1)} \perp pL_{(0)}$. But $\mathbb{Z}_p^\times \not\to (L_{(1)} \perp pL_{(0)})^{1/p}$ by Lemma 1.3.1 *i*). Hence $p\mathbb{Z}_p^\times \not\to L$ by Lemma 1.3.1 *iii*), contrary to the assumption that L is $\mathbb{Z}_p$-universal.

$r_1 = 2$: In this case, $\mathbb{Z}_p^\times \to L_{(0)}$ and $p\mathbb{Z}_p^\times \to L_{(1)}$ by Lemma 3.1.1 *i*). So $L_{(0)} \perp L_{(1)}$ is $\mathbb{Z}_p$-universal and hence L is primitively $\mathbb{Z}_p$-universal by Lemma 1.3.2 since $rk\ L \geq 5$.

$r_1 \geq 3$: By Lemma 3.1.1 *ii*), $p\mathbb{Z}_p \overset{*}{\to} L_{(1)}$. Since $\mathbb{Z}_p^\times \to L_{(0)}$ by Lemma 3.1.1 *i*), it follows that L is primitively $\mathbb{Z}_p$-universal. □

Remark 3.1.3. The conclusion of Proposition 3.1.2 does not hold when $rk\ L = 4$, as seen by Example 2.1.4.

Remark 3.1.4. The arguments in this section would be unchanged if $\mathbb{Z}_p$ were replaced by any non-dyadic local ring.

Proposition 3.1.5. *If $rk\ L = n \leq 3$, then L is $\mathbb{Z}_p$-universal if and only if $\alpha_1 = \alpha_2 = 0$ and at least one of the following holds:*

$$(-\varepsilon_1\varepsilon_2, p)_p = 1; \tag{3.1}$$

$$n = 3 \text{ and } \alpha_3 = 0. \tag{3.2}$$

Proof. The sufficiency of the stated conditions is proven in Lemma 3.1.1 *ii*). (In fact, it is shown there that under these conditions L is primitively $\mathbb{Z}_p$-universal.) To show the necessity, note first that if $\alpha_2 \geq 1$, then $q(L) \cap \mathbb{Z}_p^\times = \varepsilon_1(\mathbb{Z}_p^\times)^2$, by the Local Square

Theorem, and so L cannot be $\mathbb{Z}_p$-universal. So in order for L to be $\mathbb{Z}_p$-universal, it must be that $\alpha_2 = 0$. So suppose that $\alpha_2 = 0$ but condition (3.1) fails. Then $\langle \varepsilon_1, \varepsilon_2 \rangle$ is anisotropic, and it must be that $n = 3$ and $\alpha_3 \leq 1$, since otherwise $q(L) \cap p\mathbb{Z}_p^\times = \emptyset$, by Example 2.1.3. However, if $\alpha_3 = 1$, then it would follow from the Local Square Theorem and Example 2.1.3 that $q(L) \cap p\mathbb{Z}_p^\times = p\varepsilon_3(\mathbb{Z}_p^\times)^2$. Hence, $\alpha_3 = 0$ and (3.2) holds. This completes the proof. □

Proposition 3.1.6. [1] *If rk $L = n \geq 4$, then L is $\mathbb{Z}_p$-universal if and only if $\alpha_1 = \alpha_2 = 0$ and at least one of the following holds:*

$$\alpha_3 = 0; \tag{3.3}$$

$$(-\varepsilon_1\varepsilon_2, p)_p = 1; \tag{3.4}$$

$$\alpha_3 = \alpha_4 = 1. \tag{3.5}$$

Proof. Throughout the proof, let T denote the sublattice $\langle p^{\alpha_1}\varepsilon_1, p^{\alpha_2}\varepsilon_2, p^{\alpha_3}\varepsilon_3 \rangle$. If $\alpha_1 = \alpha_2 = 0$ and either (3.3) or (3.4) holds, then T is $\mathbb{Z}_p$-universal (in fact primitively $\mathbb{Z}_p$-universal) by the preceding proposition. If $\alpha_1 = \alpha_2 = 0$ and (3.5) holds, then $\mathbb{Z}_p^\times \to \langle \varepsilon_1, \varepsilon_2 \rangle$ and $p\mathbb{Z}_p^\times \to \langle p\varepsilon_3, p\varepsilon_4 \rangle$ by Lemma 3.1.1 *i*). This establishes the sufficiency of the stated conditions.

To prove the necessity, note first that if it is not the case that $\alpha_1 = \alpha_2 = 0$, then $\mathbb{Z}_p^\times \not\to L$ and L cannot be $\mathbb{Z}_p$-universal. So we may assume that L is $\mathbb{Z}_p$-universal and $\alpha_1 = \alpha_2 = 0$. If $\alpha_4 \geq 2$, then L $\mathbb{Z}_p$-universal implies that $\mathbb{Z}_p^\times \cup p\mathbb{Z}_p^\times \to T$ by Local Square Theorem and hence T is $\mathbb{Z}_p$-universal, which implies that either (3.3) or (3.4) holds by the preceding proposition. If $\alpha_4 = 0$ or 1, then either (3.3) or (3.5) holds. This completes the proof. □

Remark 3.1.7. The preceding proposition shows that any $\mathbb{Z}_p$-universal lattice of rank exceeding 4 over $\mathbb{Z}_p$ is split by a $\mathbb{Z}_p$-universal sublattice of rank at most 4. This in turn

[1] This statement is essentially (14) in Lemma 1 of [15].

guarantees the primitive $\mathbb{Z}_p$-universality of such a lattice, as was shown in Proposition 3.1.2. Also, as noted in the proof of Proposition 3.1.5 above, any $\mathbb{Z}_p$-universal lattice of rank less than 4 over $\mathbb{Z}_p$ is again primitively $\mathbb{Z}_p$-universal. However, as shown by Example 2.1.4, this is no longer the case for quaternary lattices. The following proposition shows that this is essentially the only situation in which there is a distinction between $\mathbb{Z}_p$-universality and primitive $\mathbb{Z}_p$-universality when p is odd.

Proposition 3.1.8. *Let L be a $\mathbb{Z}_p$-lattice with rk $L \geq 2$. If L is $\mathbb{Z}_p$-universal then L is primitively $\mathbb{Z}_p$-universal except when*

$$rk\ L = n = 4, \alpha_1 = \alpha_2 = 0, \alpha_3 = \alpha_4 = 1 \text{ and } (-\varepsilon_1\varepsilon_2, p)_p = (-\varepsilon_3\varepsilon_4, p)_p = -1.$$

In the exceptional case, L is not primitively $\mathbb{Z}_p$-universal.

Proof. Since L is $\mathbb{Z}_p$-universal, it must fall into one of the cases in Proposition 3.1.6. It was noted in the proof of that proposition that if $\alpha_1 = \alpha_2 = 0$ and either (3.3) or (3.4) holds, then L is primitively $\mathbb{Z}_p$-universal. So we need only consider the case when $\alpha_1 = \alpha_2 = 0$, and (3.3) and (3.4) fail but (3.5) holds. So $L \cong \langle \varepsilon_1, \varepsilon_2, p\varepsilon_3, p\varepsilon_4 \rangle$ with $(-\varepsilon_1\varepsilon_2, p)_p = -1$. If $(-\varepsilon_3\varepsilon_4, p)_p = 1$, then $K \cong \langle p\varepsilon_3, p\varepsilon_4 \rangle$ is isotropic and $p\mathbb{Z}_p \overset{*}{\to} K$ by Lemma 3.1.1 *ii*). Since $\mathbb{Z}_p^\times \overset{*}{\to} L$, by Lemma 3.1.1 *i*), it follows that L is primitively $\mathbb{Z}_p$-universal. That L is not primitively $\mathbb{Z}_p$-universal in the exceptional case follows exactly as in Example 2.1.4. □

3.2 $\mathbb{Z}_2$-LATTICES

This section is divided into four subsections. The first subsection gives a complete analysis of the $\mathbb{Z}_2$-universality and primitive $\mathbb{Z}_2$-universality of modular $\mathbb{Z}_2$-lattices of various ranks. The second subsection is devoted to identifying $\mathbb{Z}_2$-lattices that are $\mathbb{Z}_2$-universal, using basic computations. These results will play a key role in the proof of Proposition 3.2.12. We provide criteria to identify universal and primitively universal $\mathbb{Z}_2$-lattices in the third and fourth subsections, respectively.

Throughout this section L will denote a $\mathbb{Z}_2$-lattice. Recall that, since $\mathfrak{n}L = \mathbb{Z}_2$ whenever L is $\mathbb{Z}_2$-universal there are two cases to consider: $\mathfrak{s}L = \mathbb{Z}_2$ or $\mathfrak{s}L = \frac{1}{2}\mathbb{Z}_2$.

Note first that in order to prove that an integral $\mathbb{Z}_2$-lattice is $\mathbb{Z}_2$-universal, it suffices to show that it represents all units and elements of order one (i.e., elements of the set $2\mathbb{Z}_2^\times$). For if $\mathbb{Z}_2^\times \to L$ (or $2\mathbb{Z}_2^\times \to L$), then L represents all elements of $\mathbb{Z}_2$ of odd (or even) order. We will refer to a set of elements of $\mathbb{Z}_2$ as being *independent* if they are in distinct squareclasses. So, in order to prove that an integral $\mathbb{Z}_2$-lattice is $\mathbb{Z}_2$-universal, it suffices to show that L represents a set of four independent units and a set of four independent elements of $2\mathbb{Z}_2^\times$. Throughout this section, the letters ε, ε_i or δ will always denote elements of $\mathbb{Z}_2^\times$.

3.2.1 Modular $\mathbb{Z}_2$-lattices

In this subsection we determine which modular $\mathbb{Z}_2$-lattices are either $\mathbb{Z}_2$-universal or primitively $\mathbb{Z}_2$-universal. In particular, such a lattice L must have $nL = \mathbb{Z}$ and hence must be either an improper $\frac{1}{2}\mathbb{Z}_2$-modular lattice or a proper unimodular lattice.

Proposition 3.2.1. *Let L be an improper $\frac{1}{2}\mathbb{Z}_2$-modular $\mathbb{Z}_2$-lattice of rank n.*

i) If $n > 2$, then L is primitively $\mathbb{Z}_2$-universal.

ii) If $n = 2$, then L is primitively $\mathbb{Z}_2$-universal if and only if L is $\mathbb{Z}_2$-universal.

iii) If $n = 2$ and L is anisotropic, then $q^*(L) = \mathbb{Z}_2^\times$ and $q(L) \cap 2\mathbb{Z}_2^\times = \emptyset$.

Proof. These results follow immediately from (1.1) and Examples 2.1.1 and 2.1.2. □

Proposition 3.2.2. *Let L be a proper unimodular $\mathbb{Z}_2$-lattice of rank 2. Then L is not $\mathbb{Z}_2$-universal. Moreover,*

i) if $dL \equiv 3 \pmod{4\mathbb{Z}_2}$, then $\mathbb{Z}_2^\times \to L$ and $q(L) \cap 2\mathbb{Z}_2^\times = \emptyset$;

ii) if $dL \equiv 1 \pmod{4\mathbb{Z}_2}$, then $q(L) \cap \mathbb{Z}_2^\times \neq \emptyset$ and

$$q(L) \cap \mathbb{Z}_2^\times = \{\alpha \in \mathbb{Z}_2^\times : \alpha \equiv \varepsilon \pmod{4\mathbb{Z}_2}\}.$$

for any $\varepsilon \in q(L) \cap \mathbb{Z}_2^\times$.

Proof. i) Let $L \cong \langle \varepsilon_1, \varepsilon_2 \rangle$. Then it can be verified that $\varepsilon_1, \varepsilon_2, \varepsilon_1 + 4\varepsilon_2, \varepsilon_2 + 4\varepsilon_1$ form an independent set of four units represented by L. For example, suppose that $\varepsilon_2 \equiv \varepsilon_1 + 4\varepsilon_2 \,(\mathrm{mod}\ 8\mathbb{Z}_2)$. Then $-3\varepsilon_2 \equiv \varepsilon_1 \,(\mathrm{mod}\ 8\mathbb{Z}_2)$, and it would follow that $\varepsilon_1\varepsilon_2 \equiv 5 \,(\mathrm{mod}\ 8\mathbb{Z}_2)$, contrary to the assumption that $dL \equiv 3 \,(\mathrm{mod}\ 4\mathbb{Z}_2)$. The other verifications are similar. For the final assertion, it suffices to note that for any $x, y \in \mathbb{Z}_2^\times$, $\varepsilon_1 x^2 + \varepsilon_2 y^2 \equiv \varepsilon_1 + \varepsilon_2 \,(\mathrm{mod}\ 8\mathbb{Z}_2)$ and $\varepsilon_1 + \varepsilon_2 \equiv 0 \,(\mathrm{mod}\ 4\mathbb{Z}_2)$ since $\varepsilon_2 \equiv -\varepsilon_1 \,(\mathrm{mod}\ 4\mathbb{Z}_2)$ in this case.

ii) In this case, the underlying space V is anisotropic since $dV \neq -1$. Since L is a $\mathbb{Z}_2$-maximal lattice on V, it follows from [14, Theorem 91:1] that $q(L) = q(V) \cap \mathbb{Z}_2$. Let $L \cong \langle \varepsilon_1, \varepsilon_2 \rangle$. For any $\alpha \in \mathbb{Z}_2^\times$, by comparing the Hasse invariants of the $\mathbb{Q}_2$-spaces $\langle \varepsilon_1, \varepsilon_2 \rangle$ and $\langle \alpha, \varepsilon_1\varepsilon_2 \rangle$, we see that $\alpha \to V$ if and only if $(\alpha, -\varepsilon_1\varepsilon_2)_2 = (\varepsilon_1, \varepsilon_2)_2$. Since $-\varepsilon_1\varepsilon_2 \equiv 3 \,(\mathrm{mod}\ 4\mathbb{Z}_2)$, the value of the symbol $(\alpha, -\varepsilon_1\varepsilon_2)_2$ is determined by the congruence of α modulo $4\mathbb{Z}_2$, which verifies the assertion. □

Corollary 3.2.3. *Let $L \cong M \perp K$, with M proper unimodular of rank 2 and $\mathfrak{s}K \subseteq 2\mathbb{Z}_2$. If L is $\mathbb{Z}_2$-universal, then $\mathfrak{n}K = 2\mathbb{Z}_2$.*

Proof. If $dM \equiv 3 \,(\mathrm{mod}\ 4\mathbb{Z}_2)$ and $\mathfrak{n}K \subseteq 4\mathbb{Z}_2$, then it follows from Proposition 3.2.2 i) that $q(L) \cap 2\mathbb{Z}_2^\times = \emptyset$, and so L is not $\mathbb{Z}_2$-universal. So consider the case $dM \equiv 1 \,(\mathrm{mod}\ 4\mathbb{Z}_2)$ and let $\varepsilon \in q(M) \cap \mathbb{Z}_2^\times$. Suppose that $\mathfrak{n}K \subseteq 4\mathbb{Z}_2$ and λ is any unit represented by L. Then $\lambda = \mu + \kappa$, where $\mu \in q(M) \cap \mathbb{Z}_2^\times$ and $\kappa \in q(K) \subseteq 4\mathbb{Z}_2$. By Proposition 3.2.2 ii), $\lambda \equiv \varepsilon \,(\mathrm{mod}\ 4\mathbb{Z}_2)$. Thus, $\mathbb{Z}_2^\times \not\to L$ and L is not $\mathbb{Z}_2$-universal. □

Proposition 3.2.4. *Let L be a unimodular $\mathbb{Z}_2$-lattice of rank 3. Then the following are equivalent:*

i) L is $\mathbb{Z}_2$-universal;

ii) L is primitively $\mathbb{Z}_2$-universal;

iii) $L \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ and there exist $i, j \in \{1, 2, 3\}$ such that $\varepsilon_i \equiv -\varepsilon_j \,(\mathrm{mod}\ 4\mathbb{Z}_2)$.

Proof. ii) $\implies$ i) is clear, and iii) $\implies$ ii) follows from Proposition 3.2.2 i) and Lemma 1.3.3. So it remains to prove i) $\implies$ iii). For this, suppose that iii) is not true. So $L \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ with $\varepsilon_i \equiv \varepsilon_j \pmod{4\mathbb{Z}_2}$ for all $1 \leq i, j \leq 3$. Let $\lambda \in q(L) \cap \mathbb{Z}_2^\times$. So $\lambda = \sum_{i=1}^{3} a_i^2 \varepsilon_i$ with $a_i \in \mathbb{Z}_2$ and either one or all three of the a_i's units. If exactly one $a_i \in \mathbb{Z}_2^\times$, then $\lambda \equiv \varepsilon_i \pmod{4\mathbb{Z}_2}$; if all $a_i \in \mathbb{Z}_2^\times$, then $\lambda \equiv \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \pmod{8\mathbb{Z}_2}$. Hence, L can represent at most three squareclasses of units and so L is not $\mathbb{Z}_2$-universal. This completes the proof. □

Proposition 3.2.5. *Let L be a proper unimodular $\mathbb{Z}_2$-lattice of rank 4. Then*

i) L is $\mathbb{Z}_2$-universal;

ii) L is primitively $\mathbb{Z}_2$-universal if and only if $\{4, 8\} \subseteq q^*(L)$.

Proof. There is a splitting $L \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle$. If $\varepsilon_i \equiv -\varepsilon_j \pmod{4\mathbb{Z}_2}$ for some $i \neq j$, then L is primitively $\mathbb{Z}_2$-universal by Proposition 3.2.4 and Lemma 1.3.3.

i) It remains to show that L is $\mathbb{Z}_2$-universal when $\varepsilon_i \equiv \varepsilon_j \pmod{4\mathbb{Z}_2}$ for all $1 \leq i, j \leq 4$. By scaling L by a unit if necessary, it suffices to consider the possibilities $L \cong \langle 1, 1, a, b \rangle$, where a, b equal 1 or 5. In these cases, it is routine to show that L represents the four squareclasses of units and the four squareclasses of twice units.

ii) It suffices to prove the "*if*" statement. So assume that L is not primitively $\mathbb{Z}_2$-universal. So $\varepsilon_i \equiv \varepsilon_j \pmod{4\mathbb{Z}_2}$ for all $1 \leq i, j \leq 4$. Let $\lambda \in q^*(L)$. So $\lambda = \sum_{i=1}^{4} a_i^2 \varepsilon_i$ with all $a_i \in \mathbb{Z}_2$ and at least one $a_i \in \mathbb{Z}_2^\times$. If exactly one or three of the a_i's are in $\mathbb{Z}_2^\times$, then $\lambda \in \mathbb{Z}_2^\times$; if exactly two of the a_i's are in $\mathbb{Z}_2^\times$, then $\lambda \in 2\mathbb{Z}_2^\times$, since $\varepsilon_i \equiv \varepsilon_j \pmod{4\mathbb{Z}_2}$. So if $\lambda \in 4\mathbb{Z}_2$, it must be that $a_i \in \mathbb{Z}_2^\times$ for all $i = 1, \ldots, 4$ and it follows that $\lambda \equiv \varepsilon_1 + \cdots + \varepsilon_4 \pmod{8\mathbb{Z}_2}$. Moreover $\varepsilon_1 + \cdots + \varepsilon_4 \in 4\mathbb{Z}_2$ since $\varepsilon_i \equiv \varepsilon_j \pmod{4\mathbb{Z}_2}$. If $\varepsilon_1 + \cdots + \varepsilon_4 \in 4\mathbb{Z}_2^\times$, then $8 \overset{*}{\not\to} L$. If $\varepsilon_1 + \cdots + \varepsilon_4 \in 8\mathbb{Z}_2$, then $4 \overset{*}{\not\to} L$. This completes the proof. □

Proposition 3.2.6. *Let L be a unimodular $\mathbb{Z}_2$-lattice of rank exceeding 4. Then L is primitively $\mathbb{Z}_2$-universal.*

Proof. Follows from Proposition 3.2.5 *i*) and Lemma 1.3.3. □

Remark 3.2.7. Corollary 2 of [4] follows immediately from Lemma 3.1.1 *ii*), Proposition 3.2.5 *ii*) and Proposition 3.2.6.

3.2.2 Basic computations and observations on $\mathbb{Z}_2$-universal lattices

The goal of this subsection is to build up an inventory of integral $\mathbb{Z}_2$-lattices that are $\mathbb{Z}_2$-universal. We begin by identifying lattices that represent all units or twice units.

Lemma 3.2.8. *All $\mathbb{Z}_2$-lattices of the following types represent $\mathbb{Z}_2^\times$:*

i) $\langle \varepsilon_1, 2\varepsilon_2, \lambda\varepsilon_3 \rangle$, *where* $\lambda = 1$ *or* 4*;*

ii) $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \lambda\varepsilon_4 \rangle$, *where* $\lambda = 1$ *or* 4*;*

iii) $\langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3, 2\varepsilon_4 \rangle$.

Proof. i) It suffices to prove the result for the lattice $L \cong \langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3 \rangle$. In this case, it can be routinely verified that $\varepsilon_1, \varepsilon_1 + 2\varepsilon_2, \varepsilon_1 + 4\varepsilon_3, \varepsilon_1 + 2\varepsilon_2 + 4\varepsilon_3$ form an independent set of four units represented by L.

ii) It suffices to prove the result for the lattice $L \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3, 4\varepsilon_4 \rangle$. By Proposition 3.2.4, it suffices to consider the case $\varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon_3 \pmod{4\mathbb{Z}_2}$. Then $\varepsilon_2 + \varepsilon_3 \in 2\mathbb{Z}_2^\times$ and it follows that $\varepsilon_1, \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \varepsilon_1 + 4\varepsilon_4, \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + 4\varepsilon_4$ form an independent set of four units represented by L.

iii) Let $L \cong \langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3, 2\varepsilon_4 \rangle$. At least two of $\varepsilon_2, \varepsilon_3, \varepsilon_4$ are congruent modulo $4\mathbb{Z}_2$; without loss of generality, by re-indexing if necessary, we may assume that $\varepsilon_3 \equiv \varepsilon_4 \pmod{4\mathbb{Z}_2}$. If also $\varepsilon_2 \equiv \varepsilon_3 \pmod{4\mathbb{Z}_2}$, then $\varepsilon_2 + \varepsilon_3 \in 2\mathbb{Z}_2^\times$ and $\varepsilon_3 + \varepsilon_4 \in 2\mathbb{Z}_2^\times$. From this it follows that $\varepsilon_1, \varepsilon_1 + 2\varepsilon_2, \varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3, \varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 + 2\varepsilon_4$ form an independent set of four units represented by L. Otherwise, $\varepsilon_2 \equiv -\varepsilon_3 \pmod{4\mathbb{Z}_2}$. Then $\varepsilon_1, \varepsilon_1 + 2\varepsilon_2, \varepsilon_1 + 2\varepsilon_3, \varepsilon_1 + 2\varepsilon_3 + 2\varepsilon_4$ form an independent set of four units represented by L. □

Lemma 3.2.9. *All $\mathbb{Z}_2$-lattices of the following types represent $2\mathbb{Z}_2^\times$:*

i) $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$*;*

ii) $\langle \varepsilon_1, 2\varepsilon_2, \lambda\varepsilon_3 \rangle$, *where* $\lambda = 2$ *or* 8*;*

iii) $\langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3, \lambda\varepsilon_4 \rangle$, *where* $\lambda = 1$ *or* 4*;*

iv) $\widehat{\mathbb{A}} \perp \langle 2\varepsilon_1, \lambda\varepsilon_2 \rangle$, *where* $\lambda = 2, 4, 8$.

Proof. *i*) By Proposition 3.2.4, it suffices to consider the case $\varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon_3 \pmod{4\mathbb{Z}_2}$. Under this condition, we have $\varepsilon_i + \varepsilon_j \in 2\mathbb{Z}_2^\times$ for all $1 \leq i, j \leq 3$. Then it can be shown that $\varepsilon_1 + \varepsilon_2, \varepsilon_1 + 9\varepsilon_2, \varepsilon_1 + \varepsilon_2 + 4\varepsilon_3, \varepsilon_1 + 9\varepsilon_2 + 4\varepsilon_3$ form an independent set of four elements of $2\mathbb{Z}_2^\times$ represented by L.

ii) It suffices to consider $L \cong \langle \varepsilon_1, 2\varepsilon_2, 8\varepsilon_3 \rangle$. Let $\lambda \in \mathbb{Z}_2^\times$. By Lemma 3.2.8, there exist $a_1, a_2, a_3 \in \mathbb{Z}_2$ such that $\lambda = a_2^2\varepsilon_2 + 2a_1^2\varepsilon_1 + 4a_3^2\varepsilon_3$. So $2\lambda = (2a_1)^2\varepsilon_1 + a_2^2(2\varepsilon_2) + a_3^2(8\varepsilon_3) \in q(L)$.

iii) It suffices to consider $L \cong \langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3, 4\varepsilon_4 \rangle$. Let $\lambda \in \mathbb{Z}_2^\times$. By Lemma 3.2.8, there exist $a_1, a_2, a_3, a_4 \in \mathbb{Z}_2$ such that $\lambda = a_2^2\varepsilon_2 + 2a_1^2\varepsilon_1 + 2a_3^2\varepsilon_3 + 2a_4^2\varepsilon_4$. Then $2\lambda = (2a_1)^2\varepsilon_1 + a_2^2(2\varepsilon_2) + a_3^2(4\varepsilon_3) + a_4^2(4\varepsilon_4) \in q(L)$.

iv) Consider first $L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon_1, 4\varepsilon_2 \rangle$. Let $\lambda \in \mathbb{Z}_2^\times$. If $\lambda - \varepsilon_1 \in 8\mathbb{Z}_2$, then $2\lambda \in 2\varepsilon_1(\mathbb{Z}_2^\times)^2$ and $2\lambda \in q(L)$. If $\lambda - \varepsilon_1 \in 2\mathbb{Z}_2^\times$, then $2\lambda - 2\varepsilon_1 \in 4\mathbb{Z}_2^\times \subseteq q(\widehat{\mathbb{A}})$ and $2\lambda \in q(L)$. If $\lambda - \varepsilon_1 \in 4\mathbb{Z}_2$, then $2\lambda - 2\varepsilon_1 \in 8\mathbb{Z}_2$. So $2\lambda - 2\varepsilon_1 - 4\varepsilon_2 \in 4\mathbb{Z}_2^\times \subseteq q(\widehat{\mathbb{A}})$. So in all cases, $2\lambda \in q(L)$. It remains to consider $L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon_1, 8\varepsilon_2 \rangle$. Since $\mathbb{Z}_2^\times \to \widehat{\mathbb{A}}$, the result in this case follows from *ii*). □

Combining results from the preceding two lemmas and results for unimodular lattices from the previous subsection, and applying Lemma 1.3.3 where necessary, we obtain the $\mathbb{Z}_2$-universal lattices in the following lemma. We note that those in *i*) through *iv*) can be obtained from [15, Lemma 1]; however, that lemma is stated without proof and we have chosen to include the arguments here for the sake of completeness.

Lemma 3.2.10. *All $\mathbb{Z}_2$-lattices of the following types are $\mathbb{Z}_2$-universal.*

i) $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \lambda\varepsilon_4 \rangle$, *where* $\lambda = 1, 2, 4$;

ii) $\langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3, \lambda\varepsilon_4 \rangle$, *where* $\lambda = 2, 4, 8$;

iii) $\langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3, \lambda\varepsilon_4 \rangle$, *where* $\lambda = 2, 4$;

iv) $\langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3, \lambda\varepsilon_4 \rangle$, *where* $\lambda = 4, 8$;

v) $\widehat{\mathbb{A}} \perp \langle \varepsilon \rangle$;

vi) $\widehat{\mathbb{A}} \perp \langle 2\varepsilon_1, \lambda\varepsilon_2\rangle$, *where* $\lambda = 2, 4, 8$.

For the proof of Proposition 3.2.12, it will also be useful to identify several lattices that fail to represent $\mathbb{Z}_2^\times$ or $2\mathbb{Z}_2^\times$.

Lemma 3.2.11. $\mathbb{Z}_2^\times \not\to \langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3\rangle$, $2\mathbb{Z}_2^\times \not\to \langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3\rangle$, *and* $2\mathbb{Z}_2^\times \not\to \widehat{\mathbb{A}} \perp \langle 2\varepsilon\rangle$.

Proof. First consider $L \cong \langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3\rangle$. If $\lambda \in \mathbb{Z}_2^\times \cap q(L)$, then there exist $a_1 \in \mathbb{Z}_2^\times$ and $a_2, a_3 \in \mathbb{Z}_2$ such that $\lambda = a_1^2\varepsilon_1 + 2a_2^2\varepsilon_2 + 2a_3^2\varepsilon_3$. If $a_2, a_3 \in 2\mathbb{Z}_2$, then $\lambda \equiv \varepsilon_1 \pmod{8\mathbb{Z}_2}$. If $a_2 \in \mathbb{Z}_2^\times$ and $a_3 \in 2\mathbb{Z}_2$, then $\lambda \equiv \varepsilon_1 + 2\varepsilon_2 \pmod{8\mathbb{Z}_2}$. If $a_2 \in 2\mathbb{Z}_2$ and $a_3 \in \mathbb{Z}_2^\times$, then $\lambda \equiv \varepsilon_1 + 2\varepsilon_3 \pmod{8\mathbb{Z}_2}$. If $a_2, a_3 \in \mathbb{Z}_2^\times$, then $\lambda \equiv \varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 \pmod{8\mathbb{Z}_2}$. When $\varepsilon_2 \equiv \varepsilon_3 \pmod{4\mathbb{Z}_2}$, it follows that $\varepsilon_1 + 2\varepsilon_2 \equiv \varepsilon_1 + 2\varepsilon_3 \pmod{8\mathbb{Z}_2}$. Otherwise $\varepsilon_2 \equiv -\varepsilon_3 \pmod{4\mathbb{Z}_2}$, and $\varepsilon_1 \equiv \varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 \pmod{4\mathbb{Z}_2}$. So in either case, L represents at most three squareclasses of units. Hence, $\mathbb{Z}_2^\times \not\to L$.

Next consider $L \cong \langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3\rangle$. If $\lambda \in \mathbb{Z}_2^\times$ is such that $2\lambda \in q(L)$, then there exist $a_1 = 2b_1 \in 2\mathbb{Z}_2$ and $a_2, a_3 \in \mathbb{Z}_2$ such that $2\lambda = a_1^2\varepsilon_1 + 2a_2^2\varepsilon_2 + 4a_3^2\varepsilon_3$. From this it follows that $\lambda = 2b_1^2\varepsilon_1 + a_2^2\varepsilon_2 + 2a_3^2\varepsilon_3$. So $2\mathbb{Z}_2^\times \to L$ would imply that $\mathbb{Z}_2^\times \to \langle \varepsilon_2, 2\varepsilon_1, 2\varepsilon_3\rangle$, which we have just shown to be impossible.

Finally consider $L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon\rangle$. Suppose $2\varepsilon + 8 \to L$. Then there exist $v \in \widehat{\mathbb{A}}$ and $\mu \in \mathbb{Z}_2$ such that $2\varepsilon + 8 = q(v) + 2\mu^2\varepsilon$. It must be that $v \in 2\widehat{\mathbb{A}}$, since otherwise $q(v) \in \mathbb{Z}_2^\times$ and hence $q(v) + 2\mu^2\varepsilon \in \mathbb{Z}_2^\times$. Also, $\mu \in \mathbb{Z}_2^\times$, since otherwise $q(v) + 2\mu^2\varepsilon \in 4\mathbb{Z}_2$. So $\mu^2 \equiv 1 \pmod{8\mathbb{Z}_2}$; that is, there exists $\xi \in \mathbb{Z}_2$ such that $1 - \mu^2 = 8\xi$. Thus,

$$q(v) - 8 = 2\varepsilon(1 - \mu^2) = 16\varepsilon\xi.$$

But $\mathrm{ord}_2 q(v)$ is even, so that $\mathrm{ord}_2(q(v) - 8) = 2$ or 3, a contradiction. So $2\varepsilon + 8 \not\to L$, and the assertion is proved. □

We will now prove the 2-adic analogue of Proposition 3.1.2.

Proposition 3.2.12. *Let L be an integral $\mathbb{Z}_2$-lattice of rank $n \geq 5$. If L is $\mathbb{Z}_2$-universal, then L is primitively $\mathbb{Z}_2$-universal.*

Proof. Since L is $\mathbb{Z}_2$-universal, it must be that $\mathfrak{n}L = \mathbb{Z}_2$ and so $\mathfrak{s}L = \mathbb{Z}_2$ or $\frac{1}{2}\mathbb{Z}_2$. We first consider the case when $\mathfrak{s}L = \frac{1}{2}\mathbb{Z}_2$. So $L_{(-1)} \neq 0$ and $L \cong L_{(-1)} \perp K$, where $\mathfrak{s}K \subseteq \mathbb{Z}_2$. If $r_{-1} > 2$, or $r_{-1} = 2$ and $L_{(-1)}$ is isotropic, then L is split by $\widehat{\mathbb{H}}$ and it follows that L is primitively $\mathbb{Z}_2$-universal. So we need only consider further those lattices L for which there is a splitting of the type $L \cong \widehat{\mathbb{A}} \perp K$, where $\mathfrak{s}K \subseteq \mathbb{Z}_2$. If $\mathfrak{n}K \subseteq 4\mathbb{Z}_2$, then $\widehat{\mathbb{A}} \perp K$ cannot represent any element of $2\mathbb{Z}_2^{\times}$ and is thus not $\mathbb{Z}_2$-universal. So $2\mathbb{Z}_2 \subseteq \mathfrak{n}K$, and it follows that $\mathfrak{s}K = \mathbb{Z}_2$ or $2\mathbb{Z}_2$. If $\mathfrak{s}K = \mathbb{Z}_2$, then L primitively $\mathbb{Z}_2$-universal follows from Example 2.1.5, Lemma 3.2.10 v) and Lemma 1.3.2. So we are left to further consider only those lattices for which $\mathfrak{s}K = \mathfrak{n}K = 2\mathbb{Z}_2$. So

$$L \cong \widehat{\mathbb{A}} \perp K \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon \rangle \perp K', \text{ with } \mathfrak{s}K' \subseteq 2\mathbb{Z}_2.$$

By Lemma 3.2.11, the sublattice $\widehat{\mathbb{A}} \perp \langle 2\varepsilon \rangle$ does not represent all elements of $2\mathbb{Z}_2^{\times}$. Since L is $\mathbb{Z}_2$-universal, it follows from Lemma 1.3.1 that $\mathfrak{n}K' \supseteq 8\mathbb{Z}_2$. Hence,

$$8\mathbb{Z}_2 \subseteq \mathfrak{n}K' \subseteq \mathfrak{s}K' \subseteq 2\mathbb{Z}_2.$$

If $\mathfrak{n}K' = \mathfrak{s}K' = 2^t\mathbb{Z}_2$ for $t = 1, 2, 3$, then L is split by a sublattice $\widehat{\mathbb{A}} \perp \langle 2\varepsilon_1, 2^t\varepsilon_2 \rangle$. All such lattices are $\mathbb{Z}_2$-universal by Lemma 3.2.10 v), and it follows from Lemma 1.3.2 that L is primitively $\mathbb{Z}_2$-universal, since $n \geq 5$. Finally, consider the case when $\mathfrak{n}K' = 8\mathbb{Z}_2 = 2\mathfrak{s}K'$. Then L is split by $\widehat{\mathbb{A}} \perp \langle 2\varepsilon \rangle \perp \mathcal{P}$, where $\mathcal{P} \cong \left(\begin{smallmatrix} 8 & 4 \\ 4 & 8 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 4 \\ 4 & 0 \end{smallmatrix}\right)$. Note first that $8\mathbb{Z}_2^{\times} \xrightarrow{*} \mathcal{P}$. If $\lambda \in 4\mathbb{Z}_2^{\times}$, then for any $v \overset{*}{\in} \mathcal{P}$, $\lambda - q(v) \in 4\mathbb{Z}_2^{\times} \to \widehat{\mathbb{A}}$; hence, $\lambda \xrightarrow{*} \widehat{\mathbb{A}} \perp \mathcal{P}$. If $\lambda \in 16\mathbb{Z}_2$, then $\lambda - 8\varepsilon \in 8\mathbb{Z}_2^{\times} \xrightarrow{*} \mathcal{P}$; hence, $\lambda \xrightarrow{*} \langle 2\varepsilon \rangle \perp \mathcal{P}$. This completes the argument in the case that $\mathfrak{s}L = \frac{1}{2}\mathbb{Z}_2$.

Now we consider the case when $\mathfrak{s}L = \mathbb{Z}_2$. Then $r_{-1} = 0$ and $r_0 > 0$. We break down the argument according to the size of r_0.

$r_0 \geq 4$: L is split by a proper unimodular sublattice of rank 4, which is $\mathbb{Z}_2$-universal by Proposition 3.2.5. Since $n \geq 5$, it follows from Lemma 1.3.2 that L is primitively $\mathbb{Z}_2$-universal.

$r_0 = 3$: $L \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle \perp K$, where $\mathfrak{s}K \subseteq 2\mathbb{Z}_2$. Since $2\mathbb{Z}_2^\times \to \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ by Lemma 3.2.9, we need to consider only the case when $\mathbb{Z}_2^\times \not\to \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$, since otherwise $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ is $\mathbb{Z}_2$-universal and L is primitively $\mathbb{Z}_2$-universal by Lemma 1.3.2. Since L is $\mathbb{Z}_2$-universal, we must then have $\mathfrak{n}K \supseteq 4\mathbb{Z}_2$, by Lemma 1.3.1. So we have

$$4\mathbb{Z}_2 \subseteq \mathfrak{n}K \subseteq \mathfrak{s}K \subseteq 2\mathbb{Z}_2.$$

If $\mathfrak{n}K = \mathfrak{s}K = 2^t\mathbb{Z}_2$, for $t = 1, 2$, then L is split by $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, 2^t\varepsilon_4 \rangle$ which is $\mathbb{Z}_2$-universal. Otherwise, $\mathfrak{n}K = 4\mathbb{Z}_2 = 2\mathfrak{s}K$, in which case L is split by $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle \perp \mathcal{P}$, where $\mathcal{P} \cong \left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right)$. Then $4\mathbb{Z}_2^\times \overset{*}{\to} \mathcal{P}$. If $\lambda \in 8\mathbb{Z}_2$, then $\lambda - 4\varepsilon_1 \in 4\mathbb{Z}_2^\times$ and so $\lambda \overset{*}{\to} \langle \varepsilon_1 \rangle \perp \mathcal{P}$. That completes this subcase.

$r_0 = 2$: $L \cong \langle \varepsilon_1, \varepsilon_2 \rangle \perp K$, with $\mathfrak{s}K \subseteq 2\mathbb{Z}_2$. Since L is $\mathbb{Z}_2$-universal, we have $\mathfrak{n}K = \mathfrak{s}K = 2\mathbb{Z}_2$ by Corollary 3.2.3. So $r_1 > 0$ and $L_{(1)}$ is proper. So if $r_1 \geq 2$, L is split by a sublattice $\langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3, 2\varepsilon_4 \rangle$, which is $\mathbb{Z}_2$-universal by Lemma 3.2.10 *ii*) and the conclusion follows. So we further consider the case $r_1 = 1$; that is,

$$L \cong \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle \perp K', \text{ with } \mathfrak{s}K' \subseteq 4\mathbb{Z}_2.$$

We further assume that $2\mathbb{Z}_2^\times \not\to \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$, since otherwise $\langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$ is $\mathbb{Z}_2$-universal and there is nothing to prove. So, since L is $\mathbb{Z}_2$-universal, we must have $\mathfrak{n}K' \not\subseteq 16\mathbb{Z}_2$. So

$$8\mathbb{Z}_2 \subseteq \mathfrak{n}K' \subseteq \mathfrak{s}K' \subseteq 4\mathbb{Z}_2.$$

If $\mathfrak{n}K' = \mathfrak{s}K' = 2^t\mathbb{Z}_2$, with $t = 2, 3$, then L is split by a $\mathbb{Z}_2$-universal lattice of the type $\langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3, 2^t\varepsilon_4 \rangle$. Otherwise, $\mathfrak{n}K' = 8\mathbb{Z}_2 = 2\mathfrak{s}K'$ and L is split by a lattice of the type $\langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle \perp \mathcal{P}$, with $\mathcal{P} \cong \left(\begin{smallmatrix} 8 & 4 \\ 4 & 8 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 4 \\ 4 & 0 \end{smallmatrix}\right)$. So $8\mathbb{Z}_2^\times \overset{*}{\to} \mathcal{P}$; let $v \overset{*}{\in} \mathcal{P}$ such that $q(v) = 8$. If $\lambda \in 4\mathbb{Z}_2^\times$, then $\lambda - q(v) \in 4\mathbb{Z}_2^\times \to \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$ (since $\mathbb{Z}_2^\times \to \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$), and $\lambda \overset{*}{\to} L$. Finally if $\lambda \in 16\mathbb{Z}_2$, then $\lambda - 2^2 \cdot 2\varepsilon_3 \in 8\mathbb{Z}_2^\times$ and so $\lambda - 2^2 \cdot 2\varepsilon_3 \overset{*}{\to} \mathcal{P}$ and $\lambda \overset{*}{\to} \langle 2\varepsilon_3 \rangle \perp \mathcal{P}$.

$r_0 = 1$: $L \cong \langle \varepsilon_1 \rangle \perp K$, with $\mathfrak{s}K \subseteq 2\mathbb{Z}_2$. Since L is $\mathbb{Z}_2$-universal, $\mathfrak{n}K = \mathfrak{s}K = 2\mathbb{Z}_2$ (since otherwise $q(L) \cap 2\mathbb{Z}_2^\times = \emptyset$). So $r_1 > 0$ and $L_{(1)}$ is proper. We consider the various possibilities for r_1. If $r_1 \geq 3$, then L is split by $\langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3, 2\varepsilon_4 \rangle$ which is $\mathbb{Z}_2$-universal. If $r_1 = 2$, then $L \cong \langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3 \rangle \perp K'$, with $\mathfrak{s}K' \subseteq 4\mathbb{Z}_2$. Since L is $\mathbb{Z}_2$-universal and $\mathbb{Z}_2^\times \nrightarrow \langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3 \rangle$ by Lemma 3.2.11, it follows from Lemma 1.3.1 that $\mathfrak{n}K' = \mathfrak{s}K' = 4\mathbb{Z}_2$. So $r_2 > 0$ and $L_{(2)}$ is proper, so L is split by $\langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3, 4\varepsilon_4 \rangle$, which is $\mathbb{Z}_2$-universal. The only remaining case is when $r_1 = 1$. Then $L \cong \langle \varepsilon_1, 2\varepsilon_2 \rangle \perp K$, with $\mathfrak{s}K \subseteq 4\mathbb{Z}_2$. Since L is $\mathbb{Z}_2$-universal and $\mathbb{Z}_2^\times \nrightarrow \langle \varepsilon_1, 2\varepsilon_2 \rangle$ by Lemma 3.2.11, it follows from Lemma 1.3.1 that $\mathfrak{n}K = \mathfrak{s}K = 4\mathbb{Z}_2$. So $r_2 > 0$ and $L_{(2)}$ is proper. If $r_2 \geq 2$, then L is split by $\langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3, 4\varepsilon_4 \rangle$, which is $\mathbb{Z}_2$-universal. Finally, it remains to consider the subcase when $r_2 = 1$. Then $L \cong \langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3 \rangle \perp K''$, with $\mathfrak{s}K'' \subseteq 8\mathbb{Z}_2$. Since L is $\mathbb{Z}_2$-universal and $2\mathbb{Z}_2^\times \nrightarrow \langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3 \rangle$ by Lemma 3.2.11, it follows from Lemma 1.3.1 that $\mathfrak{n}K' = \mathfrak{s}K' = 8\mathbb{Z}_2$. So $r_3 > 0$ and $L_{(3)}$ is proper, and L is split by $\langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3, 8\varepsilon_4 \rangle$, which is $\mathbb{Z}_2$-universal. □

3.2.3 Universality criterion on $\mathbb{Z}_2$-lattices

Recall that an arbitrary $\mathbb{Z}_2$-lattice L has a Jordan splitting

$$L \cong L_1 \perp L_2 \perp \ldots \perp L_t, \tag{3.6}$$

where each L_i is $\mathfrak{s}L_i$-modular, and $\mathfrak{s}L_t \subset \ldots \subset \mathfrak{s}L_2 \subset \mathfrak{s}L_1$.[2] We will refer to L_1 as a leading Jordan component of L. Note that $\mathfrak{n}L = \mathfrak{n}L_1$ and $\mathfrak{s}L = \mathfrak{s}L_1$. While the Jordan components themselves are not unique, the number t and the ideals $\mathfrak{n}L_i$ and $\mathfrak{s}L_i$ are invariants of the lattice. A component L_i is said to be either proper or improper depending upon whether $\mathfrak{n}L_i = \mathfrak{s}L_i$ or $\mathfrak{n}L_i = 2\mathfrak{s}L_i$. The component L_i has an orthogonal basis if and only if L_i is proper (see, e.g., [14, 93:15]).

We first make some observations regarding the case $\mathfrak{s}L = \frac{1}{2}\mathbb{Z}_2$. In this case the leading Jordan component L_1 of L is improper $\frac{1}{2}\mathbb{Z}_2$-modular. So

$$L_1 \cong \widehat{\mathbb{H}} \perp \ldots \perp \widehat{\mathbb{H}} \perp \mathbb{P},$$

[2]Note that we have diverged here from the subscripting notation used so far, instead adopting the more standard convention used in [10] and [14].

where $\mathbb{P} \cong \widehat{\mathbb{H}}$ or $\widehat{\mathbb{A}}$, by (1.1). Here $\widehat{\mathbb{H}}$ and $\widehat{\mathbb{A}}$ are binary lattices corresponding to the quadratic forms xy and $x^2 + xy + y^2$, respectively. Since $q^*(\widehat{\mathbb{H}}) = \mathbb{Z}_2$, L is primitively $\mathbb{Z}_2$-universal whenever L is split by $\widehat{\mathbb{H}}$. So it will only be necessary to analyze further the lattices in which $L_1 \cong \widehat{\mathbb{A}}$. In that case $q^*(L_1) = q^*(\widehat{\mathbb{A}}) = \mathbb{Z}_2^\times$. Also, $q(L_1) = q(\widehat{\mathbb{A}}) = \{a \in \mathbb{Z}_2 : \text{ord}_2\, a \text{ is even}\}$.

Next we will discuss the case $\mathfrak{s}L = \mathbb{Z}_2$. Since L_1 is proper unimodular in this case, there exists a unit ε_1 such that $\langle \varepsilon_1 \rangle$ splits L; say $L \cong \langle \varepsilon_1 \rangle \perp K$. If $\mathfrak{n}K \subseteq 4\mathbb{Z}_2$, then L represents at most half of the units of $\mathbb{Z}_2$ (those congruent to ε_1 modulo $4\mathbb{Z}_2$), and so L is not $\mathbb{Z}_2$-universal. Hence in order for L to be $\mathbb{Z}_2$-universal we must have

$$2\mathbb{Z}_2 \subseteq \mathfrak{n}K \subseteq \mathfrak{s}K \subseteq \mathbb{Z}_2.$$

Suppose first that $\mathfrak{s}K = \mathbb{Z}_2$. Then since L_1 is proper, we have $rk\ L_1 \geq 2$, and $L \cong \langle \varepsilon_1, \varepsilon_2 \rangle \perp M$, for some ε_2. Again if $\mathfrak{n}M \subseteq 4\mathbb{Z}_2$, L fails to be $\mathbb{Z}_2$-universal, by Lemma 3.2.2. So for L to be $\mathbb{Z}_2$-universal we must have

$$2\mathbb{Z}_2 \subseteq \mathfrak{n}M \subseteq \mathfrak{s}M \subseteq \mathbb{Z}_2.$$

If $\mathfrak{s}M = \mathbb{Z}_2$, then $rk\ L_1 \geq 3$ and L is split by $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ for some ε_3. Otherwise, $\mathfrak{n}M = \mathfrak{s}M = 2\mathbb{Z}_2$, and it follows that L_2 is proper and L is split by $\langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$ for some ε_3.

If it is not the case that $\mathfrak{s}K = \mathbb{Z}_2$, then $\mathfrak{n}K = \mathfrak{s}K = 2\mathbb{Z}_2$. If this happens L_2 is proper and L is split by $\langle \varepsilon_1, 2\varepsilon_2 \rangle$ for some ε_2; say $L \cong \langle \varepsilon_1, 2\varepsilon_2 \rangle \perp N$. Now Lemma 3.2.11 implies $\mathbb{Z}_2^\times \not\to L$, so it must be that

$$4\mathbb{Z}_2 \subseteq \mathfrak{n}N \subseteq \mathfrak{s}N \subseteq 2\mathbb{Z}_2,$$

whenever L is $\mathbb{Z}_2$-universal (since $\mathbb{Z}_2^\times \not\to L$ if $\mathfrak{n}N \subseteq 8\mathbb{Z}_2$, by the Local Square Theorem). If $\mathfrak{s}N = 2\mathbb{Z}_2$, then $rk\ L_2 \geq 2$ and L is split by $\langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3 \rangle$ for some ε_3. Otherwise $\mathfrak{n}N = \mathfrak{s}N = 4\mathbb{Z}_2$, from which it then follows that L_3 is proper and L is split by $\langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3 \rangle$ for some ε_3.

We summarize the preceding discussion as follows: If $\mathfrak{s}L = \mathbb{Z}_2$ and L is $\mathbb{Z}_2$-universal, then L has a splitting of the following type:

$$L \cong T \perp T', \text{with } T \cong \langle \varepsilon_1, 2^{\alpha_2}\varepsilon_2, 2^{\alpha_3}\varepsilon_3 \rangle, \tag{3.7}$$

where $\alpha_2 = 0$ or 1, $\alpha_3 \leq \alpha_2 + 1$, and $\mathfrak{s}T' \subseteq 2^{\alpha_3}\mathbb{Z}_2$ or $T' = 0$.

Proposition 3.2.13. *Let L be a $\mathbb{Z}_2$-lattice of rank at most 3. Then L is $\mathbb{Z}_2$-universal if and only if one of the following holds:*

$$L \text{ is split by } \widehat{\mathbb{H}}; \tag{3.8}$$

$$L \cong \widehat{\mathbb{A}} \perp \langle \varepsilon \rangle; \tag{3.9}$$

$$L \cong \langle \varepsilon_1, \varepsilon_2, 2^{\alpha_3}\varepsilon_3 \rangle,\ \alpha_3 \leq 1 \text{ and } L \text{ is isotropic.} \tag{3.10}$$

Proof. To prove the forward implication suppose L is $\mathbb{Z}_2$-universal. We first consider the case that $\mathfrak{s}L = \frac{1}{2}\mathbb{Z}_2$. In this case either $L_1 \cong \widehat{\mathbb{H}}$ and (3.8) holds, or $L \cong \widehat{\mathbb{A}} \perp \langle 2^t\varepsilon \rangle$, for some non-negative integer t. If $t \geq 1$, then L is not $\mathbb{Z}_2$-universal. We can see this when $t = 1$ by Lemma 3.2.11, and when $t \geq 2$, as $\mathfrak{n}(\langle 2^t\varepsilon \rangle) \subseteq 4\mathbb{Z}_2$ it will not represent elements of order 1; hence $q(L) \cap 2\mathbb{Z}_2^\times = \emptyset$. This will leave only $t = 0$ so that $L \cong \widehat{\mathbb{A}} \perp \langle \varepsilon \rangle$, in which case (3.9) holds.

Next consider the case that $\mathfrak{s}L = \mathbb{Z}_2$. Since L is $\mathbb{Z}_2$-universal, it has the splitting (3.7), with $T' = 0$ by the preceding discussion. So L must be isotropic by Lemma 2.3.2. If $\alpha_2 = 1$, then $L \cong \langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3 \rangle$ or $\langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3 \rangle$. But Lemma 3.2.11 implies that neither of these lattices are $\mathbb{Z}_2$-universal. So $\alpha_2 = 0$ and it follows $\alpha_3 \leq \alpha_2 + 1 = 1$. Hence the case (3.10) holds.

To prove the reverse implication, first note that we have already proved L is $\mathbb{Z}_2$-universal when case (3.8) holds in Example 2.1.1, and when case (3.9) holds in Lemma 3.2.10 *v*). To complete the proof assume (3.10) holds. Without loss of generality, let $\varepsilon_1 = 1$. First consider the case $\alpha_3 = 0$. Since L is isotropic, at least one of $\varepsilon_2, \varepsilon_3$ is congruent to 3 modulo $4\mathbb{Z}_2$; (say ε_2). Then $\mathbb{Z}_2^\times$ is represented by $\langle 1, \varepsilon_2 \rangle$, by Proposition 3.2.2 *i*). Since $2\mathbb{Z}_2^\times$ is always represented by $\langle 1, \varepsilon_2, \varepsilon_3 \rangle$, we have the result.

Next, consider the case $\alpha_3 = 1$. In order to eliminate subscripts and thereby simplify the notation, we will denote ε_2 simply by ε and ε_3 by δ for the remainder of the proof. So from here on, we consider lattices $L \cong \langle 1, \varepsilon, 2\delta\rangle$. Note that $\langle 1, \varepsilon, 2\delta\rangle$ always represents $\mathbb{Z}_2^\times$. But whether it can represent all of $2\mathbb{Z}_2^\times$, depends on ε and δ. Using the criterion of (2.1), it can be checked that L is never isotropic when $\varepsilon \equiv 3 (\text{mod } 8\mathbb{Z}_2)$.

We begin by considering the case $\varepsilon \equiv 1(\text{mod } 8\mathbb{Z}_2)$. Here we have $q(\langle 1, \varepsilon\rangle) \cap \mathbb{Z}_2^\times = \{1, 5\}(\mathbb{Z}_2^\times)^2$, by Proposition 3.2.2 i). In this case, $S_2(\langle 1, \varepsilon, 2\delta\rangle) = (\varepsilon, 2\delta)_2 = 1$ and $(-1, -d(\langle 1, \varepsilon, 2\delta\rangle))_2 = (-1, -2\varepsilon\delta)_2 = (-1, -1)_2(-1, 2)_2(-1, \delta)_2 = -(-1, \delta)_2$. Using the criterion of (2.1), L is isotropic if and only if $-(-1, \delta)_2 = 1$, which is true if and only if $\delta \equiv 3(\text{mod } 4\mathbb{Z}_2)$. So there exists $\lambda \in \mathbb{Z}_2$ such that $\delta = 3+4\lambda$; thus, $2\delta+1 \equiv 7(\text{mod } 8\mathbb{Z}_2)$ and $2\delta+5 \equiv 3(\text{mod } 8\mathbb{Z}_2)$. Hence, $\mathbb{Z}_2^\times \to L$. Also, $\langle 1, \varepsilon\rangle$ represents $2\cdot 1 = 1^2+1^2$ and $2\cdot 5 = 1^2+3^2$. If $\delta \equiv 3(\text{mod } 8\mathbb{Z}_2)$, then $2 \cdot 3 \equiv 2\delta(\text{mod } 16\mathbb{Z}_2)$ and $2 \cdot 7 \equiv 2^2 + 2^2\varepsilon + 2\delta(\text{mod } 16\mathbb{Z}_2)$. So $6(\mathbb{Z}_2^\times)^2 \cup 14(\mathbb{Z}_2^\times)^2 \to L$. If $\delta \equiv 7(\text{mod } 8\mathbb{Z}_2)$, then $2 \cdot 7 \equiv 2\delta(\text{mod } 16\mathbb{Z}_2)$ and $2 \cdot 3 \equiv 2^2 + 2^2\varepsilon + 2\delta(\text{mod } 16\mathbb{Z}_2)$. So $6(\mathbb{Z}_2^\times)^2 \cup 14(\mathbb{Z}_2^\times)^2 \to L$. In either case, we conclude that $2\mathbb{Z}_2^\times \to L$, and L is $\mathbb{Z}_2$-universal.

Next we consider the case $\varepsilon \equiv 5(\text{mod } 8\mathbb{Z}_2)$. As in the previous case, $q(\langle 1, \varepsilon\rangle) \cap \mathbb{Z}_2^\times = \{1, 5\}(\mathbb{Z}_2^\times)^2$. In this case, L is isotropic if and only if $\delta \equiv 1(\text{mod } 4\mathbb{Z}_2)$. So $2\delta + 1 \cong 3$ and $2\delta + 5 \cong 7$. Hence, $\mathbb{Z}_2^\times \to L$. Also, $\langle 1, \varepsilon\rangle$ represents $2 \cdot 3 \cong 7^2 + 1^2 \cdot 5$ and $2 \cdot 7 \cong 3^2 + 1^2 \cdot 5$. If $\delta \equiv 1(\text{mod } 8\mathbb{Z}_2)$, then $2 \cdot 1 \cong 2\delta$ and $2 \cdot 5 \cong 2^2 + 2^2\varepsilon + 2\delta \to L$. If $\delta \equiv 5(\text{mod } 8\mathbb{Z}_2)$, then $2 \cdot 5 \cong 2\delta$ and $2 \cdot 1 \cong 2^2 + 2^2\varepsilon + 2\delta \to L$. In either case, we conclude that $2\mathbb{Z}_2^\times \to L$, and L is $\mathbb{Z}_2$-universal.

Finally consider the case $\varepsilon \equiv 7(\text{mod } 8\mathbb{Z}_2)$. In this case, L is isotropic regardless of the value of δ. Here the binary unimodular lattice $\langle 1, \varepsilon\rangle$ represents all elements of $\mathbb{Z}_2^\times$, by Proposition 3.2.2 *i*), and it follows that $\langle 1, \varepsilon\rangle$ represents all elements of $\mathbb{Z}_2$ of even order. Let $\mu \in \mathbb{Z}_2^\times$. If $\mu - \delta \in 8\mathbb{Z}_2$, then $2\mu \cong 2\delta \to L$. If $\mu - \delta \in 2\mathbb{Z}_2^\times$, then $2\mu - 2\delta \in 4\mathbb{Z}_2^\times$ (hence has even order) and it follows that $2\mu - 2\delta \to \langle 1, \varepsilon\rangle$. Finally, if $\mu - \delta \in 4\mathbb{Z}_2^\times$, then $\mu \cong (\delta+4)$(write $\mu-\delta = 4\gamma$; $\gamma = 1+2\rho \in \mathbb{Z}_2^\times$; then $\mu \equiv (\delta+4)(\text{mod } 8\mathbb{Z}_2)$). So there exists $\lambda \in \mathbb{Z}_2^\times$ such that $2\mu = 2(\delta + 4)\lambda^2$. Thus, $2\mu - 2\delta\lambda^2 = 8\lambda^2 \to \langle 1, \varepsilon\rangle$, since $\varepsilon \equiv 7(\text{mod } 8\mathbb{Z}_2)$. Hence, $2\mathbb{Z}_2^\times \to L$, and L is $\mathbb{Z}_2$-universal. This completes the proof. $\square$

Next we will state criteria to identify $\mathbb{Z}_2$-universal lattices of rank at least 4. Throughout the following proof we will use the list of $\mathbb{Z}_2$-universal lattices stated in Lemma 3.2.10.

Theorem 3.2.14. *Let L be a $\mathbb{Z}_2$-lattice of rank at least 4. Then L is $\mathbb{Z}_2$-universal if and only if one the following holds:*

$$L \text{ is split by } \widehat{\mathbb{H}}; \tag{3.11}$$

$$L \text{ is split by } \widehat{\mathbb{A}} \perp \langle \varepsilon \rangle; \tag{3.12}$$

$$L \cong \widehat{\mathbb{A}} \perp K \text{ and } \mathfrak{n}K = 2\mathfrak{s}K = 2\mathbb{Z}_2; \tag{3.13}$$

$$L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon \rangle \perp M \text{ and } 8\mathbb{Z}_2 \subseteq \mathfrak{n}M; \tag{3.14}$$

or L has a splitting (3.7) *and one of the following holds:*

$$\alpha_2 = \alpha_3 = 0 \text{ and either } T \text{ is isotropic or } 4\mathbb{Z}_2 \subseteq \mathfrak{n}T'; \tag{3.15}$$

$$\alpha_2 = 0,\ \alpha_3 = 1 \text{ and either } T \text{ is isotropic or } 8\mathbb{Z}_2 \subseteq \mathfrak{n}T'; \tag{3.16}$$

$$\alpha_2 = \alpha_3 = 1 \text{ and } 4\mathbb{Z}_2 \subseteq \mathfrak{n}T'; \tag{3.17}$$

$$\alpha_2 = 1,\ \alpha_3 = 2 \text{ and } 8\mathbb{Z}_2 \subseteq \mathfrak{n}T'. \tag{3.18}$$

Proof. Suppose first that L is $\mathbb{Z}_2$-universal. Consider the case that $\mathfrak{s}L = \frac{1}{2}\mathbb{Z}_2$. Then either L_1 is split by $\widehat{\mathbb{H}}$, hence (3.11) holds, or $L_1 \cong \widehat{\mathbb{A}}$, and $L \cong \widehat{\mathbb{A}} \perp K$, with $\mathfrak{s}K \subseteq \mathbb{Z}_2$. Since $q(L) \cap 2\mathbb{Z}_2^\times = \emptyset$, whenever $\mathfrak{n}K \subseteq 4\mathbb{Z}_2$, we need $2\mathbb{Z}_2 \subseteq \mathfrak{n}K \subseteq \mathfrak{s}K \subseteq \mathbb{Z}_2$. If $\mathfrak{n}K = \mathfrak{s}K = \mathbb{Z}_2$, then K is split by $\langle \varepsilon \rangle$ for some ε; hence (3.12) holds. If $\mathfrak{n}K = 2\mathfrak{s}K = 2\mathbb{Z}_2$, then (3.13) holds. If $\mathfrak{n}K = \mathfrak{s}K = 2\mathbb{Z}_2$, then K is split by $\langle 2\varepsilon \rangle$, for some ε. Thus, $L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon \rangle \perp M$. Since $2\mathbb{Z}_2^\times \not\to \widehat{\mathbb{A}} \perp \langle 2\varepsilon \rangle$, by Lemma 3.2.11 and $2\mathbb{Z}_2^\times \not\to L$ if $\mathfrak{n}M \subseteq 16\mathbb{Z}_2$, we must have $8\mathbb{Z}_2 \subseteq \mathfrak{n}M$; hence (3.14) holds.

Now consider the case when $\mathfrak{s}L = \mathbb{Z}_2$. So L has a splitting (3.7). If $\alpha_2 = \alpha_3 = 0$ and T is anisotropic, then $\mathbb{Z}_2^\times \not\to T$, by Lemma 2.3.2. So $\mathbb{Z}_2^\times \not\to L$, whenever $\mathfrak{n}T' \subseteq 8\mathbb{Z}_2$, by Local Square Theorem. Thus, $4\mathbb{Z}_2 \subseteq \mathfrak{n}T'$, and (3.15) holds. If $\alpha_2 = 0$, $\alpha_3 = 1$ and T is anisotropic, then $2\mathbb{Z}_2^\times \not\to T$, again by Lemma 2.3.2. So $2\mathbb{Z}_2^\times \not\to L$, whenever $\mathfrak{n}T' \subseteq 16\mathbb{Z}_2$. Thus, $8\mathbb{Z}_2 \subseteq \mathfrak{n}T'$ is needed, thus (3.16) holds. If $\alpha_2 = \alpha_3 = 1$, then $\mathbb{Z}_2^\times \not\to T$, by Lemma 3.2.11. Again by Local Square Theorem $\mathbb{Z}_2^\times \not\to L$, whenever $\mathfrak{n}T' \subseteq 8\mathbb{Z}_2$. Hence we need $4\mathbb{Z}_2 \subseteq \mathfrak{n}T'$, thus (3.17) holds. Finally, if $\alpha_2 = 1$, $\alpha_3 = 2$, then $2\mathbb{Z}_2^\times \not\to T$, again by Lemma 3.2.11. Then $2\mathbb{Z}_2^\times \not\to L$, whenever $\mathfrak{n}T' \subseteq 16\mathbb{Z}_2$, so we need $8\mathbb{Z}_2 \subseteq \mathfrak{n}T'$, thus (3.18) holds.

We will now establish the sufficiency of the conditions (3.11) through (3.18). Note that L is $\mathbb{Z}_2$-universal when (3.11) or (3.12) holds, follows directly from Proposition 3.2.13. For case (3.13) we only need to show that $2\mathbb{Z}_2^\times \subseteq q(L)$, since $\widehat{\mathbb{A}}$ represents all of $\mathbb{Z}_2^\times$ by itself. Here K contains an improper unimodular sublattice, so that $2\mathbb{Z}_2^\times \to K$; hence we have the result. When (3.14) holds, with $8\mathbb{Z}_2 \subseteq \mathfrak{n}M \subseteq \mathfrak{s}M \subseteq 2\mathbb{Z}_2$, we have $\mathfrak{n}M = 2^t\mathbb{Z}_2$ for $t = 1, 2$ or 3. So there is a $u \in M$ such that $q(u) \in 2^t\mathbb{Z}_2^\times$ and the sublattice $\hat{\mathbb{A}} \perp \langle 2\varepsilon \rangle \perp \mathbb{Z}_2 u$ of L is $\mathbb{Z}_2$-universal, by Lemma 3.2.10 *vi*).

We can see in (3.15) and (3.16) when the conditions on α_2 and α_3 hold and T is isotropic, then L is $\mathbb{Z}_2$-universal follows from Proposition 3.2.13. Suppose $\alpha_2 = \alpha_3 = 0$ and $4\mathbb{Z}_2 \subseteq \mathfrak{n}T'$. Then $\mathfrak{n}T' = 2^t\mathbb{Z}_2$ for $t = 0, 1$ or 2. So there exists a $u \in T'$ such that $q(u) \in 2^t\mathbb{Z}_2^\times$. Then the sublattice $T \perp \mathbb{Z}_2 u$ of L is $\mathbb{Z}_2$-universal by Lemma 3.2.10 *i*). We can use the similar argument to show that L is $\mathbb{Z}_2$-universal under the conditions stated in cases (3.16), (3.17) and (3.18), applying *ii*), *iii*) and *iv*) of Lemma 3.2.10, respectively. □

Remark 3.2.15. When L is a diagonal $\mathbb{Z}_2$-lattice of $rk\ L \geq 5$, which is universal, then by Theorem 3.2.14 we can observe that it is always split by a $\mathbb{Z}_2$-universal sublattice of rank at most 4. This in turn guarantees the primitive $\mathbb{Z}_2$-universality of L.

Using the above theorem we can also establish the following result:

Corollary 3.2.16. *Any $\mathbb{Z}_2$-universal lattice **contains** a $\mathbb{Z}_2$-universal sublattice of rank at most 4.*

We will record the following special case for later reference:

Corollary 3.2.17. *Let L be a quaternary $\mathbb{Z}_2$-lattice with $\mathfrak{s}L = \mathbb{Z}_2$. If L is $\mathbb{Z}_2$-universal, then L has a splitting*

$$L \cong \langle \varepsilon_1, 2^{\alpha_2}\varepsilon_2, 2^{\alpha_3}\varepsilon_3, 2^{\alpha_4}\varepsilon_4 \rangle, \tag{3.19}$$

where $\alpha_2 = 0$ or 1, and $\alpha_3 \leq \alpha_2 + 1$.

Remark 3.2.18. For quaternary diagonal lattices, the conditions (3.15) - (3.18) in Theorem 3.2.14 correspond to the criteria given in (15) and (16) in [15, Lemma 1].

Remark 3.2.19. There are $\mathbb{Z}_2$-universal lattices L with $rk\ L \geq 5$, such that $sL \subseteq \mathbb{Z}_2$ but which are not diagonalizable. For example, $L \cong \langle 1, 1, 1 \rangle \perp \left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)$.

3.2.4 Criteria for primitive universality of $\mathbb{Z}_2$-lattices

In this section we will establish criteria for a $\mathbb{Z}_2$-lattice to be primitively $\mathbb{Z}_2$-universal. As we have already provided the criteria for $\mathbb{Z}_2$-universality in the previous subsection it suffices to identify lattices which are $\mathbb{Z}_2$-universal but not primitively $\mathbb{Z}_2$-universal. For this purpose, it is only necessary to consider lattices of rank at most 4, in light of Proposition 3.2.12.

Proposition 3.2.20. *Let L be a $\mathbb{Z}_2$-universal lattice of rank at most 3. Then L is primitively $\mathbb{Z}_2$-universal unless*

$$L \cong \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle \text{ and } L \text{ is isotropic.}$$

In the exceptional case, L is not primitively $\mathbb{Z}_2$-universal.

Proof. Note that with the assumption that L is a $\mathbb{Z}_2$-universal lattice of rank at most 3, the possibilities for L are as in Proposition 3.2.13. Since $\widehat{\mathbb{H}}$ is primitively $\mathbb{Z}_2$-universal, whenever L is split by $\widehat{\mathbb{H}}$, result follows immediately. When $L \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$, Proposition 3.2.4 implies that L is primitively $\mathbb{Z}_2$-universal. Suppose $L \cong \widehat{\mathbb{A}} \perp \langle \varepsilon \rangle$. We know $q^*(\widehat{\mathbb{A}}) =$

$\mathbb{Z}_2^\times$. Then for any $a \in 2\mathbb{Z}_2$, $a - \varepsilon \in \mathbb{Z}_2^\times \to \widehat{\mathbb{A}}$. So, $a \xrightarrow{*} \widehat{\mathbb{A}} \perp \langle \varepsilon \rangle$. It leaves the case that $L \cong \langle 1, \varepsilon, 2\delta \rangle$ which is isotropic. We will show that no isotropic $L \cong \langle 1, \varepsilon, 2\delta \rangle$ primitively represents any element of $4\mathbb{Z}_2^\times$. Let $x, y, z \in \mathbb{Z}_2$ with at least one of $x, y, z \in \mathbb{Z}_2^\times$, and write $L(x, y, z) := x^2 + \varepsilon y^2 + 2\delta z^2$. Then $L(x, y, z) \in \mathbb{Z}_2^\times \cup 2\mathbb{Z}_2^\times$ unless $x, y \in \mathbb{Z}_2^\times$. In that case, $x^2 \equiv y^2 \equiv 1 (\text{mod } 8\mathbb{Z}_2)$ and $L(x, y, z) \equiv 1 + \varepsilon + 2\delta z^2 (\text{mod } 8\mathbb{Z}_2)$.

Now consider the cases identified in the proof of Proposition 3.2.14 in which L is isotropic. If $\varepsilon \equiv 1 (\text{mod } 8\mathbb{Z}_2)$, then $\delta \equiv 3 (\text{mod } 8\mathbb{Z}_2)$ and $L(x, y, z) \equiv 0, 2 (\text{mod } 8\mathbb{Z}_2)$. If $\varepsilon \equiv 5 (\text{mod } 8\mathbb{Z}_2)$, then $\delta \equiv 1 (\text{mod } 8\mathbb{Z}_2)$ and $L(x, y, z) \equiv 0, 6 (\text{mod } 8\mathbb{Z}_2)$. If $\varepsilon \equiv 7 (\text{mod } 8\mathbb{Z}_2)$, then $L(x, y, z) \equiv 0, 2, 6 (\text{mod } 8\mathbb{Z}_2)$. So in all cases $L(x, y, z) \notin 4\mathbb{Z}_2^\times$. Hence, L is not primitively $\mathbb{Z}_2$-universal. □

Theorem 3.2.21. *Let L be a $\mathbb{Z}_2$-universal lattice of rank 4. Then L is primitively $\mathbb{Z}_2$-universal unless one of the following holds:*

$$L \cong \widehat{\mathbb{A}} \perp \mathbb{A}; \tag{3.20}$$

$$L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon, 2^t\delta \rangle, \text{ where } t = 1 \text{ or } 3; \tag{3.21}$$

or L has a splitting (3.19) with $\alpha_2 = 0$ and one of the following holds:

$$\alpha_3 = \alpha_4 = 0 \text{ and } \varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon_3 \equiv \varepsilon_4 (\text{mod } 4\mathbb{Z}_2); \tag{3.22}$$

$$\alpha_3 = 0,\ \alpha_4 = 2 \text{ and } \varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon_3 (\text{mod } 4\mathbb{Z}_2); \tag{3.23}$$

$$\alpha_3 = \alpha_4 = 1,\ \varepsilon_1\varepsilon_2 \equiv 1 (\text{mod } 8\mathbb{Z}_2),\ \varepsilon_1 \equiv \varepsilon_3 \equiv \varepsilon_4 (\text{mod } 4\mathbb{Z}_2); \tag{3.24}$$

$$\alpha_3 = \alpha_4 = 1,\ \varepsilon_1\varepsilon_2 \equiv 3 (\text{mod } 8\mathbb{Z}_2),\ \varepsilon_3 \equiv -\varepsilon_4 (\text{mod } 4\mathbb{Z}_2); \tag{3.25}$$

$$\alpha_3 = \alpha_4 = 1,\ \varepsilon_1\varepsilon_2 \equiv 5 (\text{mod } 8\mathbb{Z}_2),\ -\varepsilon_1 \equiv \varepsilon_3 \equiv \varepsilon_4 (\text{mod } 4\mathbb{Z}_2); \tag{3.26}$$

$$\alpha_3 = 1,\ \alpha_4 = 3 \text{ and } \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle \text{ is anisotropic}; \tag{3.27}$$

or L has a splitting (3.19) with $\alpha_2 = 1$ and one of the following holds:

$$\alpha_3 = 1,\ \alpha_4 = 2,\ \varepsilon_2\varepsilon_3 \equiv 1(\text{mod } 8\mathbb{Z}_2),\ \varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon_4(\text{mod } 4\mathbb{Z}_2); \tag{3.28}$$

$$\alpha_3 = 1,\ \alpha_4 = 2,\ \varepsilon_2\varepsilon_3 \equiv 3(\text{mod } 8\mathbb{Z}_2),\ \varepsilon_1 \equiv -\varepsilon_4(\text{mod } 4\mathbb{Z}_2); \tag{3.29}$$

$$\alpha_3 = 1,\ \alpha_4 = 2,\ \varepsilon_2\varepsilon_3 \equiv 5(\text{mod } 8\mathbb{Z}_2),\ \varepsilon_1 \equiv -\varepsilon_2 \equiv \varepsilon_4(\text{mod } 4\mathbb{Z}_2); \tag{3.30}$$

$$\alpha_3 = 2,\ \alpha_4 = 3,\ \varepsilon_1\varepsilon_3 \equiv 1(\text{mod } 8\mathbb{Z}_2),\ \varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon_4(\text{mod } 4\mathbb{Z}_2); \tag{3.31}$$

$$\alpha_3 = 2,\ \alpha_4 = 3,\ \varepsilon_1\varepsilon_3 \equiv 3(\text{mod } 8\mathbb{Z}_2),\ \varepsilon_2 \equiv -\varepsilon_4(\text{mod } 4\mathbb{Z}_2); \tag{3.32}$$

$$\alpha_3 = 2,\ \alpha_4 = 3,\ \varepsilon_1\varepsilon_3 \equiv 5(\text{mod } 8\mathbb{Z}_2),\ -\varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon_4(\text{mod } 4\mathbb{Z}_2). \tag{3.33}$$

In cases (3.20) *through* (3.33)*, L is not primitively $\mathbb{Z}_2$-universal.*

Remark 3.2.22. Some of these cases are covered in [4, Proposition 3].

The proof of the Theorem 3.2.21 will be presented through the series of lemmas dealing with the individual cases in the statement. As a result of Proposition 3.2.12, we need only consider lattices of rank 4. With the assumption L is $\mathbb{Z}_2$-universal to establish primitive $\mathbb{Z}_2$-universality it suffices to show that $4\mathbb{Z}_2 \overset{*}{\to} L$. Also for a fixed positive integer k , there are four unit squareclasses of elements in $2^k\mathbb{Z}_2^\times$. Hence to prove $2^k\mathbb{Z}_2^\times \overset{*}{\to} L$, it suffices to find four elements of $q^*(L) \cap 2^k\mathbb{Z}_2^\times$ that are distinct modulo squares.

Lemma 3.2.23. *Let $L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon, 2^t\delta\rangle$, where $t \geq 1$. Then L is primitively $\mathbb{Z}_2$-universal if and only if $t = 2$.*

Proof. By (3.14), it suffices to consider $1 \leq t \leq 3$. Take first the case $t = 1$. So $L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon, 2\delta\rangle$. Write $\varepsilon + \delta = 2\lambda$, for some $\lambda \in \mathbb{Z}_2$. Suppose $a \in q^*(L) \cap 4\mathbb{Z}_2$. Then there exists $v \in \widehat{\mathbb{A}}$ and $x, y \in \mathbb{Z}_2$ such that;

$$a = q(v) + 2\varepsilon x^2 + 2\delta y^2, \tag{3.34}$$

where at least one of x, y is in $\mathbb{Z}_2^\times$ or $v \overset{*}{\in} \widehat{\mathbb{A}}$. If $v \overset{*}{\in} \widehat{\mathbb{A}}$, then $q(v) \in \mathbb{Z}_2^\times$ and right hand side of (3.34) would be in $\mathbb{Z}_2^\times$. So $v \in 2\widehat{\mathbb{A}}$ and at least one of x, y is in $\mathbb{Z}_2^\times$; hence they both must lie in $\mathbb{Z}_2^\times$, since otherwise the right hand side of (3.34) is in $2\mathbb{Z}_2^\times$. So, there exist $k, l \in \mathbb{Z}_2$ such that $x^2 = 1+8k$, $y^2 = 1+8l$. Substituting this in (3.34) and solving for $q(v)$ gives

$$q(v) = a - 2(\varepsilon + \delta) - 16(k+l). \tag{3.35}$$

If $\lambda \in 4\mathbb{Z}_2$, then $a \overset{*}{\nrightarrow} L$ for any $\alpha \in 8\mathbb{Z}_2^\times$ (since the right hand side of (3.35) is in $8\mathbb{Z}_2^\times$ and $\widehat{\mathbb{A}}$ does not represent any element of odd order). If $\lambda \in 2\mathbb{Z}_2^\times$, then $a \overset{*}{\nrightarrow} L$ for any $a \in 16\mathbb{Z}_2$. If $\lambda \in \mathbb{Z}_2^\times$, then $a = 4(\lambda+2) \overset{*}{\nrightarrow} L$. Hence L is not primitively $\mathbb{Z}_2$-universal when $t = 1$.

Next consider the case $t = 3$. So $L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon, 8\delta\rangle$. Assume that for $a \in \mathbb{Z}_2$, $4a \overset{*}{\rightarrow} \widehat{\mathbb{A}} \perp \langle 2\varepsilon, 8\delta\rangle$. Then for some $\lambda \in \mathbb{Z}_2$, $4a = 4\lambda + 2\varepsilon x^2 + 8\delta y^2$; $y \in \mathbb{Z}_2^\times$, $x \in 2\mathbb{Z}_2$; $x = 2x_0$. So $4a = 4\lambda + 8\varepsilon x_0^2 + 8\delta y^2$. This will reduce to $a = \lambda + 2\varepsilon x_0^2 + 2\delta y^2$. Thus, for any $a \in 4\mathbb{Z}_2$ such that $a \overset{*}{\nrightarrow} \widehat{\mathbb{A}} \perp \langle 2\varepsilon, 2\delta\rangle$, we have $4a \overset{*}{\nrightarrow} \widehat{\mathbb{A}} \perp \langle 2\varepsilon, 8\delta\rangle$. Thus, $\widehat{\mathbb{A}} \perp \langle 2\varepsilon, 8\delta\rangle$ is not primitively $\mathbb{Z}_2$-universal follows by the case $t = 1$, proved above.

It remains to prove the case $t = 2$. So $L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon, 4\delta\rangle$. First note that 4δ, $4\delta + 8\varepsilon$, $4\delta + 16$, $4\delta + 8\varepsilon + 16$ are in distinct squareclasses of $4\mathbb{Z}_2^\times$, and the vector corresponds to 4δ guarantees the primitive representation. So, $4\mathbb{Z}_2^\times$ is primitively represented by L. Now, take any $\alpha \in 8\mathbb{Z}_2$. Then $\alpha - 4\delta \in 4\mathbb{Z}_2^\times$. Hence it is represented by $\widehat{\mathbb{A}}$ which implies $8\mathbb{Z}_2$ is primitively represented by $\widehat{\mathbb{A}} \perp 4\delta$. Thus L is primitively $\mathbb{Z}_2$-universal. □

Proof of Theorem 3.2.21 when $\mathfrak{s}L = \frac{1}{2}\mathbb{Z}_2$*:*

Since L is assumed to be $\mathbb{Z}_2$-universal, it holds one of (3.11) through (3.14). The result is clear if L is split by either $\widehat{\mathbb{H}}$ or $\widehat{\mathbb{A}} \perp \langle\varepsilon\rangle$, since they are primitively $\mathbb{Z}_2$-universal lattices. If case (3.13) holds, either $L \cong \widehat{\mathbb{A}} \perp \mathbb{H}$ or $L \cong \widehat{\mathbb{A}} \perp \mathbb{A}$. If $L \cong \widehat{\mathbb{A}} \perp \mathbb{H}$, then L is primitively $\mathbb{Z}_2$-universal since $2\mathbb{Z}_2^\times \overset{*}{\rightarrow} \mathbb{H}$. If $L \cong \widehat{\mathbb{A}} \perp \mathbb{A}$, Then L is not primitively $\mathbb{Z}_2$-universal, by Example 2.1.5. In the case (3.14), we have $L \cong \widehat{\mathbb{A}} \perp \langle 2\varepsilon, 2^t\delta\rangle$, $t \geq 1$. This is covered by Lemma 3.2.23. □

For the remainder of this subsection, L will denote a $\mathbb{Z}_2$-lattice with the splitting (3.19).

Lemma 3.2.24. *Let $\alpha_i = 0$ for all $2 \le i \le 4$. Then L is primitively $\mathbb{Z}_2$-universal if and only if there exist $1 \le i, j \le 4$, such that $\varepsilon_i \not\equiv \varepsilon_j (\text{mod } 4\mathbb{Z}_2)$.*

Proof. If there exist i, j such that $\varepsilon_i \not\equiv \varepsilon_j (\text{mod } 4\mathbb{Z}_2)$, then L contains a ternary $\mathbb{Z}_2$-universal sublattice, by Lemma 2.3.1 and Proposition 3.2.13. Hence L is primitively $\mathbb{Z}_2$-universal. Conversely, assume that $\varepsilon_i \equiv \varepsilon_j (\text{mod } 4\mathbb{Z}_2)$ for all $1 \le i, j \le 4$. By scaling if necessary we may assume without loss of generality that $\varepsilon_1 = 1$ and $\varepsilon_i \equiv 1 (\text{mod } 4\mathbb{Z}_2)$ for $2 \le i \le 4$. So $\varepsilon_i \equiv 1$ or $5 (\text{mod } 8\mathbb{Z}_2)$. If all of $\varepsilon_2, \varepsilon_3, \varepsilon_4$ are congruent to 1 modulo $8\mathbb{Z}_2$, then $S_2(\langle 1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle) = S_2(\langle 1,1,1,1 \rangle) = 1 = -(-1,-1)_2$ and $d(\langle 1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle) = d(\langle 1,1,1,1 \rangle) = 1$. If exactly one of $\varepsilon_2, \varepsilon_3, \varepsilon_4$ is congruent to 1 modulo $8\mathbb{Z}_2$; (say $\varepsilon_2 \equiv 1 (\text{mod } 8\mathbb{Z}_2)$), then $S_2(\langle 1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle) = S_2(\langle 1,1,5,5 \rangle) = (5,5)_2 = 1 = -(-1,-1)_2$ and $d(\langle 1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle) = d(\langle 1,1,5,5 \rangle) = 1$. So in both cases it follows from (2.2) that L is anisotropic, hence not primitively $\mathbb{Z}_2$-universal. If exactly two of $\varepsilon_2, \varepsilon_3, \varepsilon_4$ are congruent to 1 modulo $8\mathbb{Z}_2$, then $L \cong \langle 1,1,1,5 \rangle$. Then $q(v) \not\equiv 4 (\text{mod } 8\mathbb{Z}_2)$ for any $v \overset{*}{\in} L$; thus L is not primitively $\mathbb{Z}_2$-universal. If all of $\varepsilon_2, \varepsilon_3, \varepsilon_4$ are congruent to 5 modulo $8\mathbb{Z}_2$, then $L \cong \langle 1,5,5,5 \rangle$ and we can reduce this to the previous case by scaling by 5. □

Lemma 3.2.25. *Let $\alpha_2 = \alpha_3 = 0$ and $\alpha_4 = 2$. Then L is primitively $\mathbb{Z}_2$-universal if and only if there exist $1 \le i, j \le 3$ such that $\varepsilon_i \not\equiv \varepsilon_j (\text{mod } 4\mathbb{Z}_2)$.*

Proof. If there exist $1 \le i, j \le 3$ such that $\varepsilon_i \not\equiv \varepsilon_j (\text{mod } 4\mathbb{Z}_2)$, then $T \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle$ is isotropic by Lemma 2.3.1 and so $\mathbb{Z}_2$-universal by Proposition 3.2.13. Hence L is primitively $\mathbb{Z}_2$-universal. Conversely, assume that $\varepsilon_i \equiv \varepsilon_j (\text{mod } 4\mathbb{Z}_2)$ for all $1 \le i, j \le 3$. Without loss of generality, we may assume that $\varepsilon_1 = 1$ and so $\varepsilon_2 \equiv \varepsilon_3 \equiv 1 (\text{mod } 4\mathbb{Z}_2)$. Since all squares are congruent to 0 or 1 modulo $4\mathbb{Z}_2$, it follows that $q(v) \not\equiv 0 (\text{mod } 4\mathbb{Z}_2)$ for any $v \overset{*}{\in} T$. Now consider $a = 4(\varepsilon_4 - \varepsilon_2\varepsilon_3)$. If $a \in q^*(L)$, then there would exist $v \in T$ and $b \in \mathbb{Z}_2$ such that $a = q(v) + 4\varepsilon_4 b^2$, where either $v \overset{*}{\in} T$ or $b \in \mathbb{Z}_2^\times$. But we just showed that $v \overset{*}{\notin} T$ as

$a - 4\varepsilon_4 b^2 \in 4\mathbb{Z}_2$. So $b \in \mathbb{Z}_2^\times$ and thus $b^2 \equiv 1(\text{mod } 8\mathbb{Z}_2)$; say $b^2 = 1 + 8\hat{b}$, where $\hat{b} \in \mathbb{Z}_2$. Then $a - 4\varepsilon_4 b^2 = 4(\varepsilon_4 - \varepsilon_2\varepsilon_3) - 4\varepsilon_4(1 + 8\hat{b}) = 4(-\varepsilon_2\varepsilon_3 - 8\varepsilon_4\hat{b}) \in -\varepsilon_2\varepsilon_3\hat{\mathbb{Q}}_2^2$. But Lemma 2.3.2 implies that $-\varepsilon_2\varepsilon_3\hat{\mathbb{Q}}_2^2 \cap q(T) = \emptyset$. Thus, $a \notin q^*(L)$ and hence L is not primitively $\mathbb{Z}_2$-universal. □

Next we state a lemma which is a slight refinement of [9, Proposition 3.1]. We will be using this lemma in the proofs of the next two results.

Lemma 3.2.26. *Let $T \cong \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$. Then T is anisotropic implies $q^*(T) \cap 8\mathbb{Z}_2 = \emptyset$.*

Proof. Suppose T is anisotropic and $q^*(T) \cap 8\mathbb{Z}_2 \neq \emptyset$. Then $\varepsilon_1 x^2 + \varepsilon_2 y^2 + 2\varepsilon_3 z^2 \equiv 0(\text{mod } 8\mathbb{Z}_2)$ for some $x, y, z \in \mathbb{Z}_2$, which can only occur if $x, y \in \mathbb{Z}_2^\times$. Then it must be $\varepsilon_2 y^2 + 2\varepsilon_3 z^2 \in \mathbb{Z}_2^\times$. So, $\varepsilon_1 x^2 \equiv -\varepsilon_2 y^2 - 2\varepsilon_3 z^2(\text{mod } 8\mathbb{Z}_2)$ which reduces to $x^2 \equiv \varepsilon_1^{-1}(-\varepsilon_2 y^2 - 2\varepsilon_3 z^2)(\text{mod } 8\mathbb{Z}_2)$. Thus, by Local Square Theorem, $-\varepsilon_1^{-1}(\varepsilon_2 y^2 + 2\varepsilon_3 z^2) = \lambda^2$; $\lambda \in \mathbb{Z}_2^\times$. So, $-(\varepsilon_2 y^2 + 2\varepsilon_3 z^2) = \varepsilon_1 \lambda^2$. This implies $\varepsilon_1 \lambda^2 + \varepsilon_2 y^2 + 2\varepsilon_3 z^2 = 0$ which is a contradiction since T is anisotropic. □

Lemma 3.2.27. *Let $\alpha_2 = 0$, $\alpha_3 = \alpha_4 = 1$. Then L is primitively $\mathbb{Z}_2$-universal except in the following cases:*

i) $\varepsilon_1\varepsilon_2 \equiv 1(\text{mod } 8\mathbb{Z}_2)$ *and* $\varepsilon_1 \equiv \varepsilon_3 \equiv \varepsilon_4(\text{mod } 4\mathbb{Z}_2)$*;*

ii) $\varepsilon_1\varepsilon_2 \equiv 3(\text{mod } 8\mathbb{Z}_2)$ *and* $\varepsilon_3 \equiv -\varepsilon_4(\text{mod } 4\mathbb{Z}_2)$*;*

iii) $\varepsilon_1\varepsilon_2 \equiv 5(\text{mod } 8\mathbb{Z}_2)$ *and* $-\varepsilon_1 \equiv \varepsilon_3 \equiv \varepsilon_4(\text{mod } 4\mathbb{Z}_2)$*.*

In the exceptional cases, L is not primitively $\mathbb{Z}_2$-universal.

Proof. If $T \cong \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_i \rangle$ is isotropic for either $i = 3$ or $i = 4$, then T is $\mathbb{Z}_2$-universal, by Lemma 3.2.13, and it follows that L is primitively $\mathbb{Z}_2$-universal since it is split by a $\mathbb{Z}_2$-universal sublattice. Using (2.1), it is routine to check that this occurs in all cases in which we assert that L is primitively $\mathbb{Z}_2$-universal, except when $\varepsilon_1\varepsilon_2 \equiv 3(\text{mod } 8\mathbb{Z}_2)$ and $\varepsilon_3 \equiv \varepsilon_4(\text{mod } 4\mathbb{Z}_2)$. For instance, suppose $\varepsilon_1\varepsilon_2 \equiv 1(\text{mod } 8\mathbb{Z}_2)$

and $-\varepsilon_1 \equiv \varepsilon_3 (\text{mod } 4\mathbb{Z}_2)$. Consider $T \cong \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$. After scaling by ε_1 we have $T' \cong \langle \varepsilon_1^2, \varepsilon_1\varepsilon_2, 2\varepsilon_1\varepsilon_3 \rangle \cong \langle 1, 1, 2\varepsilon_1\varepsilon_3 \rangle$. Here $-\varepsilon_1 \equiv \varepsilon_3 (\text{mod } 4\mathbb{Z}_2)$ implies $\varepsilon_1\varepsilon_3 \equiv -1 (\text{mod } 4\mathbb{Z}_2)$. So $S_2(\langle 1, 1, 2\varepsilon_1\varepsilon_3 \rangle) = 1$ and $(-1, -dT')_2 = (-1, -2\varepsilon_1\varepsilon_3)_2 = (-1, -1)_2(-1, 2)_2(-1, \varepsilon_1\varepsilon_3)_2 = 1$ since $\varepsilon_1\varepsilon_3 \equiv -1 (\text{mod } 4\mathbb{Z}_2)$. Therefore T' (and hence T) is isotropic by (2.1).

Now consider the remaining case when $\varepsilon_1\varepsilon_2 \equiv 3 (\text{mod } 8\mathbb{Z}_2)$ and $\varepsilon_3 \equiv \varepsilon_4 (\text{mod } 4\mathbb{Z}_2)$. For that case, it follows that $\varepsilon_1 + \varepsilon_2, \varepsilon_1 + 9\varepsilon_2, \varepsilon_1 + 25\varepsilon_2, \varepsilon_1 + 49\varepsilon_2$ are four elements of $4\mathbb{Z}_2^\times \cap q^*(\langle \varepsilon_1, \varepsilon_2 \rangle)$ that are distinct modulo squares; hence $4\mathbb{Z}_2^\times \overset{*}{\to} \langle \varepsilon_1, \varepsilon_2 \rangle$. Then for any $\lambda \in 8\mathbb{Z}_2$, since $\varepsilon_3 \equiv \varepsilon_4 (\text{mod } 4\mathbb{Z}_2)$, we have $\lambda - 2\varepsilon_3 - 2\varepsilon_4 \in 4\mathbb{Z}_2^\times$, and it follows that $\lambda \overset{*}{\to} L$; hence, L is primitively $\mathbb{Z}_2$-universal.

It remains to show that L is not primitively $\mathbb{Z}_2$-universal in the exceptional cases. In all cases $i)$ through $iii)$, we have $T \cong \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_i \rangle$ anisotropic for both $i = 3$ and $i = 4$ and $\varepsilon_1\varepsilon_2\varepsilon_3 - \varepsilon_4 \equiv 0 (\text{mod } 4\mathbb{Z}_2)$. For example, let $\varepsilon_1\varepsilon_2 \equiv 1 (\text{mod } 8\mathbb{Z}_2)$ and $\varepsilon_1 \equiv \varepsilon_3 \equiv \varepsilon_4 (\text{mod } 4\mathbb{Z}_2)$. Consider $T \cong \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$. After scaling T by ε_1, we get $T' \cong \langle 1, 1, 2\varepsilon_1\varepsilon_3 \rangle$. Then $S_2(\langle 1, 1, 2\varepsilon_1\varepsilon_3 \rangle) = 1$ and $(-1, -dT')_2 = (-1, -2\varepsilon_1\varepsilon_3)_2 = (-1, -1)_2(-1, 2)_2(-1, \varepsilon_1\varepsilon_3)_2 = -1$ since $\varepsilon_1\varepsilon_3 \equiv 1 (\text{mod } 4\mathbb{Z}_2)$. So the condition (2.1) fails and so T' (and hence T) is anisotropic. Similarly one can check other cases. Since T is anisotropic, T does not represent any element of $-(2\varepsilon_1\varepsilon_2\varepsilon_3)\dot{\mathbb{Q}}_2^2$, by Lemma 2.3.2. We will show that $\lambda = -2(\varepsilon_1\varepsilon_2\varepsilon_3 - \varepsilon_4)$ is not primitively represented by L. On the contrary, suppose that $\lambda \overset{*}{\to} L$; so $\lambda \overset{*}{=} q(v) + 2\varepsilon_4 w^2$, for some $v \in T$, $w \in \mathbb{Z}_2$. Since $\lambda \in 8\mathbb{Z}_2$, it must be that $w \in \mathbb{Z}_2^\times$, by Lemma 3.2.26. So $w^2 \equiv 1 (\text{mod } 8\mathbb{Z}_2)$ and it follows that $\lambda - 2\varepsilon_4 w^2 \equiv -2\varepsilon_1\varepsilon_2\varepsilon_3 (\text{mod } 16\mathbb{Z}_2)$. By the Local Square Theorem, it then follows that $\lambda - 2\varepsilon_4 w^2 \in -2\varepsilon_1\varepsilon_2\varepsilon_3(\mathbb{Z}_2^\times)^2$, and we have reached a contradiction. So $\lambda \overset{*}{\not\to} L$ and L is not primitively $\mathbb{Z}_2$-universal. □

Remark 3.2.28. In other words, the Lemma 3.2.27 states that; *When $\alpha_2 = 0$, $\alpha_3 = \alpha_4 = 1$, L is primitively $\mathbb{Z}_2$-universal if and only if none of the conditions (3.24) through (3.26) holds.*

Lemma 3.2.29. *Let $\alpha_2 = 0$, $\alpha_3 = 2$, $\alpha_4 = 3$ and $T \cong \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$. Then L is primitively*

$\mathbb{Z}_2$*-universal if and only if* T *is isotropic.*

Proof. When $\langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$ is isotropic, the result follows immediately by Proposition 3.2.13 and Lemma 1.3.2. To prove the converse, assume that $W \cong \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$ is anisotropic, with $dW = 2\delta$; $\delta \in \mathbb{Z}_2^\times$. Take $a = 8(-\delta + \varepsilon_4)$. Suppose $a \in q^*(L)$. Then there exist $x, y, z, w \in \mathbb{Z}_2$ such that $a = \varepsilon_1 x^2 + \varepsilon_2 y^2 + 2\varepsilon_3 z^2 + 8\varepsilon_4 w^2$, with at least one of $x, y, z, w \in \mathbb{Z}_2^\times$. By Lemma 3.2.26, it must be that $w \in \mathbb{Z}_2^\times$. So, $a - 8\varepsilon_4 w^2 = \varepsilon_1 x^2 + \varepsilon_2 y^2 + 2\varepsilon_3 z^2$, where $a - 8\varepsilon_4 w^2 = 8(-\delta + \varepsilon_4) - 8\varepsilon_4(1 + 8k) = -8(\delta + 8\varepsilon_4 k) = 4(-2)(\delta + 8\varepsilon_4 k) \in -2\delta\dot{\mathbb{Q}}_2^2$, which is a contradiction since $-dW \not\to \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$, by Lemma 2.3.2. ∎

Lemma 3.2.30. *For any* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathbb{Z}_2^\times$*, the following are equivalent:*

a) $K \cong \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3, 2\varepsilon_4 \rangle$ *is primitively* $\mathbb{Z}_2$*-universal;*

b) $M \cong \langle \varepsilon_1, 2\varepsilon_3, 4\varepsilon_2, 8\varepsilon_4 \rangle$ *is primitively* $\mathbb{Z}_2$*-universal;*

c) $N \cong \langle \varepsilon_3, 2\varepsilon_1, 2\varepsilon_2, 4\varepsilon_4 \rangle$ *is primitively* $\mathbb{Z}_2$*-universal.*

Proof. First note that K, M and N are $\mathbb{Z}_2$-universal, by (3.16), (3.18) and (3.17), respectively. To prove $a) \implies b)$, for any $a \in 4\mathbb{Z}_2^\times$; $a - 8\varepsilon_4 \in 4\mathbb{Z}_2^\times$. So, $a - 8\varepsilon_4 \to \langle \varepsilon_1, 4\varepsilon_2, 2\varepsilon_3 \rangle$; hence $a \xrightarrow{*} M$. So it will suffice to show that $8\mathbb{Z}_2 \xrightarrow{*} M$.

Now suppose that $\langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$ is isotropic, and hence $\mathbb{Z}_2$-universal, by Proposition 3.2.13. Consider $a \in 8\mathbb{Z}_2$, and write $a = 8a_0$. Then $2a_0 - 2\varepsilon_4 \in \mathbb{Z}_2 \to \langle \varepsilon_1, \varepsilon_2, 2\varepsilon_3 \rangle$. So there exist $x, y, z \in \mathbb{Z}_2$ such that $2a_0 - 2\varepsilon_4 = \varepsilon_1 x^2 + \varepsilon_2 y^2 + 2\varepsilon_3 z^2$. Then $8a_0 - 8\varepsilon_4 = 4\varepsilon_1 x^2 + 4\varepsilon_2 y^2 + 2 \cdot 4\varepsilon_3 z^2 = \varepsilon_1(2x)^2 + 4\varepsilon_2 y^2 + 2\varepsilon_3(2z)^2$. Then $a = \varepsilon_1(2x)^2 + 4\varepsilon_2 y^2 + 2\varepsilon_3(2z)^2 + 8\varepsilon_4$ and so $a \xrightarrow{*} M$.

Next suppose that $\langle \varepsilon_1, \varepsilon_2, 2\varepsilon_4 \rangle$ is isotropic, and hence $\mathbb{Z}_2$-universal. Consider first $a = 8a_0 \in 8\mathbb{Z}_2^\times$. Since $\langle \varepsilon_1, \varepsilon_2, 2\varepsilon_4 \rangle$ is $\mathbb{Z}_2$-universal, there exists $x, y, z \in \mathbb{Z}_2$ such that $2a_0 = \varepsilon_1 x^2 + \varepsilon_2 y^2 + 2\varepsilon_4 w^2$, and at least one of $y, w \in \mathbb{Z}_2^\times$. So $a = \varepsilon_1(2x)^2 + 4\varepsilon_2 y^2 + 8\varepsilon_4 w^2$ and so $a \xrightarrow{*} \langle \varepsilon_1, 4\varepsilon_2, 8\varepsilon_4 \rangle$. Lastly, if $a \in 16\mathbb{Z}_2$, then $a - 2^2 \cdot 2\varepsilon_3 \in 8\mathbb{Z}_2^\times$ and so $a - 2^2 \cdot 2\varepsilon_3 \xrightarrow{*} \langle \varepsilon_1, 4\varepsilon_2, 8\varepsilon_4 \rangle$, giving $a \xrightarrow{*} M$.

As noted in the proof of the Lemma 3.2.27, this covers all cases in which K is primitively $\mathbb{Z}_2$-universal, except when $\varepsilon_1\varepsilon_2 \equiv 3(\text{mod } 8\mathbb{Z}_2)$ and $\varepsilon_3 \equiv \varepsilon_4(\text{mod } 4\mathbb{Z}_2)$. For this case, we first note that $\{8\varepsilon_4, 8\varepsilon_4+16\varepsilon_1, 8\varepsilon_4+16\varepsilon_2, 8\varepsilon_4+32\varepsilon_3\}$ is a set of four independent elements of $8\mathbb{Z}_2^\times$ which are primitively represented by M. Next, take any $a \in 16\mathbb{Z}_2^\times$; say $a = 16\delta$. As shown in the proof of Lemma 3.2.27, we see that $4\mathbb{Z}_2^\times \xrightarrow{*} \langle\varepsilon_1, \varepsilon_2\rangle$, so there exist $x, y \in \mathbb{Z}_2^\times$ such that $4\delta = \varepsilon_1 x^2+\varepsilon_2 y^2$. Then $a = 16\delta = \varepsilon_1(2x)^2+4\varepsilon_2 y^2$ gives a primitive representation of a by $\langle\varepsilon_1, 4\varepsilon_2\rangle$. It follows that $16\mathbb{Z}_2^\times \xrightarrow{*} \langle\varepsilon_1, 4\varepsilon_2\rangle$. To complete this case, consider $a \in 32\mathbb{Z}_2$. Then $a - 2^2 \cdot 2\varepsilon_3 - 8\varepsilon_4 = a - 8(\varepsilon_3 + \varepsilon_4)$ is an element of $16\mathbb{Z}_2^\times$, since $\varepsilon_3 \equiv \varepsilon_4(\text{mod } 4\mathbb{Z}_2)$. Thus, with the argument which we have just shown $a - 2^2 \cdot 2\varepsilon_3 - 8\varepsilon_4 \xrightarrow{*} \langle\varepsilon_1, 4\varepsilon_2\rangle$. Hence, $32\mathbb{Z}_2 \xrightarrow{*} M$.

To prove $b) \implies c)$, let $a \in \mathbb{Z}_2$. Then there exist $x, y, z, w \in \mathbb{Z}_2$ such that $2a = \varepsilon_1 x^2 + 4\varepsilon_2 y^2 + 2\varepsilon_3 z^2 + 8\varepsilon_4 w^2$, where $x \in 2\mathbb{Z}_2$ (say $x = 2\hat{x}$, $\hat{x} \in \mathbb{Z}_2$) and at least one of $y, z, w \in \mathbb{Z}_2^\times$. Then $a = 2\varepsilon_1\hat{x}^2 + 2\varepsilon_2 y^2 + \varepsilon_3 z^2 + 4\varepsilon_4 w^2$. It follows that $a \xrightarrow{*} N$.

Finally, to prove $c) \implies a)$, let $a \in \mathbb{Z}_2$. Then there exist $x, y, z, w \in \mathbb{Z}_2$ such that $2a = 2\varepsilon_1 x^2 + 2\varepsilon_2 y^2 + \varepsilon_3 z^2 + 4\varepsilon_4 w^2$, where $z \in 2\mathbb{Z}_2$ (say $z = 2\hat{z}$, $\hat{z} \in \mathbb{Z}_2$) and at least one of $x, y, w \in \mathbb{Z}_2^\times$. Then $a = \varepsilon_1 x^2+\varepsilon_2 y^2+2\varepsilon_3\hat{z}^2+2\varepsilon_4 w^2$. It follows that $a \xrightarrow{*} K$. This completes the proof. □

We will now proceed to the completion of the proof of Theorem 3.2.21.

Proof of Theorem 3.2.21 when $\mathfrak{s}L = \mathbb{Z}_2$*:*

Since L is assumed to be $\mathbb{Z}_2$-universal, one of the cases (3.15) through (3.18) holds, by Theorem 3.2.14.

Case I: $\alpha_1 = \alpha_2 = \alpha_3 = 0$. If T is isotropic or if $\alpha_4 = 1$, then L is primitively $\mathbb{Z}_2$-universal by Proposition 3.2.13, Lemma 1.3.3 and Lemma 3.2.8 *i*). If $\alpha_4 = 0$, then L fails to be primitively $\mathbb{Z}_2$-universal exactly when (3.22) holds, by Lemma 3.2.24. If $\alpha_4 = 2$, then L fails to be primitively $\mathbb{Z}_2$-universal exactly when (3.23) holds, by Lemma 3.2.25.

Case II: $\alpha_1 = \alpha_2 = 0,\ \alpha_3 = 1$. If T is isotropic or if $\alpha_4 = 2$, then L is primitively $\mathbb{Z}_2$-universal by Proposition 3.2.13, Lemma 1.3.3 and Lemma 3.2.8 *i*). If $\alpha_4 = 1$, then L fails to be primitively $\mathbb{Z}_2$-universal exactly when one of (3.24) through (3.26) holds, by Lemma 3.2.27. If $\alpha_4 = 3$, then L fails to be primitively $\mathbb{Z}_2$-universal exactly when (3.27) holds, by Lemma 3.2.29.

Case III: $\alpha_1 = 0,\ \alpha_2 = \alpha_3 = 1$. First assume that $\alpha_4 = 1$. Then for any $a \in 4\mathbb{Z}_2$, $a - 2\varepsilon_4 \in 2\mathbb{Z}_2^\times \to \langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3\rangle$, by Lemma 3.2.9 *ii*). Thus, $a \xrightarrow{*} L$. If $\alpha_4 = 2$, then L fails to be primitively $\mathbb{Z}_2$-universal exactly when one of (3.28) through (3.30) holds, by Lemmas 3.2.27 and 3.2.30.

Case IV: $\alpha_1 = 0,\ \alpha_2 = 1\ \alpha_3 = 2$. Assume first that $\alpha_4 = 2$. Note that $2\mathbb{Z}_2^\times \to \langle \varepsilon_1, 2\varepsilon_2, 2\varepsilon_3\rangle$, and this implies $4\mathbb{Z}_2^\times \xrightarrow{*} \langle 2\varepsilon_1, 4\varepsilon_2, 4\varepsilon_3\rangle$. Next for any $a \in 8\mathbb{Z}_2$, $a - 4\varepsilon_4 \in 4\mathbb{Z}_2^\times$. Since $\mathbb{Z}_2^\times \to \langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3\rangle$, by Lemma 3.2.8 *i*), we have $a - 4\varepsilon_4 \in 4\mathbb{Z}_2^\times \to \langle \varepsilon_1, 2\varepsilon_2, 4\varepsilon_3\rangle$. Hence $a \xrightarrow{*} L$. If $\alpha_4 = 3$, then L fails to be primitively $\mathbb{Z}_2$-universal exactly when one of (3.31) through (3.33) holds, by Lemmas 3.2.27 and 3.2.30. This completes the proof of Theorem 3.2.21. □

3.3 PROOFS OF THEOREMS

We conclude the chapter by supplying proofs of the theorems stated in the introduction. As the theorems are stated there in the traditional language of quadratic forms, we will first review the connections between quadratic forms and quadratic lattices. A nondegenerate integral quadratic form $f = f(X_1, \dots, X_n)$ of rank n can be written as

$$f = \sum_{1 \le i,j \le n} a_{ij} X_i X_j, \text{ where } a_{ij} = a_{ji}, a_{ii} \in \mathbb{Z}, 2a_{ij} \in \mathbb{Z} \text{ for } i \ne j. \tag{3.36}$$

Let M_f denote the symmetric matrix (a_{ij}), and associate to f a quadratic $\mathbb{Z}$-lattice L and basis $\mathcal{B}$ for L such that the Gram matrix of L with respect to $\mathcal{B}$ is M_f. An integer is (primitively) represented by the form f if and only if it is (primitively) represented by the associated lattice L. Our main tool for proving Theorems 1.1 through 1.3 is the following result that relates the positive integers primitively represented by a positive definite quadratic $\mathbb{Z}$-lattice to those integers that are primitively represented by all local completions L_p of the lattice: *Let L be a positive definite integral $\mathbb{Z}$-lattice of rank $n \geq 4$. Then there is an integer N with the following property: If $a \geq N$ is an integer that is primitively represented by L_p for all primes p, then a is primitively represented by L.* (see, e.g., [5, Theorem 1.6, page 204]). In particular, such a lattice L is almost primitively universal if and only if L_p is primitively $\mathbb{Z}_p$-universal for all primes p. From this, the proofs of Theorems 0.2.1 and 0.2.2 are now immediate.

Proof of Theorem 0.2.1 Follows from Corollary 2.2.3 □

Proof of Theorem 0.2.2 Follows from Propositions 3.1.2 and 3.2.12. □

The form (3.36) is said to be *classically integral* if $a_{ij} \in \mathbb{Z}$ for all i, j. In this case, the discriminant $df = \det L$ is an integer, and

$$\mathrm{ord}_p \mathfrak{v} L_p = \mathrm{ord}_p \mathfrak{v} L = \mathrm{ord}_p dL = \mathrm{ord}_p df$$

for all primes p. In particular, for a classically integral form f and positive integer t,

$$p^t \mid df \text{ if and only if } \mathfrak{v} L_p \subseteq p^t \mathbb{Z}_p.$$

Proof of Theorem 0.2.3 Let L be a positive definite $\mathbb{Z}$-lattice for which $\mathfrak{s}L \subseteq \mathbb{Z}$, $rk\ L = n \geq 4$ and, for all primes p, $p^{n-2} \nmid dL$. Moreover, it is assumed that L represents an odd integer, and that dL is even when $n = 4$. If $\mathfrak{s}L_p \subseteq p\mathbb{Z}_p$ for some prime p, then $\mathfrak{v}L_p \subseteq p^n \mathbb{Z}_p$ and it would follow that $p^n \mid dL$, contrary to assumption. Hence, $\mathfrak{s}L = \mathbb{Z}$. When $p = 2$, the assumption that L represents an odd integer guarantees that $\mathfrak{n}L = \mathbb{Z}$ as well. So for each prime p, L_p has a splitting of the type $L_p \cong L_{(0)} \perp K$, where $L_{(0)}$ is diagonalizable and $\mathfrak{s}K \subseteq p\mathbb{Z}_p$ or $K = 0$. Since $p^{n-2} \nmid dL$, it follows that $rk\ K \leq n - 3$ and so $r_0 = rk\ L_{(0)} \geq 3$. So, if p is odd, L_p is primitively $\mathbb{Z}_p$-universal by Lemma 4.1(ii). So we need only consider further the case $p = 2$. If $r_0 \geq 4$, then L_2 is split by $N \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle$, which is $\mathbb{Z}_2$-universal by Lemma 3.2.10. The assumption that dL is even when $n = 4$ rules out the possibility that $L = N$; so $n \geq 5$ and it follows from Theorem 0.2.2 that L_2 is primitively $\mathbb{Z}_2$-universal. So to complete the proof, we consider the case $r_0 = 3$. So

$$L_2 \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle \perp K, \text{ with } \mathfrak{s}K \subseteq 2\mathbb{Z}_2 \text{ and } rk\ K = n - 3.$$

If $\mathfrak{s}K \subseteq 4\mathbb{Z}_2$, then $\mathfrak{v}L_2 = \mathfrak{v}K \subseteq 2^{2(n-3)}\mathbb{Z}_2$; but $2(n-3) \geq n-2$ since $n \geq 4$, thus contradicting the assumption that $p^{n-2} \nmid dL$. So $\mathfrak{s}L_2 = 2\mathbb{Z}_2$. This leaves two possibilities: $\mathfrak{n}K = 2\mathbb{Z}_2$ or $\mathfrak{n}K = 4\mathbb{Z}_2$. First consider the case $\mathfrak{n}K = 4\mathbb{Z}_2$. Then $rk\ K \geq 2$ (since $\mathfrak{n}K \neq \mathfrak{s}K$) and $n \geq 5$. Since $\mathfrak{n}K = 4\mathbb{Z}_2$, there exists $\varepsilon_4 \in \mathbb{Z}_2^\times$ such that $4\varepsilon_4 \to K$. So L_2 contains a sublattice $\langle \varepsilon_1, \varepsilon_2, \varepsilon_3, 4\varepsilon_4 \rangle$, which is $\mathbb{Z}_2$-universal by Lemma 3.2.10. So L_2 is $\mathbb{Z}_2$-universal and hence primitively $\mathbb{Z}_2$-universal by Theorem 0.2.2. Finally, consider the case $\mathfrak{n}K = \mathfrak{s}K = 2\mathbb{Z}_2$. Then there exists $\varepsilon_4 \in \mathbb{Z}_2^\times$ such that $2\varepsilon_4 \to K$. So L_2 contains a sublattice $N \cong \langle \varepsilon_1, \varepsilon_2, \varepsilon_3, 2\varepsilon_4 \rangle$, which is $\mathbb{Z}_2$-universal by Lemma 3.2.10. If $n \geq 5$, it follows from Theorem 0.2.2 that L_2 is primitively $\mathbb{Z}_2$-universal. If $n = 4$, then $L_2 = N$. Since $\mathbb{Z}_2^\times \to \langle \varepsilon_2, \varepsilon_3, 2\varepsilon_4 \rangle$ by Lemma 3.2.8, it follows from Lemma 1.3.3 that L_2 is primitively $\mathbb{Z}_2$-universal. This completes the proof. $\square$

The proof of Theorem 0.2.4 relies on several fundamental results from spinor genus theory. For general background on the spinor genus, the reader is referred, e.g., to [14, §102A] or [5, Chapter 11]. For a $\mathbb{Z}$-lattice L, the genus, spinor genus and isometry class of L will be denoted by $\operatorname{gen} L$, $\operatorname{spn} L$ and $\operatorname{cls} L$, respectively. If $\mathcal{S}$ denotes one of the objects $\operatorname{gen} L$, $\operatorname{spn} L$ or $\operatorname{cls} L$, the notations $a \to \mathcal{S}$ or $a \xrightarrow{*} \mathcal{S}$ will mean that there exists a lattice $K \in \mathcal{S}$ such that $a \to K$ or $a \xrightarrow{*} K$, respectively.

Proof of Theorem 0.2.4 Let L be an indefinite integral $\mathbb{Z}$-lattice of rank $n \geq 5$ such that every integer is represented by $\operatorname{gen} L$. Then L_p is $\mathbb{Z}_p$-universal for all primes p. So, by Propositions 3.1.2 and 3.2.12, L_p is primitively $\mathbb{Z}_p$-universal. Let $0 \neq a \in \mathbb{Z}$. Then $a \xrightarrow{*} L_p$

for all p, and it follows as in [14, Example 102:5] that $a \stackrel{*}{\rightarrow} \operatorname{gen} L$. Since $n \geq 4$, it then follows from [5, Theorem 7.1, page 227] that $a \stackrel{*}{\rightarrow} \operatorname{spn} L$. Since L is indefinite and $n \geq 3$, $\operatorname{spn} L = \operatorname{cls} L$ by [14, Theorem 104:5]. Hence $a \stackrel{*}{\rightarrow} \operatorname{cls} L$ and we conclude that $a \stackrel{*}{\rightarrow} L$, as desired. □

CHAPTER 4

ALMOST PRIMITIVELY UNIVERSAL CLASSICALLY INTEGRAL POSITIVE DEFINITE QUATERNARY QUADRATIC FORMS

As an application of the local results stated in Chapter 3, we now complete the determination begun by Budarina [4] of the almost primitively universal forms among the universal classically integral quaternary quadratic forms. Recall that a positive definite integral quadratic form (or a corresponding $\mathbb{Z}$-lattice) is said to be *primitively universal* if it primitively represents all positive integers, and almost primitively universal if it primitively represents all sufficiently large positive integers (that is, primitively represents all but at most finitely many positive integers). It is known that a positive definite quadratic form in at least four variables is almost primitively universal if and only if it is locally primitively universal for all primes (e.g., see [5, Theorem 1.6, page 204]).

4.1 PREVIOUS RESULTS

Using the Fifteen Theorem of Conway and Schneeberger [6], which is stated as Theorem 4.1.1 below, Bhargava produced the complete list of classically integral positive definite quaternary quadratic forms [2, Table 5], which appears here as Figure 4.1.

Theorem 4.1.1. [1] [15-Theorem, J.H. Conway and W. Schneeberger, 1993] *If a positive definite classically integral quadratic form represents every positive integer up to 15, then it represents every positive integer.*

Theorem 4.1.2. [M. Bhargava, 2000] *If a positive definite classically integral quadratic form represents the nine critical numbers 1, 2, 3, 5, 6, 7, 10, 14, 15, then it represents every positive integer.*

[1] A generalization of this theorem to the representation of forms by forms can be found in [12].

1: 1 1 1 0 0 0	16: 1 2 8 0 0 0	28: 2 3 5 2 0 0	48: 2 3 8 0 0 0	72: 2 4 9 0 0 0
2: 1 1 2 0 0 0	16: 2 2 4 0 0 0	28: 2 4 4 0 2 0	48: 2 4 6 0 0 0	72: 2 4 10 4 0 0
3: 1 1 3 0 0 0	16: 2 3 3 2 0 0	28: 2 4 5 4 2 0	48: 2 5 5 2 0 0	72: 2 5 8 4 0 0
3: 1 2 2 2 0 0	17: 1 2 9 2 0 0	30: 2 3 5 0 0 0	49: 2 3 9 2 2 0	74: 2 4 10 2 2 0
4: 1 1 4 0 0 0	17: 1 3 6 2 0 0	30: 2 4 4 2 0 0	49: 2 4 7 0 0 2	76: 2 4 10 0 2 0
4: 1 2 2 0 0 0	17: 2 3 4 0 2 2	31: 2 3 6 2 2 0	49: 2 5 6 0 2 2	77: 2 5 9 4 2 0
4: 2 2 2 2 2 0	18: 1 2 9 0 0 0	31: 2 4 5 0 2 2	50: 2 4 7 2 2 0	78: 2 4 10 2 0 0
5: 1 1 5 0 0 0	18: 1 3 6 0 0 0	32: 2 4 4 0 0 0	51: 2 3 9 0 2 0	78: 2 5 8 2 0 0
5: 1 2 3 2 0 0	18: 2 2 5 2 0 0	32: 2 4 5 4 0 0	52: 2 3 9 2 0 0	80: 2 4 10 0 0 0
6: 1 1 6 0 0 0	18: 2 3 3 0 0 0	33: 2 3 6 0 2 0	52: 2 5 6 2 0 2	80: 2 4 11 4 0 0
6: 1 2 3 0 0 0	18: 2 3 4 2 0 2	34: 2 3 6 2 0 0	52: 2 5 6 4 0 0	80: 2 5 8 0 0 0
6: 2 2 2 2 0 0	19: 1 2 10 2 0 0	34: 2 4 5 2 2 0	53: 2 5 6 2 2 0	82: 2 4 11 2 2 0
7: 1 1 7 0 0 0	19: 2 3 4 2 2 0	34: 2 4 6 4 0 2	54: 2 3 9 0 0 0	82: 2 5 9 4 0 0
7: 1 2 4 2 0 0	20: 1 2 10 0 0 0	35: 2 4 5 0 0 2	54: 2 4 7 2 0 0	85: 2 5 9 0 2 0
7: 2 2 3 2 0 2	20: 2 2 5 0 0 0	36: 2 3 6 0 0 0	54: 2 5 6 0 0 2	86: 2 4 11 2 0 0
8: 1 2 4 0 0 0	20: 2 2 6 2 2 0	36: 2 4 5 0 2 0	54: 2 5 7 4 2 2	87: 2 5 10 4 2 0
8: 1 3 3 2 0 0	*20: 2 3 4 0 0 2*	36: 2 4 6 4 2 0	55: 2 3 10 2 2 0	88: 2 4 11 0 0 0
8: 2 2 2 0 0 0	20: 2 4 4 4 2 0	36: 2 5 5 4 2 2	55: 2 5 6 0 2 0	88: 2 4 12 4 0 0
8: 2 2 3 2 2 0	22: 1 2 11 0 0 0	37: 2 5 5 4 2 0	55: 2 5 7 4 0 2	88: 2 5 9 2 0 0
9: 1 2 5 2 0 0	22: 2 2 6 2 0 0	38: 2 4 5 2 0 0	56: 2 4 7 0 0 0	90: 2 4 12 2 2 0
9: 1 3 3 0 0 0	22: 2 3 4 2 0 0	38: 2 4 6 0 2 2	56: 2 4 8 4 0 0	90: 2 5 9 0 0 0
9: 2 2 3 0 0 2	22: 2 3 5 0 2 2	39: 2 3 7 0 2 0	57: 2 3 10 0 2 0	92: 2 4 13 4 2 0
10: 1 2 5 0 0 0	23: 1 2 12 2 0 0	40: 2 3 7 2 0 0	58: 2 3 10 2 0 0	92: 2 5 10 4 0 0
10: 2 2 3 2 0 0	23: 2 3 5 2 0 2	40: 2 4 5 0 0 0	58: 2 4 8 2 2 0	93: 2 5 10 2 2 0
10: 2 2 4 2 0 2	24: 1 2 12 0 0 0	40: 2 4 6 2 0 2	58: 2 5 6 2 0 0	94: 2 4 12 2 0 0
11: 1 2 6 2 0 0	24: 2 2 6 0 0 0	40: 2 4 6 4 0 0	58: 2 5 7 0 2 2	95: 2 5 10 0 2 0
11: 1 3 4 2 0 0	24: 2 2 7 2 2 0	41: 2 4 7 4 0 2	60: 2 3 10 0 0 0	96: 2 4 12 0 0 0
12: 1 2 6 0 0 0	24: 2 3 4 0 0 0	42: 2 3 7 0 0 0	60: 2 4 9 4 2 0	96: 2 4 13 4 0 0
12: 1 3 4 0 0 0	24: 2 4 4 0 2 2	42: 2 4 6 0 0 2	60: 2 5 6 0 0 0	98: 2 4 13 2 2 0
12: 2 2 3 0 0 0	24: 2 4 4 4 0 0	42: 2 4 6 2 2 0	61: 2 5 7 2 0 2	98: 2 5 10 2 0 0
12: 2 2 4 0 0 2	25: 1 2 13 2 0 0	42: 2 5 5 4 0 0	62: 2 4 8 2 0 0	100: 2 4 13 0 2 0
12: 2 3 3 0 2 2	*25: 2 3 5 0 0 2*	43: 2 3 8 2 2 0	62: 2 5 7 4 0 0	100: 2 4 14 4 2 0
13: 2 2 5 2 0 2	25: 2 3 5 2 2 0	44: 2 4 6 0 2 0	63: 2 5 7 0 0 2	100: 2 5 10 0 0 0
13: 2 3 3 2 2 0	26: 1 2 13 0 0 0	45: 2 4 7 0 2 2	63: 2 5 7 2 2 0	102: 2 4 13 2 0 0
14: 1 2 7 0 0 0	26: 2 2 7 2 0 0	45: 2 5 5 0 2 0	64: 2 4 8 0 0 0	104: 2 4 13 0 0 0
14: 1 3 5 2 0 0	26: 2 4 4 2 2 0	45: 2 5 6 4 2 2	66: 2 4 9 2 2 0	104: 2 4 14 4 0 0
14: 2 2 4 2 0 0	27: 1 2 14 2 0 0	46: 2 3 8 2 0 0	68: 2 4 9 0 2 0	106: 2 4 14 2 2 0
15: 1 2 8 2 0 0	27: 2 3 5 0 2 0	46: 2 4 6 2 0 0	68: 2 4 10 4 2 0	108: 2 4 14 0 2 0
15: 1 3 5 0 0 0	27: 2 4 5 4 0 2	46: 2 5 6 4 0 2	68: 2 5 7 2 0 0	110: 2 4 14 2 0 0
15: 2 2 5 0 0 2	28: 1 2 14 0 0 0	47: 2 4 7 2 0 2	70: 2 4 9 2 0 0	112: 2 4 14 0 0 0
15: 2 3 3 0 2 0	28: 2 2 7 0 0 0	47: 2 5 6 4 2 0	70: 2 5 7 0 0 0	

Figure 4.1. 204 universal quaternary quadratic forms up to equivalence [2]

From Theorem 4 and Proposition 3 of [4], Budarina was able to conclude that among the forms in this list having odd determinant, 39 are almost primitively universal and 23 are not almost primitively universal. However, for the 142 forms in the list having even determinant, she was able only to conclude from those results that 74 of these forms are almost primitively universal and one is not almost primitively universal. This leaves 67 forms for which no conclusion was reached. Among the remaining forms, an additional 15 diagonal forms were subsequently shown to be primitively universal by Earnest, Kim and Meyer [8]. Their result is stated as Theorem 4.1.3 below, which relates with the Figure 4.2.

Theorem 4.1.3. [A.G. Earnest, J.Y. Kim, N.D. Meyer, 2014] *There are 96 inequivalent positive definite diagonal quaternary integral quadratic forms that are strictly regular (see Figure 4.2). Among these, 34 are in one-class genera and 27 are strictly universal (primitively universal).*

⟨1, 1, 1, 1⟩∗	⟨1, 2, 2, 3⟩ ∗ †	⟨1, 3, 3, 3⟩∗	⟨1, 4, 8, 20⟩
⟨1, 1, 1, 2⟩ ∗ †	⟨1, 2, 2, 4⟩∗	⟨1, 3, 3, 6⟩∗	⟨1, 4, 8, 24⟩
⟨1, 1, 1, 3⟩ ∗ †	⟨1, 2, 2, 6⟩ ∗ †	⟨1, 3, 3, 9⟩∗	⟨1, 4, 8, 28⟩
⟨1, 1, 1, 4⟩∗	⟨1, 2, 2, 7⟩†	⟨1, 3, 4, 4⟩	⟨1, 4, 8, 32⟩
⟨1, 1, 1, 5⟩∗	⟨1, 2, 2, 8⟩∗	⟨1, 3, 4, 8⟩	⟨1, 4, 8, 40⟩
⟨1, 1, 1, 8⟩∗	⟨1, 2, 3, 3⟩†	⟨1, 3, 4, 12⟩	⟨1, 4, 8, 64⟩
⟨1, 1, 2, 2⟩∗	⟨1, 2, 3, 4⟩†	⟨1, 3, 6, 6⟩∗	⟨1, 4, 12, 12⟩
⟨1, 1, 2, 3⟩ ∗ †	⟨1, 2, 3, 5⟩†	⟨1, 3, 6, 9⟩	⟨1, 5, 5, 5⟩∗
⟨1, 1, 2, 4⟩ ∗ †	⟨1, 2, 3, 6⟩	⟨1, 3, 6, 12⟩	⟨1, 8, 8, 8⟩∗
⟨1, 1, 2, 5⟩†	⟨1, 2, 3, 7⟩†	⟨1, 3, 6, 15⟩	⟨2, 3, 3, 6⟩∗
⟨1, 1, 2, 6⟩ ∗ †	⟨1, 2, 3, 8⟩	⟨1, 3, 6, 18⟩	⟨2, 3, 3, 9⟩
⟨1, 1, 2, 7⟩†	⟨1, 2, 4, 4⟩ ∗ †	⟨1, 3, 9, 9⟩∗	⟨2, 3, 6, 6⟩∗
⟨1, 1, 2, 8⟩	⟨1, 2, 4, 5⟩†	⟨1, 3, 9, 18⟩	⟨2, 3, 6, 12⟩∗
⟨1, 1, 2, 10⟩	⟨1, 2, 4, 6⟩ ∗ †	⟨1, 3, 9, 27⟩	⟨3, 3, 4, 12⟩
⟨1, 1, 2, 16⟩	⟨1, 2, 4, 7⟩†	⟨1, 3, 9, 54⟩	⟨3, 4, 4, 8⟩
⟨1, 1, 3, 3⟩∗	⟨1, 2, 4, 8⟩	⟨1, 4, 4, 4⟩∗	⟨3, 4, 4, 12⟩
⟨1, 1, 3, 4⟩†	⟨1, 2, 4, 9⟩†	⟨1, 4, 4, 8⟩∗	⟨3, 4, 8, 8⟩
⟨1, 1, 3, 5⟩†	⟨1, 2, 4, 11⟩†	⟨1, 4, 4, 12⟩	⟨3, 4, 8, 12⟩
⟨1, 1, 3, 6⟩†	⟨1, 2, 4, 12⟩†	⟨1, 4, 4, 16⟩	⟨3, 4, 8, 16⟩
⟨1, 1, 3, 9⟩∗	⟨1, 2, 4, 13⟩†	⟨1, 4, 4, 20⟩	⟨3, 4, 8, 20⟩
⟨1, 1, 4, 4⟩∗	⟨1, 2, 4, 14⟩†	⟨1, 4, 4, 32⟩	⟨3, 4, 12, 12⟩
⟨1, 1, 4, 8⟩∗	⟨1, 2, 4, 16⟩	⟨1, 4, 8, 8⟩	⟨3, 4, 12, 24⟩
⟨1, 1, 4, 12⟩	⟨1, 2, 6, 16⟩	⟨1, 4, 8, 12⟩	⟨3, 4, 12, 36⟩
⟨1, 2, 2, 2⟩ ∗ †	⟨1, 2, 8, 8⟩∗	⟨1, 4, 8, 16⟩	⟨3, 8, 12, 24⟩

Figure 4.2. Strictly regular diagonal quaternary forms; † = Primitively universal [8]

In summary, that leaves 52 forms for which no conclusion has so far been reached regarding almost primitive universality, and it is these forms which will be analyzed here. These 52 forms are listed in Table 4.1 below. Each form is described using a notation $(a\ b\ c\ d\ e\ f\ )$, where a, b, c, d, e, f are corresponding integer coefficients of the form. So, $(a\ b\ c\ d\ e\ f\ )$ indicates the quaternary form $x^2 + ay^2 + bz^2 + cw^2 + dzw + eyw + fyz$. The first column of Table 4.1 contains an identifying number signifying the position of the form in the list [2, Table 5], and the second column gives the determinant. For reference, the third column gives the identifying symbol for this form as it appears in [4] (the notation Q_d^k denotes the kth form in Bhargava's list with determinant d).

Table 4.1. 52 Universal forms that remain to be considered

Index #	Det	Sym	$(a\ b\ c\ d\ e\ f)$		Index #	Det	Sym	$(a\ b\ c\ d\ e\ f)$
5	4	Q_4^1	(1 1 4 0 0 0)		132	52	Q_{52}^1	(2 3 9 2 0 0)
6	4	Q_4^2	(1 2 2 0 0 0)		133	52	Q_{52}^2	(2 5 6 2 0 2)
7	4	Q_4^3	(2 2 2 2 2 0)		134	52	Q_{52}^3	(2 5 6 4 0 0)
31	12	Q_{12}^4	(2 2 4 0 0 2)		144	56	Q_{56}^2	(2 4 8 4 0 0)
42	16	Q_{16}^1	(1 2 8 0 0 0)		150	60	Q_{60}^1	(2 3 10 0 0 0)
43	16	Q_{16}^2	(2 2 4 0 0 0)		151	60	Q_{60}^2	(2 4 9 4 2 0)
44	16	Q_{16}^3	(2 3 3 2 0 0)		152	60	Q_{60}^3	(2 5 6 0 0 0)
55	20	Q_{20}^1	(1 2 10 0 0 0)		158	64	Q_{64}^1	(2 4 8 0 0 0)
56	20	Q_{20}^2	(2 2 5 0 0 0)		160	68	Q_{68}^1	(2 4 9 0 2 0)
57	20	Q_{20}^3	(2 2 6 2 2 0)		161	68	Q_{68}^2	(2 4 10 4 2 0)
66	24	Q_{24}^1	(1 2 12 0 0 0)		162	68	Q_{68}^3	(2 5 7 2 0 0)
68	24	Q_{24}^3	(2 2 7 2 2 0)		166	72	Q_{72}^2	(2 4 10 4 0 0)
71	24	Q_{24}^6	(2 4 4 4 0 0)		167	72	Q_{72}^3	(2 5 8 4 0 0)
81	28	Q_{28}^1	(1 2 14 0 0 0)		169	76	Q_{76}^1	(2 4 10 0 2 0)
83	28	Q_{28}^3	(2 3 5 2 0 0)		173	80	Q_{80}^1	(2 4 10 0 0 0)
85	28	Q_{28}^5	(2 4 5 4 2 0)		174	80	Q_{80}^2	(2 4 11 4 0 0)
91	32	Q_{32}^2	(2 4 5 4 0 0)		175	80	Q_{80}^3	(2 5 8 0 0 0)
97	36	Q_{36}^1	(2 3 6 0 0 0)		182	88	Q_{88}^2	(2 4 12 4 0 0)
98	36	Q_{36}^2	(2 4 5 0 2 0)		183	88	Q_{88}^3	(2 5 9 2 0 0)
99	36	Q_{36}^3	(2 4 6 4 2 0)		186	92	Q_{92}^1	(2 4 13 4 2 0)
100	36	Q_{36}^4	(2 5 5 4 2 2)		187	92	Q_{92}^2	(2 5 10 4 0 0)
105	40	Q_{40}^1	(2 3 7 2 0 0)		192	96	Q_{96}^2	(2 4 13 4 0 0)
108	40	Q_{40}^4	(2 4 6 4 0 0)		195	100	Q_{100}^1	(2 4 13 0 2 0)
115	44	Q_{44}^1	(2 4 6 0 2 0)		196	100	Q_{100}^2	(2 4 14 4 2 0)
124	48	Q_{48}^1	(2 3 8 0 0 0)		200	104	Q_{104}^2	(2 4 14 4 0 0)
126	48	Q_{48}^3	(2 5 5 2 0 0)		202	108	Q_{108}^1	(2 4 14 0 2 0)

4.2 FORMS THAT ARE ALMOST PRIMITIVELY UNIVERSAL

Let L be a $\mathbb{Z}$-lattice corresponding to one of the forms listed in Table 4.1. Then L is almost primitively universal if and only if L_p is primitively $\mathbb{Z}_p$-universal for all primes p, where L_p denotes the p-adic completion of L. We will show that this is the case for the 24 lattices listed in Table 4.2 below. Here we use the result stated in [10]; *Let f be a quadratic form with determinant d. If $p \nmid 2d$, then f is primitively $\mathbb{Z}_p$-universal.* So, we only need to check whether a given form is primitively $\mathbb{Z}_p$-universal at each $p \mid 2d$, where d is the determinant of the quadratic form. Further, in all of the following cases when $p^2 \nmid d$ for odd prime p, we can conclude that the quadratic form is $\mathbb{Z}_p$-universal, by (3.3).

Table 4.2. Almost primitively universal forms

Index #	Det	Sym	L_2
56	20	Q_{20}^{2}	$\langle 1,5,2,2\rangle$
66	24	Q_{24}^{1}	$\langle 1,1,2,\ 2^2\cdot 3\rangle$
68	24	Q_{24}^{3}	$\langle 1,3,7,\ 2^3\cdot 7\rangle$
71	24	Q_{24}^{6}	$\langle 1,2\cdot 3,2\cdot 7,2\cdot 7\rangle$
81	28	Q_{28}^{1}	$\langle 1,1,2,2\cdot 7\rangle$
83	28	Q_{28}^{3}	$\langle 1,3,2,2\cdot 5\rangle$
91	32	Q_{32}^{2}	$\langle 1,5,2,\ 2^4\cdot 5\rangle$
105	40	Q_{40}^{1}	$\langle 1,3,2,\ 2^2\cdot 7\rangle$
108	40	Q_{40}^{4}	$\langle 1,2,2\cdot 3,2\cdot 7\rangle$
126	48	Q_{48}^{3}	$\langle 1,5,2,\ 2^3\cdot 7\rangle$
134	52	Q_{52}^{3}	$\langle 1,5,2,2\rangle$
144	56	Q_{56}^{2}	$\langle 1,2,2,2\cdot 7\rangle$
150	60	Q_{60}^{1}	$\langle 1,3,2,2\cdot 5\rangle$
152	60	Q_{60}^{3}	$\langle 1,5,2,2\cdot 3\rangle$
162	68	Q_{68}^{3}	$\langle 1,5,2,2\cdot 5\rangle$
166	72	Q_{72}^{2}	$\langle 1,2,2\cdot 5,2\cdot 5\rangle$
167	72	Q_{72}^{3}	$\langle 1,5,2,2^2\cdot 5\rangle$
173	80	Q_{80}^{1}	$\langle 1,2,2\cdot 5,2^2\rangle$
175	80	Q_{80}^{3}	$\langle 1,5,2,2^3\rangle$
182	88	Q_{88}^{2}	$\langle 1,2\cdot 3,2\cdot 7,2\cdot 7\rangle$
183	88	Q_{88}^{3}	$\langle 1,5,2,2^2\cdot 7\rangle$
187	92	Q_{92}^{2}	$\langle 1,5,2,2\cdot 3\rangle$
192	96	Q_{96}^{2}	$\langle 1,5,2,2^4\cdot 7\rangle$
200	104	Q_{104}^{2}	$\langle 1,2,2\cdot 3,2\cdot 7\rangle$

Observe first that the only determinant occurring in Table 4.2 that is divisible by the square of an odd prime is 72, which occurs for lattices # 166 and # 167. In both of these cases, L_3 has a splitting of the type $T \perp \langle 3^2 u \rangle$, where T is a ternary unimodular $\mathbb{Z}_3$-lattice and $u \in \mathbb{Z}_3^\times$. Hence, no lattice L appearing in Table 4.2 has a p-adic completion L_p for which the exceptional case of Proposition 3.1.8 holds for any odd prime p. Since all lattices L in Table 4.2 are universal, it follows that L_p is $\mathbb{Z}_p$-universal for all primes p. Hence, L_p is primitively $\mathbb{Z}_p$-universal for all lattices L in Table 4.2 and all odd primes p, by Proposition 3.1.8. Thus, in all the cases obstructions could come only from $\mathbb{Z}_2$.

So we need only further consider the case $p = 2$. Note that by Corollary 3.2.17, L_2 has a splitting of the type (3.19) for every L in [2, Table 5]. In particular, for every L in Table 4.2 such a splitting is given in the fourth column of the table, where the units ε_i are taken from $\{1, 3, 5, 7\}$. Since L_2 is $\mathbb{Z}_2$-universal for each L in Table 4.2, it follows from Theorem 3.2.21 that L_2 is primitively $\mathbb{Z}_2$-universal unless it has a splitting in one of the exceptional cases enumerated in (3.22) through (3.33).

We will now proceed to analyze the splittings of L_2 occurring in Table 4.2. This will be done by considering a list of representative examples and identifying all lattices in the table having splittings of the same type.

Example 4.2.1. Consider lattice Q_{20}^2 (# 56) given by $L \cong \langle 1, 2, 2, 5 \rangle$. The splitting (3.19) for L_2 has $\alpha_2 = 0,\ \alpha_3 = \alpha_4 = 1$ and $\varepsilon_1 = \varepsilon_3 = \varepsilon_4 = 1,\ \varepsilon_2 = 5$. Here $\varepsilon_1\varepsilon_2 \equiv 5 (\text{mod } 8\mathbb{Z}_2)$, but $\varepsilon_3 \not\equiv -\varepsilon_1 (\text{mod } 4\mathbb{Z}_2)$. Hence this splitting does not fall into the exceptional case (3.26) of Theorem 3.2.21. Therefore, L_2 is primitively $\mathbb{Z}_2$-universal.

The lattices # 81, 83, 134, 150, 152, 162, 187 will follow similar arguments. In these cases, the splitting of L_2 has $\alpha_2 = 0,\ \alpha_3 = \alpha_4 = 1$, but does not fall into any of the exceptional cases (3.24) through (3.26) of Theorem 3.2.21.

Example 4.2.2. Consider lattice Q_{24}^1 (# 66) given by $L \cong \langle 1, 1, 2, 12 \rangle$. The splitting (3.19) for L_2 has $\alpha_2 = 0,\ \alpha_3 = 1, \alpha_4 = 2$. No splittings of this type occur among the exceptional cases in Theorem 3.2.21. Therefore, L_2 is primitively $\mathbb{Z}_2$-universal.

The lattices # 68, 71, 91, 105, 108, 144, 166, 167, 182, 183, 192, 200 will follow a similar argument, since in all these cases the splitting of L_2 has $\alpha_2 = 0,\ \alpha_3 = 1,\ \alpha_4 = 2$.

Example 4.2.3. Consider lattice Q^1_{80} (# 173) given by $L \cong \langle 1, 2, 4, 10\rangle$. The splitting (3.19) for L_2 has $\alpha_2 = \alpha_3 = 1, \alpha_4 = 2$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_4 = 1,\ \varepsilon_3 = 5$. Here $\varepsilon_2\varepsilon_3 \equiv 5 (\text{mod } 8\mathbb{Z}_2)$, but $\varepsilon_1 \not\equiv -\varepsilon_2 (\text{mod } 4\mathbb{Z}_2)$. Hence this splitting does not fall into the exceptional case (3.30) of Theorem 3.2.21. Therefore, L_2 is primitively $\mathbb{Z}_2$-universal.

Example 4.2.4. Consider lattice Q^3_{80} (# 175) given by $L \cong \langle 1, 2, 5, 8\rangle$. The splitting (3.19) for L_2 has $\alpha_2 = 0,\ \alpha_3 = 1, \alpha_4 = 3$. Since $\langle 1, 5, 2\rangle$ is isotropic over $\mathbb{Z}_2$, L_2 does not fall in to the exceptional case (3.27) of Theorem 3.2.21. Therefore, L_2 is primitively $\mathbb{Z}_2$-universal.

The lattice # 126 will follow a similar argument.

This covers all forms in Table 4.2, and we summarize the results in the following statement.

Proposition 4.2.5. *The forms in Table 4.2 are almost primitively universal.*

A form that is almost primitively universal and is alone in its genus (that is, has class number 1) is in fact primitively universal. This yields the following three primitively universal quaternary forms that have not been previously identified.

Proposition 4.2.6. *The following forms are primitively universal:*

$$x^2 + 2y^2 + 3z^2 + 5w^2 + 2yw; \tag{4.1}$$

$$x^2 + 2y^2 + 3z^2 + 5w^2 + 2zw; \tag{4.2}$$

$$x^2 + 2y^2 + 4z^2 + 5w^2 + 4zw. \tag{4.3}$$

Proof. The form (4.1) is Q^2_{27} appearing in [4, Proposition 6]. The forms (4.2) and (4.3) are # 83 and # 91, respectively, in Table 4.2. These forms are almost primitively universal and have class number 1. □

4.3 FORMS THAT ARE NOT ALMOST PRIMITIVELY UNIVERSAL

In order to show that a lattice L is not almost primitively universal, it suffices to find one prime p for which L_p is not primitively $\mathbb{Z}_p$-universal. For the remaining forms given in Table 4.1, but not in Table 4.2, we will show that this is always the case for $p = 2$. The forms to be considered are given in Table 4.3, along with the splitting of L_2 for the corresponding lattice L.

Table 4.3. Forms which are not almost primitively universal

Index #	Det	Sym	L_2
5	4	Q_4^1	$\langle 1,1,1,2^2\rangle$
6	4	Q_4^2	$\langle 1,1,2,2\rangle$
7	4	Q_4^3	$\langle 3,7,7,\ 2^2\cdot 3\rangle$
31	12	Q_{12}^4	$\langle 3,7,7,4\rangle$
42	16	Q_{16}^1	$\langle 1,1,2,2^3\rangle$
43	16	Q_{16}^2	$\langle 1,2,2,2^2\rangle$
44	16	Q_{16}^3	$\langle 1,3,2,\ 2^3\cdot 3\rangle$
55	20	Q_{20}	$\langle 1,1,2,2\cdot 5\rangle$
57	20	Q_{20}^3	$\langle 3,7,7,\ 2^2\cdot 7\rangle$
85	28	Q_{28}^5	$\langle 1,5,5,\ 2^2\cdot 7\rangle$
97	36	Q_{36}^1	$\langle 1,3,2,2\cdot 3\rangle$
98	36	Q_{36}^2	$\langle 1,5,5,2^2\rangle$
99	36	Q_{36}^3	$\langle 3,3,3,\ 2^2\cdot 3\rangle$
100	36	Q_{36}^4	$\langle 1,5,5,2^2\rangle$
115	44	Q_{44}	$\langle 3,7,7,2^2\rangle$
124	48	Q_{48}	$\langle 1,3,2,2^3\rangle$
132	52	Q_{52}	$\langle 1,3,2,2\cdot 7\rangle$
133	52	Q_{52}^2	$\langle 1,5,5,\ 2^2\cdot 5\rangle$
151	60	Q_{60}^2	$\langle 1,1,1,\ 2^2\cdot 7\rangle$
158	64	Q_{64}^1	$\langle 1,2,2^2,2^3\rangle$
160	68	Q_{68}	$\langle 1,1,1,2^2\rangle$
161	68	Q_{68}^2	$\langle 3,7,7,\ 2^2\cdot 3\rangle$
169	76	Q_{76}	$\langle 3,7,7,2^2\rangle$
174	80	Q_{80}^2	$\langle 1,3,2,\ 2^3\cdot 7\rangle$
186	92	Q_{92}	$\langle 1,5,5,\ 2^2\cdot 7\rangle$
195	100	Q_{100}^1	$\langle 1,5,5,2^2\rangle$
196	100	Q_{100}^2	$\langle 3,3,3,\ 2^2\cdot 3\rangle$
202	108	Q_{108}	$\langle 3,7,7,2^2\rangle$

Example 4.3.1. Consider lattice Q_4^1 (# 5) given by $L \cong \langle 1,1,1,4\rangle$. The splitting (3.19)

for L_2 has $\alpha_2 = \alpha_3 = 0$, $\alpha_4 = 2$, and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$. Here L_2 falls into the exceptional case (3.23) of Theorem 3.2.21. Therefore, L_2 is not primitively $\mathbb{Z}_2$-universal.

The lattices # 7, 31, 57, 85, 98, 99, 100, 115, 133, 151, 160, 161, 169, 186, 195, 196, 202 will follow the same argument since in all of these cases the splitting of L_2 falls into the exceptional case (3.23) of Theorem 3.2.21.

Example 4.3.2. Consider lattice Q_{20}^1 (# 55) given by $L \cong \langle 1, 1, 2, 10 \rangle$. The splitting (3.19) for L_2 has $\alpha_2 = 0$, $\alpha_3 = \alpha_4 = 1$, and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, $\varepsilon_4 = 5$. Here L_2 falls into the exceptional case (3.24) of Theorem 3.2.21. Therefore, L_2 is not primitively $\mathbb{Z}_2$-universal.

The lattice # 6 will follow a similar argument, and # 97, 132 fall into the exceptional case (3.25) of Theorem 3.2.21.

Example 4.3.3. Consider lattice Q_{16}^1 (# 42) given by $L \cong \langle 1, 1, 2, 8 \rangle$. The splitting (3.19) for L_2 has $\alpha_2 = 0$, $\alpha_3 = 1$, $\alpha_4 = 3$. Since $\langle 1, 1, 2 \rangle$ is anisotropic, L_2 falls into the exceptional case (3.27) of Theorem 3.2.21. Therefore, L_2 is not primitively $\mathbb{Z}_2$-universal.

The lattices # 44, 124, 174 will follow similar arguments.

Example 4.3.4. Consider lattice Q_{16}^2 (# 43) given by $L \cong \langle 1, 2, 2, 4 \rangle$. The splitting (3.19) for L_2 has $\alpha_2 = \alpha_3 = 1$, $\alpha_4 = 2$, and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$. Here L_2 falls into the exceptional case (3.28) of Theorem 3.2.21. Therefore, L_2 is not primitively $\mathbb{Z}_2$-universal.

Example 4.3.5. Consider lattice Q_{64}^1 (# 158) given by $L \cong \langle 1, 2, 4, 8 \rangle$. The splitting (3.19) for L_2 has $\alpha_2 = 1$, $\alpha_3 = 2$, $\alpha_4 = 3$, and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 1$. Here L_2 falls into the exceptional case (3.31) of Theorem 3.2.21. Therefore, L_2 is not primitively $\mathbb{Z}_2$-universal.

This covers all forms in Table 4.3, and we summarize the results in the following statement.

Proposition 4.3.6. *The forms in Table 4.3 are not almost primitively universal.*

Remark 4.3.7. Proposition 2.2.1 and the Corollary 2.2.3 provide an alternate approach for showing that L_2 is not primitively $\mathbb{Z}_2$-universal for some of the forms in Table 4.3.

Let L be a $\mathbb{Z}$-lattice on V, where V is a quaternary quadratic space over $\mathbb{Q}_p$. Suppose $dV_p = \dot{\mathbb{Q}}_p^2$ and $S_pV_p = -(-1,-1)_p$. Then V_p is anisotropic, by (2.2). Now, Proposition 2.2.1 implies L is not almost primitively universal. This works well over dyadic fields.

Example 4.3.8. Consider lattice Q_4^1 (# 5) given by $L \cong \langle 1,1,1,4\rangle$. Then $V_2 \cong \langle 1,1,1,4\rangle$. We can see $dV_2 = 4\dot{\mathbb{Q}}_2^2 = \dot{\mathbb{Q}}_2^2$ and $S_2V_2 = S_2(\langle 1,1,1,4\rangle) = 1 = -(-1) = -(-1,-1)_2$. So, V_2 is anisotropic. Thus, L is not almost primitively universal.

Example 4.3.9. Consider the non-diagonal lattice Q_{36}^4 (# 100) given by $L \cong \langle 1\rangle \perp \left(\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 5 \end{smallmatrix}\right)$. Then $V_2 \cong \langle 1,1\rangle \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 5 \end{smallmatrix}\right) \cong \langle 1,1,2,2\cdot 9\rangle \cong \langle 1,1,2,2\rangle$. We can see $dV_2 = \dot{\mathbb{Q}}_2^2$ and $S_2V_2 = S_2(\langle 1,1,2,2\rangle) = (2,2)_2 = 1 = -(-1) = -(-1,-1)_2$. So, V_2 is anisotropic. Thus, L is not almost primitively universal.

Following this method, we can conclude the lattices # 5, 6, 7, 42, 43, 44, 97, 98, 99, 100, 158, 160, 161, 195, 196 are not almost primitively universal.

CHAPTER 5

PRIMITIVE COUNTERPART TO THE 15-THEOREM

In this chapter, we will study the primitive universality of the forms in the list [2, Table 5] of universal positive definite classically integral quaternary quadratic forms (see Figure 4.1). Among these forms, 34 have been previously proven to be primitively universal. For the remaining 170 forms, we will search the range 1 to 400 for positive integers that are not primitively represented by the form. The smallest such positive integer, if one exists, is called the *primitive truant* of the form. Any form which has a primitive truant is ruled out for primitive universality; those for which none is found by our search remain as candidates for primitive universality. Three of these candidates were proven to be primitively universal in Proposition 4.2.6. In this chapter, we will supply proofs of primitive universality for an additional 25 of the candidates.

A primitive counterpart to the Fifteen Theorem would have the following form: *there exists a finite set S of positive integers such that every positive definite classically integral quadratic form (regardless of rank) that primitively represents all the integers in S is primitively universal.* In particular, any such set S must contain all of the primitive truants of the quaternary forms found by our search. If the remaining candidates for primitive universality that we have identified can be proven to be primitively universal, then the list consisting of Bhargava's nine critical numbers and the set of primitive truants that we have found would constitute a complete set S for quaternary forms. This leads us to formulate Conjecture 0.2.5. Of course there is no guarantee that additional primitive truants will not appear for forms of higher rank; however, in Bhargava's original work it was seen that no new truants (without considering primitivity) occur beyond rank 4.

5.1 PREVIOUS RESULTS AND METHODOLOGY

We have already introduced and discussed about the Fifteen Theorem and the results of studies related to this topic in Chapter 4. Budarina proved that, among 113 almost primitively universal forms listed in [4], ten forms are primitively universal (see Figure 5.1).

Later, in their study on strictly universal forms which are diagonal, Earnest, Kim and Meyer [8] identified 27 primitively universal forms (see Figure 4.2) in [2, Table 5]. With the above two results we have a total of 34 primitively universal forms identified so far, leaving 170 forms to be checked for primitive universality.

We used the mathematical software SAGE [18] to check whether each of these forms have any primitive truant. Each form was checked up to 400. Out of these 170 forms 97 of them gave primitive truants, leaving 73 forms without any primitive truant up to 400. Interestingly, the set of primitive truants up to 400 was { 4, 8, 9, 12, 16, 24, 25, 32, 48, 49, 64 }, a small set with just 11 elements and out of them 9, 25, 49, 64 appeared only once while other values repeated several times.

		Bhargava's notation
Q_2^1:	${x_1}^2 + {x_2}^2 + {x_3}^2 + 2{x_4}^2$	(1 1 2 0 0 0)
Q_3^1:	${x_1}^2 + {x_2}^2 + {x_3}^2 + 3{x_4}^2$	(1 1 3 0 0 0)
Q_3^2:	${x_1}^2 + {x_2}^2 + 2{x_3}^2 + 2{x_4}^2 + 2x_3x_4$	(1 2 2 2 0 0)
Q_5^2:	${x_1}^2 + {x_2}^2 + 2{x_3}^2 + 3{x_4}^2 + 2x_3x_4$	(1 2 3 2 0 0)
Q_6^2:	${x_1}^2 + {x_2}^2 + 2{x_3}^2 + 3{x_4}^2$	(1 2 3 0 0 0)
Q_7^3:	${x_1}^2 + 2{x_2}^2 + 2{x_3}^2 + 3{x_4}^2 + 2x_2x_3 + 2x_3x_4$	(2 2 3 2 0 2)
Q_8^2:	${x_1}^2 + {x_2}^2 + 3{x_3}^2 + 3{x_4}^2 + 2x_3x_4$	(1 3 3 2 0 0)
Q_8^4:	${x_1}^2 + 2{x_2}^2 + 2{x_3}^2 + 3{x_4}^2 + 2x_2x_4 + 2x_3x_4$	(2 2 3 2 2 0)
Q_{12}^5:	${x_1}^2 + 2{x_2}^2 + 3{x_3}^2 + 3{x_4}^2 + 2x_2x_3 + 2x_2x_4$	(2 3 3 0 2 2)
Q_{15}^4:	${x_1}^2 + 2{x_2}^2 + 3{x_3}^2 + 3{x_4}^2 + 2x_2x_4$	(2 3 3 0 2 0)

Figure 5.1. 10 primitively universal quaternary forms identified by Budarina [4]

The remainder of this chapter consists of a discussion of the existence of a primitive

Fifteen Theorem and theoretical arguments to prove that some of the remaining 73 forms are primitively universal. So far we have been able to prove 28 of them to be primitively universal.

For convenience, we will list all remaining 170 forms at the end of this section for later reference. The notations and the symbols for each form are the same as we described in Chapter 4, Section 4.1. Here we have included the primitive truant (if any) obtained up to 400, in the fifth column, and if we could not find any primitive truant up to 400, we have listed the class number of the form which we will be using for further analysis of that form.

Table 5.1. Primitive truants for universal quaternary forms - Part I

Index #	Det	Sym	(a b c d e f)	Class #	Primitive truant up to 400
1	1	Q_1^1	(1 1 1 0 0 0)		8
5	4	Q_4^1	(1 1 4 0 0 0)		32
6	4	Q_4^2	(1 2 2 0 0 0)		16
7	4	Q_4^3	(2 2 2 2 2 0)		32
8	5	Q_5^1	(1 1 5 0 0 0)		4
10	6	Q_6^1	(1 1 6 0 0 0)		4
12	6	Q_6^3	(2 2 2 2 0 0)	2	None
13	7	Q_7^1	(1 1 7 0 0 0)		4
14	7	Q_7^2	(1 2 4 2 0 0)	2	None
20	9	Q_9^1	(1 2 5 2 0 0)		8
21	9	Q_9^2	(1 3 3 0 0 0)		9
22	9	Q_9^3	(2 2 3 0 0 2)		8
24	10	Q_{10}^2	(2 2 3 2 0 0)	2	None
25	10	Q_{10}^3	(2 2 4 2 0 2)	2	None
26	11	Q_{11}^1	(1 2 6 2 0 0)		None
27	11	Q_{11}^2	(1 3 4 2 0 0)	2	None
31	12	Q_{12}^4	(2 2 4 0 0 2)		24
33	13	Q_{13}^1	(2 2 5 2 0 2)		4
34	13	Q_{13}^2	(2 3 3 2 2 0)	2	None
36	14	Q_{14}^2	(1 3 5 2 0 0)	2	None
37	14	Q_{14}^3	(2 2 4 2 0 0)	3	None
38	15	Q_{15}^1	(1 2 8 2 0 0)	2	None
40	15	Q_{15}^3	(2 2 5 0 0 2)	2	None
42	16	Q_{16}^1	(1 2 8 0 0 0)		64
43	16	Q_{16}^2	(2 2 4 0 0 0)		32
44	16	Q_{16}^3	(2 3 3 2 0 0)		64

Table 5.2. Primitive truants for universal quaternary forms - Part II

Index #	Det	Sym	(a b c d e f)	Class #	Primitive truant up to 400
45	17	Q^1_{17}	(1 2 9 2 0 0)		8
46	17	Q^2_{17}	(1 3 6 2 0 0)	2	None
47	17	Q^3_{17}	(2 3 4 0 2 2)	2	None
48	18	Q^1_{18}	(1 2 9 0 0 0)		8
50	18	Q^3_{18}	(2 2 5 2 0 0)	3	None
52	18	Q^5_{18}	(2 3 4 2 0 2)	2	None
53	19	Q^1_{19}	(1 2 10 2 0 0)		8
54	19	Q^2_{19}	(2 3 4 2 2 0)	3	None
55	20	Q^1_{20}	(1 2 10 0 0 0)		8
56	20	Q^2_{20}	(2 2 5 0 0 0)		12
57	20	Q^3_{20}	(2 2 6 2 2 0)		12
58	20	Q^4_{20}	(2 3 4 0 0 2)	2	None
59	20	Q^5_{20}	(2 4 4 4 2 0)	2	None
60	22	Q^1_{22}	(1 2 11 0 0 0)		8
61	22	Q^2_{22}	(2 2 6 2 0 0)	4	None
62	22	Q^3_{22}	(2 3 4 2 0 0)	3	None
63	22	Q^4_{22}	(2 3 5 0 2 2)	4	None
64	23	Q^1_{23}	(1 2 12 2 0 0)		8
65	20	Q^2_{23}	(2 3 5 2 0 2)	2	None
66	24	Q^1_{24}	(1 2 12 0 0 0)		8
68	24	Q^3_{24}	(2 2 7 2 2 0)	2	None
70	24	Q^5_{24}	(2 4 4 0 2 2)	2	None
71	24	Q^6_{24}	(2 4 4 4 0 0)	2	None
72	25	Q^1_{25}	(1 2 13 2 0 0)		8
73	25	Q^2_{25}	(2 3 5 0 0 2)		25
74	25	Q^3_{25}	(2 3 5 2 2 0)		8
75	26	Q^1_{26}	(1 2 13 0 0 0)		8
76	26	Q^2_{26}	(2 2 7 2 0 0)	4	None
77	26	Q^3_{26}	(2 4 4 2 2 0)	4	None
78	27	Q^1_{27}	(1 2 14 2 0 0)		8
79	27	Q^2_{27}	(2 3 5 0 2 0)	1	None
80	27	Q^3_{27}	(2 4 5 4 0 2)	4	None
81	28	Q^1_{28}	(1 2 14 0 0 0)		8
83	28	Q^3_{28}	(2 3 5 2 0 0)	1	None
84	28	Q^4_{28}	(2 4 4 0 2 0)	4	None
85	28	Q^5_{28}	(2 4 5 4 2 0)		24
87	30	Q^2_{30}	(2 4 4 2 0 0)	4	None
88	31	Q^1_{31}	(2 3 6 2 2 0)	3	None
89	31	Q^2_{31}	(2 4 5 0 2 2)	6	None
91	32	Q^2_{32}	(2 4 5 4 0 0)	1	None

Table 5.3. Primitive truants for universal quaternary forms - Part III

Index #	Det	Sym	$(a\ b\ c\ d\ e\ f)$	Class #	Primitive truant up to 400
92	33	Q_{33}^{1}	(2 3 6 0 2 0)		8
93	34	Q_{34}^{1}	(2 3 6 2 0 0)	5	None
94	34	Q_{34}^{2}	(2 4 5 2 2 0)	5	None
95	34	Q_{34}^{3}	(2 4 6 4 0 2)	5	None
96	32	Q_{32}^{2}	(2 4 5 0 0 2)	4	None
97	36	Q_{36}^{1}	(2 3 6 0 0 0)		16
98	36	Q_{36}^{2}	(2 4 5 0 2 0)		32
99	36	Q_{36}^{3}	(2 4 6 4 2 0)		32
100	36	Q_{36}^{4}	(2 5 5 4 2 2)		4
101	37	Q_{37}^{1}	(2 5 5 4 2 0)		4
102	38	Q_{38}^{1}	(2 4 5 2 0 0)	5	None
103	38	Q_{38}^{2}	(2 4 6 0 2 2)	6	None
104	39	Q_{39}^{1}	(2 3 7 0 2 0)	3	None
105	40	Q_{40}^{1}	(2 3 7 2 0 0)	3	None
107	40	Q_{40}^{3}	(2 4 6 2 0 2)	3	None
108	40	Q_{40}^{4}	(2 4 6 4 0 0)	2	None
109	40	Q_{40}^{5}	(2 4 7 4 0 2)	5	None
111	42	Q_{42}^{2}	(2 4 6 0 0 2)	6	None
112	42	Q_{42}^{3}	(2 4 6 2 2 0)	3	None
113	42	Q_{42}^{4}	(2 5 5 4 0 0)		4
114	43	Q_{43}^{1}	(2 3 8 2 2 0)	6	None
115	44	Q_{44}^{1}	(2 4 6 0 2 0)		24
116	45	Q_{45}^{1}	(2 4 7 0 2 2)	4	None
117	45	Q_{45}^{2}	(2 5 5 0 2 0)		4
118	45	Q_{45}^{3}	(2 2 6 4 2 2)		4
119	46	Q_{46}^{1}	(2 3 8 2 0 0)	8	None
120	46	Q_{46}^{2}	(2 4 6 2 0 0)	8	None
121	46	Q_{46}^{3}	(2 5 6 4 0 2)		4
122	47	Q_{47}^{1}	(2 4 7 2 0 2)	9	None
123	47	Q_{47}^{2}	(2 5 6 4 2 0)		4
124	48	Q_{48}^{1}	(2 3 8 0 0 0)		48
126	48	Q_{48}^{3}	(2 5 5 2 0 0)		4
127	49	Q_{49}^{1}	(2 5 5 2 2 0)		8
128	49	Q_{49}^{2}	(2 4 7 0 0 2)		49
129	49	Q_{49}^{3}	(2 5 6 0 2 2)		4
130	50	Q_{50}^{1}	(2 4 7 2 2 0)	6	None
131	51	Q_{41}^{1}	(2 3 9 0 2 0)		8
132	52	Q_{52}^{1}	(2 3 9 2 0 0)		8
133	52	Q_{52}^{2}	(2 5 6 2 0 2)		4
134	52	Q_{52}^{3}	(2 5 6 4 0 0)		4

Table 5.4. Primitive truants for universal quaternary forms - Part IV

Index #	Det	Sym	$(a\ b\ c\ d\ e\ f)$	Class #	Primitive truant up to 400
135	53	Q_{53}^1	(2 5 6 2 2 0)		4
136	54	Q_{54}^1	(2 3 9 0 0 0)		8
137	54	Q_{54}^2	(2 4 7 2 0 0)	6	None
138	54	Q_{54}^3	(2 5 6 0 0 2)		4
139	54	Q_{54}^4	(2 5 7 4 2 2)		4
140	55	Q_{55}^1	(2 3 10 2 2 0)		8
141	55	Q_{55}^2	(2 5 6 0 2 0)		4
142	55	Q_{55}^3	(2 5 7 4 0 2)		4
144	56	Q_{56}^2	(2 4 8 4 0 0)	3	None
145	57	Q_{57}^1	(2 3 10 0 2 0)		8
146	58	Q_{58}^1	(2 3 10 2 0 0)		8
147	58	Q_{58}^2	(2 4 8 2 2 0)	8	None
148	58	Q_{58}^3	(2 5 6 2 0 0)		4
149	58	Q_{58}^4	(2 5 7 0 2 2)		4
150	60	Q_{60}^1	(2 3 10 0 0 0)		8
151	60	Q_{60}^2	(2 4 9 4 2 0)		24
152	60	Q_{60}^3	(2 5 6 0 0 0)		4
153	61	Q_{61}^1	(2 5 7 2 0 2)		4
154	62	Q_{62}^1	(2 4 8 2 0 0)	10	None
155	62	Q_{62}^2	(2 5 7 4 0 0)		4
156	63	Q_{63}^1	(2 5 7 0 0 2)		4
157	63	Q_{63}^2	(2 5 7 2 2 0)		4
158	64	Q_{64}^1	(2 4 8 0 0 0)		64
159	66	Q_{66}^1	(2 4 9 2 2 0)	6	None
160	68	Q_{68}^1	(2 4 9 0 2 0)		24
161	68	Q_{68}^1	(2 4 10 4 2 0)		24
162	68	Q_{68}^2	(2 5 7 2 0 0)		4
163	70	Q_{70}^1	(2 4 9 2 0 0)	6	None
164	70	Q_{70}^2	(2 5 7 0 0 0)		4
166	72	Q_{72}^2	(2 4 10 4 0 0)	3	None
167	72	Q_{72}^3	(2 5 8 4 0 0)		4
168	74	Q_{74}^1	(2 4 10 2 2 0)	10	None
169	76	Q_{76}^1	(2 4 10 0 2 0)		24
170	77	Q_{77}^1	(2 5 9 4 2 0)		4
171	78	Q_{78}^1	(2 4 10 2 0 0)	7	None
172	78	Q_{78}^2	(2 5 8 2 0 0)		4
173	80	Q_{80}^1	(2 4 10 0 0 0)		24
174	80	Q_{80}^2	(2 4 11 4 0 0)		24
175	80	Q_{80}^3	(2 5 8 0 0 0)		4
176	82	Q_{82}^1	(2 4 11 2 2 0)	11	None

Table 5.5. Primitive truants for universal quaternary forms - Part V

Index #	Det	Sym	$(a\ b\ c\ d\ e\ f)$	Class #	Primitive truant up to 400
177	82	Q_{82}^2	(2 5 9 4 0 0)		4
178	85	Q_{80}^2	(2 5 9 0 2 0)		4
179	86	Q_{86}^1	(2 4 11 2 0 0)	11	None
180	87	Q_{87}^1	(2 5 10 4 2 0)		4
182	88	Q_{88}^2	(2 4 12 4 0 0)	4	None
183	88	Q_{88}^3	(2 5 9 2 0 0)		4
184	90	Q_{90}^1	(2 4 12 2 2 0)	7	None
185	90	Q_{90}^2	(2 5 9 0 0 0)		4
186	92	Q_{92}^1	(2 4 13 4 2 0)		24
187	92	Q_{92}^2	(2 5 10 4 0 0)		4
188	93	Q_{93}^1	(2 5 10 2 2 0)		4
189	94	Q_{94}^1	(2 4 12 2 0 0)	16	None
190	95	Q_{80}^2	(2 5 10 0 2 0)		4
192	96	Q_{96}^2	(2 4 13 4 0 0)	3	None
193	98	Q_{98}^1	(2 4 13 2 2 0)	11	None
194	98	Q_{98}^2	(2 5 10 2 0 0)		4
195	100	Q_{100}^1	(2 4 13 0 2 0)		24
196	100	Q_{100}^2	(2 4 14 4 2 0)		24
197	100	Q_{100}^3	(2 5 10 0 0 0)		4
198	102	Q_{102}^1	(2 4 13 2 0 0)	9	None
200	104	Q_{104}^3	(2 4 14 4 0 0)	4	None
201	106	Q_{106}^1	(2 4 14 2 2 0)	15	None
202	108	Q_{108}^1	(2 4 14 0 2 0)		24
203	110	Q_{110}^1	(2 4 14 2 0 0)	10	None

Remark 5.1.1. Note that when a quadratic form is not almost primitively universal, clearly it is not primitively universal. In [4], Budarina has listed 24 forms which are not almost primitively universal. In Chapter 4, we stated another 28 forms in Proposition 4.3.6, which are not almost primitively universal. So one can see these 52 forms are not primitively universal right away. However, these arguments will not give any method to find the value of the primitive truant, which emphasize the importance of utilizing a computational method such as SAGE [18].

In [8], Earnest, Kim and Meyer have analyzed all the diagonal forms in [2, Table 5]. They have identified 27 forms out of those 96 forms as primitively universal, which on the other hand implies that the remaining 69 diagonal forms are not primitively universal. But one should note that these forms can be almost primitively universal. So we have included these forms in our analysis in Chapter 4, to come up with Proposition 4.3.6.

5.2 SOME TERNARY LATTICES

Here we will analyze some ternary lattices that are frequently used in our calculations, to find what they can primitively represent locally at each prime p. Recall that we only have to check $p = 2$ and the primes which divide the determinant of the lattice. Most of the ternary lattices can represent a large set of positive integers, but no ternary positive definite integral quadratic $\mathbb{Z}$-lattice is universal. We will use the following lemma to prove that[1].

Lemma 5.2.1. *Let (V, q) be a regular quadratic space over $\mathbb{Q}$ with dimension 3. Let S be the set of all finite primes. Then for any $p \in S$, V_p is universal over $\mathbb{Q}_p$ if and only if $S_pV \cdot (dV, -1)_p = (-1, -1)_p$.*

Proof. Follows from Lemma 2.3.2 and (2.1). □

Proposition 5.2.2. *If a regular quadratic space (V, q) over $\mathbb{Q}$ is positive definite, then there exists $p \in S$ such that V_p is not universal over $\mathbb{Q}_p$ when dimension of V is 3.*

Proof. The Lemma 5.2.1 implies that, if $S_pV \cdot (dV, -1)_p \neq (-1, -1)_p$ for some $p \in S$, then V_p is not universal over $\mathbb{Q}_p$. Suppose $S_pV \cdot (dV, -1)_p = (-1, -1)_p$ for all $p \in S$. Then

$$1 = \prod_{p \,\in\, \Omega} S_pV \cdot (dV, -1)_p = (\prod_{p \,\in\, S} S_pV \cdot (dV, -1)_p) S_\infty V \cdot (dV, -1)_\infty$$

where $\Omega = S \cup \{\infty\}$ and $S_\infty V = 1$ and $(dV, -1)_\infty = 1$ since $V \cong \langle 1, \dots, 1\rangle$ over $\mathbb{R}$. Hence it follows:

$$1 = \prod_{p \,\in\, S} S_pV \cdot (dV, -1)_p = \prod_{p \,\in\, S} (-1, -1)_p = (-1, -1)_2 \prod_{p \ odd} (-1, -1)_p = (-1, -1)_2 = -1$$

and we arrive at a contradiction. □

[1]For a proof using only simple congruence properties, see [7].

Another useful theorem we will be using frequently is [10, Theorem 6.28], stated below:

Theorem 5.2.3. *Let L be a lattice on the regular space V, and let $M \subseteq L$ be an $\mathfrak{U}$-modular sublattice. Then M splits L (i.e., $L = M \perp L'$ for some sublattice L') if and only if $B(M, L) \subseteq \mathfrak{U}$. In particular, every unimodular sublattice of an integral lattice L splits L.*

This theorem helps to find a corresponding diagonal form of a lattice or a simplified form of a lattice which will make it easier for our computations.

Example 5.2.4. Consider the ternary lattice $K \cong \left(\begin{smallmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{smallmatrix}\right)$. Note that the *det* $K = 10$. So we are interested in finding the diagonal or simplified form in $\mathbb{Z}_2$ and $\mathbb{Z}_5$. Let us consider $\mathbb{Z}_2$ first. Note that the binary lattice $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right) \cong \mathbb{A}$ has a determinant 3. So $\mathbb{A}$ is a unimodular sublattice of K_2, and by Theorem 5.2.3, $\mathbb{A}$ splits K_2; say $K_2 \cong \mathbb{A} \perp M$. Now using discriminant, $d(\mathbb{A})d(M) = 3d(M) \in 10(\mathbb{Z}_2^\times)^2$ implies $d(M) \in 30(\mathbb{Z}_2^\times)^2 = 2 \cdot 7(\mathbb{Z}_2^\times)^2$; so, $K_2 \cong \mathbb{A} \perp \langle 2 \cdot 7 \rangle$. Note that in $\mathbb{Z}_2$, $\mathbb{A}$ cannot be simplified further.

Next consider $\mathbb{Z}_5$. Again we can see $\mathbb{A}$ is a unimodular sublattice in $\mathbb{Z}_5$, hence $K_5 \cong \mathbb{A} \perp M$, for some M. Note that $d(\mathbb{A})d(M) = 3d(M) \in 10(\mathbb{Z}_5^\times)^2$ implies $d(M) \in 30(\mathbb{Z}_5^\times)^2 = 6 \cdot 5(\mathbb{Z}_5^\times)^2 = 5(\mathbb{Z}_5^\times)^2$. Then $K \cong \mathbb{A} \perp \langle 5 \rangle$. But 2 is again a unit in $\mathbb{Z}_5$, so $\langle 2 \rangle$ splits $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. Then $\mathbb{A} \cong \langle 2 \rangle \perp \langle \beta \rangle$ such that $2d(\langle \beta \rangle) \in 3(\mathbb{Z}_5^\times)^2$, which implies $d(\langle \beta \rangle) \in 6(\mathbb{Z}_5^\times)^2 = 1(\mathbb{Z}_5^\times)^2$. So $K_5 \cong \langle 1, 2, 5 \rangle$.

Now we will provide local computations of the representations of some ternary lattices which will give enough background to proceed with our proofs in the next section.

Example 5.2.5. Let $K \cong \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{smallmatrix}\right) = \langle 1 \rangle \perp \mathbb{A}$, with respect to basis $\{v_1, v_2, v_3\}$. Note that the *det* $K = 3$, so K_p is primitively $\mathbb{Z}_p$-universal for $p \neq 2, 3$. It is easy to see $K_3 \cong \langle 1, 2, 2 \cdot 3 \rangle$, and is $\mathbb{Z}_3$-universal by Proposition 3.1.6, case (3.4). We can check that it does not fall in to the exceptional case in Proposition 3.1.8. Thus, K_3 is primitively $\mathbb{Z}_3$-universal.

We will use the argument in [10, page 168] to analyze K_2. Let $M = \mathbb{Z}_2(v_1+v_2)+\mathbb{Z}_2 v_3 \cong \left(\begin{smallmatrix} 3 & 1 \\ 1 & 2 \end{smallmatrix}\right) \cong \langle 3, 7\rangle$. So M is unimodular. Now by Theorem 5.2.3, M splits K_2. This will give $K_2 \cong \langle 3, 7, 7\rangle$.

Now we will determine what K_2 primitively represents. Among the elements of $\mathbb{Z}_2^\times$, it is easy to see that $\{1, 3, 7\} \xrightarrow{*} K_2$, but $5 \stackrel{*}{\not\to} K_2$ by Lemma 2.3.2. Also, $2\mathbb{Z}_2^\times \xrightarrow{*} K_2$ by Lemma 3.2.9. Suppose $a = 3x_1^2 + 7x_2^2 + 7x_3^2 \in 2\mathbb{Z}_2$, where $x_1, x_2, x_3 \in \mathbb{Z}_2$ with at least one $x_i \in \mathbb{Z}_2^\times$. Then it must be that exactly two of the x_i's are units. If $x_1, x_2 \in \mathbb{Z}_2^\times$ (or $x_1, x_3 \in \mathbb{Z}_2^\times$), then $a \equiv 3 + 7 + 0 \equiv 2 (\text{mod } 4\mathbb{Z}_2)$. If $x_2, x_3 \in \mathbb{Z}_2^\times$, then $a \equiv 0 + 7 + 7 \equiv 2 (\text{mod } 4\mathbb{Z}_2)$. In both cases, $a \notin 4\mathbb{Z}_2$. Thus,

$$q^*(K_2) = q^*(\langle 3, 7, 7\rangle) = (\mathbb{Z}_2^\times)^2 \cup 3(\mathbb{Z}_2^\times)^2 \cup 7(\mathbb{Z}_2^\times)^2 \cup 2\mathbb{Z}_2^\times.$$

Example 5.2.6. Let $K \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Note that the *det* $K = 5$, so we only have to check $\mathbb{Z}_2$ and $\mathbb{Z}_5$. Here $K_2 \cong \langle 1, 3, 7\rangle$ is primitively $\mathbb{Z}_2$-universal, by Lemma 3.2.4.

Now, in $\mathbb{Z}_5$, we have $K_5 \cong \langle 1, 2, 2 \cdot 5\rangle$, which will give $q^*(K_5) = \mathbb{Z}_5^\times \cup 5\Delta(\mathbb{Z}_5^\times)^2$, where Δ is a non-square unit. (In most of the computations the fact $\mathbb{Z}_5^\times \subseteq q^*(K_5)$ is sufficient.)

Example 5.2.7. Let $K \cong \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 4 \end{pmatrix}$. Note that the *det* $K = 24$, so we only have to check $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Here $K_3 \cong \langle 2, 4, 3\rangle \cong \langle 1, 2, 3\rangle$ is primitively $\mathbb{Z}_2$-universal, by Propositions 3.1.6 and 3.1.8.

Now, in $\mathbb{Z}_2$, we have $K_2 \cong \langle 2\rangle \perp \left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)$. Clearly $q^*(K_2) \cap \mathbb{Z}_2^\times = \emptyset$, and we know $q^*(\left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)_2) \cap 4\mathbb{Z}_2 = 4\mathbb{Z}_2^\times$.

Note that if $M \cong \langle 1\rangle \perp \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, then by scaling M by 2 we obtain K_2. By Example 5.2.5 we know $q^*(M) \cap \mathbb{Z}_2^\times = (\mathbb{Z}_2^\times)^2 \cup 3(\mathbb{Z}_2^\times)^2 \cup 7(\mathbb{Z}_2^\times)^2$. This implies $q^*(K_2) = 2(\mathbb{Z}_2^\times)^2 \cup 6(\mathbb{Z}_2^\times)^2 \cup 14(\mathbb{Z}_2^\times)^2 \cup 4\mathbb{Z}_2^\times$.

Example 5.2.8. Let $K \cong \langle 1, 2, 4\rangle$. Since the *det* $K = 8$, K_p is primitively $\mathbb{Z}_p$-universal for all odd primes p. By Lemma 3.2.8 *i*), $\mathbb{Z}_2^\times \xrightarrow{*} K_2$. Then by [5, Theorem 5.1, page 143], $n \xrightarrow{*} \textit{gen } K$ for all odd n. It follows that $n \xrightarrow{*} K$ for all odd n since K has class number 1.

5.3 SOME PROOFS OF PRIMITIVE UNIVERSALITY

Now we are ready to present the outline of our proofs of 28 forms which we identified as primitively universal, among the remaining 73 forms listed in Section 5.1 without any primitive truant up to 400. For convenience, we will be presenting some selected proofs, which will give the idea of our methodology. The following proposition contains these 28 forms. Note that we have already stated and proved three out of these 28 forms which are of class number 1, in Proposition 4.2.6. For completeness, those three forms are restated here.

Proposition 5.3.1. *The following forms are primitively universal:*

$$x^2 + 2y^2 + 2z^2 + 2w^2 + 2zw; \tag{5.1}$$

$$x^2 + y^2 + 2z^2 + 4w^2 + 2zw; \tag{5.2}$$

$$x^2 + 2y^2 + 2z^2 + 3w^2 + 2zw; \tag{5.3}$$

$$x^2 + 2y^2 + 2z^2 + 4w^2 + 2yz + 2zw; \tag{5.4}$$

$$x^2 + y^2 + 2z^2 + 6w^2 + 2zw; \tag{5.5}$$

$$x^2 + y^2 + 3z^2 + 4w^2 + 2zw; \tag{5.6}$$

$$x^2 + y^2 + 3z^2 + 5w^2 + 2zw; \tag{5.7}$$

$$x^2 + 2y^2 + 2z^2 + 4w^2 + 2zw; \tag{5.8}$$

$$x^2 + 2y^2 + 2z^2 + 5w^2 + 2zw; \tag{5.9}$$

$$x^2 + 2y^2 + 3z^2 + 4w^2 + 2yz; \tag{5.10}$$

$$x^2 + 2y^2 + 4z^2 + 4w^2 + 4zw + 2yw; \tag{5.11}$$

$$x^2 + 2y^2 + 2z^2 + 6w^2 + 2zw; \tag{5.12}$$

$$x^2 + 2y^2 + 3z^2 + 5w^2 + 2yz + 2zw; \tag{5.13}$$

$$x^2 + 2y^2 + 4z^2 + 4w^2 + 2yz + 2yw; \tag{5.14}$$

$$x^2 + 2y^2 + 4z^2 + 4w^2 + 4zw; \tag{5.15}$$

$$x^2 + 2y^2 + 4z^2 + 4w^2 + 2yw + 2zw; \tag{5.16}$$

$$x^2 + 2y^2 + 3z^2 + 5w^2 + 2yw; \tag{5.17}$$

$$x^2 + 2y^2 + 3z^2 + 5w^2 + 2zw; \tag{5.18}$$

$$x^2 + 2y^2 + 4z^2 + 4w^2 + 2yw; \tag{5.19}$$

$$x^2 + 2y^2 + 4z^2 + 5w^2 + 4zw : \tag{5.20}$$

$$x^2 + 2y^2 + 4z^2 + 5w^2 + 2yz; \tag{5.21}$$

$$x^2 + 2y^2 + 3z^2 + 7w^2 + 2zw; \tag{5.22}$$

$$x^2 + 2y^2 + 4z^2 + 6w^2 + 4zw; \tag{5.23}$$

$$x^2 + 2y^2 + 4z^2 + 6w^2 + 2yz; \tag{5.24}$$

$$x^2 + 2y^2 + 4z^2 + 8w^2 + 4zw; \tag{5.25}$$

$$x^2 + 2y^2 + 4z^2 + 10w^2 + 4zw; \tag{5.26}$$

$$x^2 + 2y^2 + 4z^2 + 12w^2 + 4zw; \tag{5.27}$$

$$x^2 + 2y^2 + 4z^2 + 13w^2 + 4zw. \tag{5.28}$$

Remark 5.3.2. Note that the forms (5.1) through (5.28) correspond to the lattices # 12, 14, 24, 25, 26, 27, 36, 37, 50, 58, 59, 61, 65, 70, 71, 77, 79, 83, 84, 91, 96, 105, 108, 111, 144, 166, 182 and 192, respectively, which are given in the Tables 5.1 - 5.5.

The main argument we use here is somewhat similar to the method used by Bhargava in [2]. But to preserve the primitive representation, here we have to find a ternary sublattice which splits the quaternary lattice L. We can identify four ternary sublattices of L, not necessarily distinct, denoted by $T_{(i)}$, where $i = 1, 2, 3, 4$. Here $T_{(i)}$ denotes the ternary sublattice of L whose Gram matrix is obtained from the Gram matrix of L by deleting the *i-th* row and *i-th* column.

In order to transfer information about the local primitive representations of a lattice to information about global primitive representations, the lattice must be unique in its genus (i.e., its class number equals 1). So we are interested in analyzing further the ternary lattices $T_{(i)}$ with class number 1, at each prime p, to check what are the primitive exclusions of $(T_{(i)})_p$ (since no ternary lattice is universal, such exclusions always exist). Sometimes these exclusions can be covered by other ternary sublattices of L, which do not necessarily split L (see Example 5.3.5). Otherwise, we need to check whether there is a way to cover these exclusions primitively using $T_{(i)}$ and the complement of $T_{(i)}$ in L. Note that when $L \cong \langle \beta \rangle \perp T_{(i)}$, for some $\beta \in \mathbb{N}$, then $\alpha - \beta \to T_{(i)}$ implies $\alpha \xrightarrow{*} \langle \beta \rangle \perp T_{(i)} \cong L$, where primitivity is guaranteed by the vector associated with β. In either way, if we are able to primitively represent every positive integer using L, then we can conclude that L is primitively $\mathbb{Z}_2$-universal.

Also, when $T_{(i)}$ has class number 1, then for any $\alpha \in \mathbb{N}$, $\alpha \overset{*}{\nrightarrow} T_{(i)}$ implies that there exists some $p \in S$ such that $\alpha \overset{*}{\nrightarrow} (T_{(i)})_p$. We will be using these arguments directly throughout our proof from now on.

Example 5.3.3. Consider lattice Q_6^3 (# 12) given by $L \cong \langle 1 \rangle \perp \begin{pmatrix} 2&0&0 \\ 0&2&1 \\ 0&1&2 \end{pmatrix}$. Here the ternary lattice, $T_{(2)} \cong \begin{pmatrix} 1&0&0 \\ 0&2&1 \\ 0&1&2 \end{pmatrix}$ has class number 1. Observe first that $(T_{(2)})_p$ is primitively $\mathbb{Z}_p$-universal for $p \neq 2, 3$ since $det\ (T_{(2)}) = 3$. Next, by Example 5.2.5, we have $(T_{(2)})_3$ is primitively $\mathbb{Z}_3$-universal, and $q^*((T_{(2)})_2) = q^*(\langle 3, 7, 7 \rangle) = (\mathbb{Z}_2^\times)^2 \cup 3(\mathbb{Z}_2^\times)^2 \cup 7(\mathbb{Z}_2^\times)^2 \cup 2\mathbb{Z}_2^\times$. With these computations, we know that for any $\alpha \in \mathbb{N}$, if $\alpha \overset{*}{\nrightarrow} T_{(2)}$, then either $\alpha \in 4\mathbb{Z}_2$ or $\alpha \in 5(\mathbb{Z}_2^\times)^2$.

Now take any $\alpha \in \mathbb{N}$. Note that $L \cong \langle 2 \rangle \perp T_{(2)}$. To prove that L is primitively universal, it suffices to show either $\alpha \xrightarrow{*} T_{(2)}$ or $\alpha - 2 \rightarrow T_{(2)}$. Suppose $\alpha \not\xrightarrow{*} T_{(2)}$ (i.e. $\alpha \not\xrightarrow{*} (T_{(2)})_2$). If $\alpha \in 4\mathbb{Z}_2$; say $\alpha = 4\alpha_0$, then $\alpha - 2 = 4\alpha_0 - 2 \in 2\mathbb{Z}_2^\times \xrightarrow{*} (T_{(2)})_2$. If $\alpha \in 5(\mathbb{Z}_2^\times)^2$, then $\alpha - 2 \in 3(\mathbb{Z}_2^\times)^2 \xrightarrow{*} (T_{(2)})_2$. In both cases $\alpha - 2 \xrightarrow{*} T_{(2)}$, hence $\alpha \xrightarrow{*} L$.

Example 5.3.4. Consider lattice Q_{10}^2 (# 24) given by $L \cong \langle 1 \rangle \perp \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Here the ternary lattice, $T_{(2)} \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ has class number 1. By Example 5.2.6, $(T_{(2)})_p$ is primitively $\mathbb{Z}_p$-universal except when $p = 5$ and we have $\mathbb{Z}_5^\times \subseteq q^*((T_{(2)})_5)$.

Now take any $\alpha \in \mathbb{N}$. Note that $L \cong \langle 2 \rangle \perp T_{(2)}$. To prove that L is primitively universal, it suffices to show either $\alpha \xrightarrow{*} T_{(2)}$ or $\alpha - 2 \rightarrow T_{(2)}$. Suppose $\alpha \not\xrightarrow{*} T_{(2)}$. Then $\alpha \in 5\mathbb{Z}_5$; say $\alpha = 5\alpha_0$. So $\alpha - 2 = 5\alpha_0 - 2 \in \mathbb{Z}_5^\times \xrightarrow{*} (T_{(2)})_5$. Thus $\alpha - 2 \xrightarrow{*} T_{(2)}$, hence $\alpha \xrightarrow{*} L$.

Example 5.3.5. Consider lattice Q_{10}^3 (# 25) given by $L \cong \langle 1 \rangle \perp \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$. Here the ternary lattices $T_{(1)} \cong \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$ and $T_{(3)} \cong \langle 1, 2, 4 \rangle$ have class number 1. First note that, by Example 5.2.8, all positive odd integers are primitively represented by $T_{(3)}$. Therefore, in order to prove L is primitively universal, we only have to show further that all positive even integers are primitively represented by L.

Since $det\ T_{(1)} = 10$, $(T_{(1)})_p$ is primitively $\mathbb{Z}_p$-universal except possibly when $p = 2, 5$. Using Example 5.2.4, we have $(T_{(1)})_2 \cong \mathbb{A} \perp \langle 2 \cdot 7 \rangle$. By Propositions 3.2.13 and 3.2.20, $\widehat{\mathbb{A}} \perp \langle 7 \rangle$ is primitively $\mathbb{Z}_p$-universal. It follows after scaling that $2\mathbb{Z}_2 \xrightarrow{*} (T_{(1)})_2$. Also, $(T_{(1)})_5 \cong \langle 1, 2, 5 \rangle$ and hence $\mathbb{Z}_5^\times \xrightarrow{*} (T_{(1)})_5$.

Now let α be an even positive integer. If $5 \nmid \alpha$, then $\alpha \xrightarrow{*} (T_{(1)})_p$ for all p. Hence $\alpha \xrightarrow{*} T_{(1)}$ since $T_{(1)}$ has class number 1. So $\alpha \xrightarrow{*} L$. If $5 \mid \alpha$, then $\alpha - 4 \in \mathbb{Z}_5^\times \xrightarrow{*} T_{(1)}$. So $\alpha = 2^2 + (\alpha - 4) \xrightarrow{*} \langle 1 \rangle \perp T_{(1)} \cong L$.

Example 5.3.6. Consider lattice Q_{96}^2 (# 192) given by $L \cong \langle 1 \rangle \perp \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 13 \end{pmatrix}$. Here the ternary lattices $T_{(2)} \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 13 \end{pmatrix}$ and $T_{(4)} \cong \langle 1, 2, 4 \rangle$ have class number 1. First note that, by Example 5.2.8, all positive odd integers are primitively represented by $T_{(4)}$. Here $(T_{(2)})_3 \cong$

$\langle 1,1,3\rangle$, hence $\mathbb{Z}_3^\times \subseteq q^*((T_{(2)})_3)$, and $(T_{(2)})_2 \cong \langle 1,5,7\cdot 2^4\rangle$. It can be checked by direct computation that $(\mathbb{Z}_2^\times)^2 \cup 5(\mathbb{Z}_2^\times)^2 \cup 6(\mathbb{Z}_2^\times)^2 \cup 14(\mathbb{Z}_2^\times)^2 \cup 4(\mathbb{Z}_2^\times)^2 \cup 20(\mathbb{Z}_2^\times)^2 \cup 8(\mathbb{Z}_2^\times)^2 \cup 40(\mathbb{Z}_2^\times)^2 \cup 48(\mathbb{Z}_2^\times)^2 \cup 80(\mathbb{Z}_2^\times)^2 \cup 112(\mathbb{Z}_2^\times)^2 \subseteq q^*((T_{(2)})_2)$, and $8\mathbb{Z}_2 \subseteq q((T_{(2)})_2)$.

Now let α be an even positive integer. If $\alpha \stackrel{*}{\not\to} T_{(2)}$, there are three possibilities to consider:

i) $4 \mid \alpha$, $3 \nmid \alpha$;

ii) $\alpha \in 2(\mathbb{Z}_2^\times)^2 \cup 10(\mathbb{Z}_2^\times)^2$, $3 \nmid \alpha$;

iii) $3 \mid \alpha$.

Consider the first case i). Write $\alpha = 4\alpha_0$ and break further into subcases depending upon whether α_0 is even or odd.

If α_0 is even, then by separately considering the cases $\alpha_0 \equiv 0,2,4,6 (\text{mod } 8\mathbb{Z}_2)$, it can be shown that $\alpha - 2\cdot 3^2 \stackrel{*}{\to} (T_{(2)})_2$ (for example, if $\alpha_0 \equiv 2 (\text{mod } 8\mathbb{Z}_2)$, then writing $\alpha_0 = 2+8k$ produces

$$\alpha - 2\cdot 3^2 = -10 + 32k = 2(-5+16k) \in 6(\mathbb{Z}_2^\times)^2 \stackrel{*}{\to} (T_{(2)})_2.)$$

Also, $\alpha - 2\cdot 3^2 \in \mathbb{Z}_3^\times$ since $3 \nmid \alpha$. So $\alpha - 2\cdot 3^2 \stackrel{*}{\to} (T_{(2)})_p$ for all p, and it follows that $\alpha - 2\cdot 3^2 \stackrel{*}{\to} T_{(2)}$ when $\alpha \geq 19$, since $T_{(2)}$ has class number 1. So $\alpha \stackrel{*}{\to} \langle 2\rangle \perp T_{(2)} \cong L$ for $\alpha \geq 19$.

On the other hand, if α_0 is odd, then $\alpha \stackrel{*}{\to} T_{(2)}$ when $\alpha_0 \equiv 1,5 (\text{mod } 8\mathbb{Z}_2)$. When $\alpha_0 \equiv 3,7 (\text{mod } 8\mathbb{Z}_2)$, it can be shown that $\alpha - 2\cdot 6^2 \stackrel{*}{\to} (T_{(2)})_p$ for all p (for example, if $\alpha_0 \equiv 3 (\text{mod } 8\mathbb{Z}_2)$, then writing $\alpha_0 = 3+8k$ produces

$$\alpha - 2\cdot 6^2 = -60 + 32k = 4(-15+8k) \in 4(\mathbb{Z}_2^\times)^2 \stackrel{*}{\to} (T_{(2)})_2.)$$

Hence $\alpha - 2\cdot 6^2 \stackrel{*}{\to} T_{(2)}$ and $\alpha \stackrel{*}{\to} \langle 2\rangle \perp T_{(2)} \cong L$ whenever $\alpha \geq 73$.

Now consider the subcase of case ii) when $\alpha \in 2(\mathbb{Z}_2^\times)^2$. So $3 \nmid \alpha$ and $\alpha = 2+16k$ for some $k \in \mathbb{N}$. Then

$$\alpha - 2\cdot 3^2 = 8(-2+2k) \in 8\mathbb{Z}_2 \to T_{(2)}.$$

Hence $\alpha - 2 \cdot 3^2 \to (T_{(2)})_p$ for all p and thus $\alpha - 2 \cdot 3^2 \to T_{(2)}$ whenever $\alpha \geq 19$, since $T_{(2)}$ has class number 1. So $\alpha = 2 \cdot 3^2 + q(v)$ for some $v \in T_{(2)}$. This gives a representation of α by L. The g.c.d. of the coefficients of v cannot be divisible by 3 since $3 \nmid \alpha$. Therefore, the representation must be primitive and $\alpha \xrightarrow{*} L$. The proof for the subcase of ii) when $\alpha \in 10(\mathbb{Z}_2^\times)^2$ is similar.

The proof for the remaining case iii) follows in a similar manner. Combining these cases and using the fact that L primitively represents the positive integers up to 400, we can conclude that L is primitively universal.

Although ternary lattices can cover large sets of positive integers, in some cases it is not possible to find a ternary sublattice of L with class number 1 that works well. That is when we dig further in to binary sublattices of L which have class number 1. There we get more primitive exclusions, but the advantage is that in the complement we have two orthogonal components to work with as shown in the following example:

Example 5.3.7. Consider lattice Q_{11}^2 (# 27) given by $L \cong \langle 1 \rangle \perp \begin{pmatrix} 1&0&0 \\ 0&3&1 \\ 0&1&4 \end{pmatrix}$. Here we will use the binary lattice $K \cong \begin{pmatrix} 3&1 \\ 1&4 \end{pmatrix}$, which has class number 1, and we write $L \cong \langle 1, 1 \rangle \perp K$. The determinant of K is 11, and we only need to check $\mathbb{Z}_2$ and $\mathbb{Z}_{11}$. It is not hard to see $K_2 \cong \langle 1, 3 \rangle$; hence $\mathbb{Z}_2^\times \subseteq q^*(K_2)$ by Proposition 3.2.2 *i*), and $K_{11} \cong \langle 1, 11 \rangle$; hence $q^*(K_{11}) = (\mathbb{Z}_{11}^\times)^2 \cup 11(\mathbb{Z}_{11}^\times)^2$.

Now take any $\alpha \in \mathbb{N}$. Suppose $\alpha \not\xrightarrow{*} K$. We divide the argument in to four cases.

Case i): $2 \mid \alpha,\ 11 \mid \alpha$. Let $\alpha = 2\alpha_0$ and $\alpha \equiv 0 (\text{mod } 11\mathbb{Z}_{11})$. Then $\alpha - 3^2 - 2^2 = \alpha - 13 \equiv 9 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 3^2 - 2^2 \in (\mathbb{Z}_{11}^\times)^2$. Also $\alpha - 3^2 - 2^2 = 2\alpha_0 - 13 \in \mathbb{Z}_2^\times$. Thus, $\alpha - 3^2 - 2^2 \xrightarrow{*} K_2$ and K_{11}. So, $\alpha - 3^2 - 2^2 \xrightarrow{*} K$. Hence $\alpha \xrightarrow{*} \langle 1, 1 \rangle \perp K \cong L$.

Case ii): $2 \nmid \alpha,\ 11 \mid \alpha$. Let $\alpha = 2\alpha_0 + 1$ and $\alpha \equiv 0 (\text{mod } 11\mathbb{Z}_{11})$. Then $\alpha - 1 - 1 = \alpha - 2 \equiv 9 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 1 - 1 \in (\mathbb{Z}_{11}^\times)^2$. Also $\alpha - 1 - 1 = 2\alpha_0 + 1 - 2 \in \mathbb{Z}_2^\times$. Thus, $\alpha - 1 - 1 \xrightarrow{*} K_2$ and K_{11}. So, $\alpha - 1 - 1 \xrightarrow{*} K$. Hence $\alpha \xrightarrow{*} \langle 1, 1 \rangle \perp K \cong L$.

Case iii): $2 \mid \alpha$, $11 \nmid \alpha$. This consists of two subcases. First let $\alpha = 2\alpha_0$ and $\alpha \in \Delta(\mathbb{Z}_{11}^{\times})^2$. So, $\alpha \equiv 2, 6, 7, 8, 10 (\text{mod } 11\mathbb{Z}_{11})$. In order to have a primitive representation we need to find λ such that $\alpha - \lambda$ is odd so that $\alpha - \lambda \xrightarrow{*} K_2$. If $\alpha \equiv 2, 6, 10 (\text{mod } 11\mathbb{Z}_{11})$, then $\alpha - 1 \equiv 1, 5, 9 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 1 \in (\mathbb{Z}_{11}^{\times})^2$. If $\alpha \equiv 7 (\text{mod } 11\mathbb{Z}_{11})$, then $\alpha - 3^2 \equiv 9 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 3^2 \in (\mathbb{Z}_{11}^{\times})^2$. If $\alpha \equiv 8 (\text{mod } 11\mathbb{Z}_{11})$, then $\alpha - 2^2 - 1 \equiv 3 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 2^2 - 1 \in (\mathbb{Z}_{11}^{\times})^2$. In each of the subcases there exist $\lambda_1, \lambda_2 \in \mathbb{Z}^2$ such that $\alpha - \lambda_1 - \lambda_2 \xrightarrow{*} K_2$ and K_{11}. Thus, $\alpha - \lambda_1 - \lambda_2 \xrightarrow{*} K$. Hence $\alpha \xrightarrow{*} \langle 1, 1 \rangle \perp K \cong L$.

Next let $\alpha = 2\alpha_0$ and $\alpha \in (\mathbb{Z}_{11}^{\times})^2$. So, $\alpha \equiv 1, 3, 4, 5, 9 (\text{mod } 11\mathbb{Z}_{11})$. If $\alpha \equiv 4, 5 (\text{mod } 11\mathbb{Z}_{11})$, then $\alpha - 1 \equiv 3, 4 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 1 \in (\mathbb{Z}_{11}^{\times})^2$. If $\alpha \equiv 3, 9 (\text{mod } 11\mathbb{Z}_{11})$, then $\alpha - 2^2 - 1 \equiv 9, 4 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 2^2 - 1 \in (\mathbb{Z}_{11}^{\times})^2$. If $\alpha \equiv 1 (\text{mod } 11\mathbb{Z}_{11})$, then $\alpha - 3^2 \equiv 3 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 3^2 \in (\mathbb{Z}_{11}^{\times})^2$. In each of the subcases there exist $\lambda_1, \lambda_2 \in \mathbb{Z}^2$ such that $\alpha - \lambda_1 - \lambda_2 \xrightarrow{*} K_2$ and K_{11}. Thus, $\alpha - \lambda_1 - \lambda_2 \xrightarrow{*} K$. Hence $\alpha \xrightarrow{*} \langle 1, 1 \rangle \perp K \cong L$.

Case iv): $2 \nmid \alpha$, $11 \nmid \alpha$. Let $\alpha = 2\alpha_0 + 1$ and $\alpha \in \Delta(\mathbb{Z}_{11}^{\times})^2$. So, $\alpha \equiv 2, 6, 7, 8, 10 (\text{mod } 11\mathbb{Z}_{11})$. In order to have a primitive representation we need to find λ such that $\alpha - \lambda$ is odd so that $\alpha - \lambda \xrightarrow{*} K_2$. If $\alpha \equiv 2, 7, 8 (\text{mod } 11\mathbb{Z}_{11})$, then $\alpha - 2^2 \equiv 9, 3, 4 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 2^2 \in (\mathbb{Z}_{11}^{\times})^2$. If $\alpha \equiv 6 (\text{mod } 11\mathbb{Z}_{11})$, then $\alpha - 1 - 1 \equiv 4 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 1 - 1 \in (\mathbb{Z}_{11}^{\times})^2$. If $\alpha \equiv 10 (\text{mod } 11\mathbb{Z}_{11})$, then $\alpha - 2^2 \equiv 5 (\text{mod } 11\mathbb{Z}_{11})$; so $\alpha - 2^2 \in (\mathbb{Z}_{11}^{\times})^2$. In each of these there exist $\lambda_1, \lambda_2 \in \mathbb{Z}^2$ such that $\alpha - \lambda_1 - \lambda_2 \xrightarrow{*} K_2$ and K_{11}. Thus, $\alpha - \lambda_1 - \lambda_2 \xrightarrow{*} K$. Hence $\alpha \xrightarrow{*} \langle 1, 1 \rangle \perp K \cong L$.

Outline of the proof of the Proposition 5.3.1: The proof for the forms (5.17), (5.18) and (5.20) is already given in Proposition 4.2.6.

The result is proved for the form (5.1) in Example 5.3.3. A similar argument can be applied to the forms (5.5), (5.13), (5.26), (5.27) with the use of ternary sublattice $T_{(1)}$, and to the forms (5.7) (5.9), (5.12), with the use of ternary sublattice $T_{(2)}$ of the corresponding lattice L.

The result is proved for the form (5.3) in Example 5.3.4, and a similar argument can be applied to the form (5.10), with the use of ternary sublattice $T_{(4)}$ of the corresponding lattice L.

The result is proved for the form (5.4) in Example 5.3.5. A similar argument, which is even easier, can be applied to the forms (5.8), (5.11), (5.15), (5.16), (5.23), with the use of ternary sublattice $T_{(1)}$ of each of the corresponding lattices L, since each of these $T_{(1)}$'s has the property $q^*((T_{(1)})_2) = 2\mathbb{Z}_2$. Proof of (5.15) will follow a similar reasoning, using the result of Example 5.2.7.

Although the proof of (5.14), which can be performed using $T_{(1)}$ looks similar to that of (5.1), since $q^*((T_{(1)})_2) = 2\mathbb{Z}_2$ and there is no other ternary sublattice which can be proven to represent odd numbers primitively, we need to consider three different cases here. For any $\alpha \in \mathbb{N}$, if $\alpha \overset{*}{\not\to} T_{(1)}$, then $2 \mid \alpha,\ 3 \mid \alpha$ or $2 \nmid \alpha,\ 3 \mid \alpha$ or $2 \nmid \alpha,\ 3 \nmid \alpha$. In all of these cases we can prove $\alpha \overset{*}{\to} L$.

In the proof of (5.22), which can be performed using $T_{(2)}$, one can check that $q^*((T_{(2)})_2) = \mathbb{Z}_2^\times \cup 4\mathbb{Z}_2$. With the property that $\mathbb{Z}_5^\times \subseteq q^*((T_{(2)})_5)$, and as $T_{(4)}$ represents all positive odd integers, we can complete the proof by considering three different cases. For any positive even integer, α, if $\alpha \overset{*}{\not\to} T_{(1)}$, then $4 \mid \alpha,\ 5 \mid \alpha$ or $4 \nmid \alpha,\ 5 \mid \alpha$ or $4 \nmid \alpha,\ 5 \nmid \alpha$. In all of these cases we can prove $\alpha \overset{*}{\to} L$.

The result is proved for the form (5.6) in Example 5.3.7. This form is equipped with the binary sublattices of class number 1. Same procedure can be used to prove the primitive universality of the forms (5.2), (5.19), (5.21), (5.24), (5.25).

The sketch of the proof of the form (5.28) is given in Example 5.3.6. □

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