Southern Illinois University Carbondale
OpenSIUC

Theses | Theses and Dissertations

5-1-2015

A Family of Circles in a Window

Ethan Taylor Lightfoot
Southern Illinois University Carbondale, etlightfoot@gmail.com

Follow this and additional works at: http://opensiuc.lib.siu.edu/theses

Recommended Citation
Lightfoot, Ethan Taylor, "A Family of Circles in a Window" (2015). *Theses.* Paper 1647.

This Open Access Thesis is brought to you for free and open access by the Theses and Dissertations at OpenSIUC. It has been accepted for inclusion in Theses by an authorized administrator of OpenSIUC. For more information, please contact opensiuc@lib.siu.edu.

A FAMILY OF CIRCLES IN A WINDOW

by

Ethan Lightfoot

B.S., Southeast Missouri State University, 2013

B.S.E.H.S, Southeast Missouri State University, 2013

A Thesis
Submitted in Partial Fulfillment of the Requirements for the
Master of Science Degree

Department of Mathematics
in the Graduate School
Southern Illinois University Carbondale
May 2015

Copyright by ETHAN LIGHTFOOT, 2015
All Rights Reserved

THESIS

A FAMILY OF CIRCLES IN A WINDOW

By

Ethan Lightfoot

A Thesis Submitted in Partial

Fulfillment of the Requirements

for the Degree of

Master

in the field of Mathematics

Approved by:

Dr. Jerzy Kocik, Chair

Dr. Dubravka Ban

Dr. John Mcsorely

Graduate School
Southern Illinois University Carbondale
April 9th, 2015

AN ABSTRACT OF THE THESIS OF

ETHAN LIGHTFOOT, for the Master of Science degree in Mathematics, presented on April 9th, 2015, at Southern Illinois University Carbondale.

TITLE: A FAMILY OF CIRCLES IN A WINDOW

MAJOR PROFESSOR: Dr. J. Kocik

For Ford Circles on the real line, $[0,1]$, G.T. Williams and D.H. Browne discovered that this infinite arrangement of circles has an area-sum $\pi + \pi\frac{\zeta(3)}{\zeta(4)}$, where $\zeta(s)$ is the Riemann-Zeta function from complex analysis and number theory. The purpose of this paper is to explore their findings in detail and provide alternative methods to prove the statements found in the paper. Then we will attempt to show similar results on the Apollonian Window circle packing using inversion through circles and the results of Williams and Browne.

DEDICATION

For my Potato. Without you the world is simple and grey.

ACKNOWLEDGMENTS

I would like to thank Dr. Kocik for his assistance and insights leading to the writing of this paper. Your enthusiasm ignites my own passion for mathematics each day.

Also, many thanks to Dr. Ban for letting me use her lecture notes as a reference for the number theoretic part of the paper.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

INTRODUCTION

This paper provides an algebraic and number theoretic approach to a classical problem of geometry and complex analysis. We begin with a simple Ford circle arrangement and through tools of algebra and number theory arrive at the beautiful result for the area-sum of the arrangement as

$$\pi + \frac{\pi\zeta(3)}{\zeta(4)}$$

The goal of this paper is to explore the findings in [1] and establish proofs for the theorems presented in it. We then add information on our own findings of a similar structure found in the Apollonian Window. The intended audience is a reader with basic proof technical ability and a willingness to learn and explore this topic. Any definitions or concepts needed by the reader are explicitly stated in the paper.

Though we are exploring another's paper, we try to provide a more contemporary proof to the originals. Some of the proofs of the statements found in [1] are not mentioned in the paper in order to stay on task. Nevertheless, the proofs are intended to be accessible to all readers.

Chapter 1 deals with Ford circles and the corresponding Stern-Brocot Array generated by this construction. This chapter contains basic theorems and relationships of the circles in the arrangement as well as properties of the values found in the Stern-Brocot Array.

Chapter 2 deals with the Stern-Brocot Tree, which we obtain through the Stern-Brocot Array. The Stern-Brocot Tree is used to prove many of the big statements made in the original paper [1].

Chapter 3 deals with the determination of the area-sum of the original Ford circles. This section has many definitions and theorems from number theory, all of which are clearly defined for the reader.

Chapter 4 deals with recent attempts at determining whether the area-sum for parts of the Apollonian Window. We attempt to translate our results from [1] and use geometric inversion from [6] to find the area-sum of these parts of the Window.

CHAPTER 1

A FAMILY OF INTEGERS FROM FORD CIRCLES

1.1 FORD CIRCLES

A Family of Integers and a Theorem on Circles [1], begins with an introduction on what are more commonly known as Ford circles [2]. This goes as follows:

Definition. Between two tangent unit circles lying on a line L, a third circle is inscribed tangent to both circles and to L. Then two more circles are inscribed in the newly-created spaces, each tangent to two circles and to L. Now four circles are inscribed similarily along L, and so on $ad\ inf$.

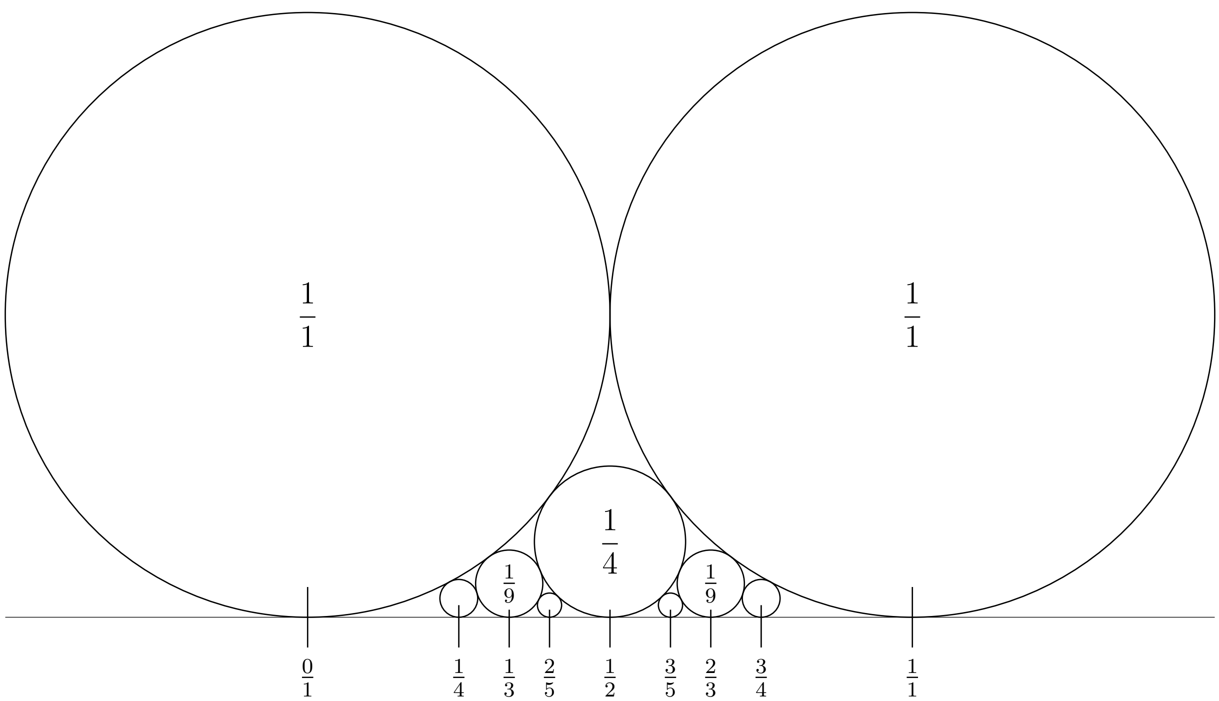

Figure 1.1. Ford Circles

These types of circles are really quite amazing and are key to our understanding of a similar disk packing such as the Apollonian Window. In order to get a grasp on the area of the resulting figure, a pattern must be established for determining the radii of the succesively generated circles.

Theorem 1.1.1 (Williams and Browne)**.** *If two circles of radii* $\frac{1}{r^2}$ *and* $\frac{1}{s^2}$ *are tangent to each other and to L, the radius of the smaller circle to both circles and to L is* $\frac{1}{(r+s)^2}$*, that of the larger circle being* $\frac{1}{(r-s)^2}$*.*

They proceed as follows:

This is readily established by comparing the projections of the three lines of center on L. Thus, in the figure, the first inscribed circle has a radius of $\frac{1}{4}$*, the next two have radii of* $\frac{1}{9}$*, and so on.*

Since the authors do not prove Theorem 1.1.1, we will prove this theorem using Descartes' formula for four mutually tangent circles [6].

Theorem 1.1.2 (Descartes' Formula)**.** *If four circles are tangent to each other at six distinct points, and the circles have curvatures* k_i *(for* $i = 1, ..., 4$*), then*

$$(k_1 + k_2 + k_3 + k_4)^2 = 2(k_1^2 + k_2^2 + k_3^2 + k_4^2) \tag{1.1}$$

where the curvature of a circle is the reciprocal of its radius. ($k_i = \pm\frac{1}{r_i}\ \forall\ i \in [4]$)

The positive curvature of a circle is the interior curvature, while the negative is the exterior. A straight line is a degenerate circle with curvature zero [3]. Here is a proof of Theorem 1.1.1.

Proof. In the Ford Circles in Figure 1.1, describe the curvatures as $k_1 = \dfrac{1}{\frac{1}{r^2}} = r^2$, $k_2 = s^2$, $k_3 = 0$ (line L), and $k_4 = x^2$. Using Theorem 1.1.2 we have

$$k_4 = k_1 + k_2 \pm 2\sqrt{k_1 k_2}$$

This equation has a $\pm$ as any three mutually tangent circles produce two different circles which are tangent to them. The $+$ corresponds to the circle which is internally tangent to the three circles. The $-$ corresponds to the externally tangent circle. For $+$, we have,

$$k_4 = k_1 + k_2 + 2\sqrt{k_1 k_2} = \left(\sqrt{k_1}\right)^2 + \left(\sqrt{k_2}\right)^2 + 2\sqrt{k_1 k_2} = \left(\sqrt{k_1} + \sqrt{k_2}\right)^2$$

Thus, we have $k_4 = x^2 = \left(\sqrt{r^2} + \sqrt{s^2}\right)^2$. Therefore, $x = r + s$.
Similarly, $k_4 = k_1 + k_2 - 2\sqrt{k_1 k_2}$ gives us $x = r - s$. $\square$

1.2 THE FAMILY OF INTEGERS

Now that we have established a pattern for the radii of the generated circles, we can define an array categorizing the circles present at various iterations.

Definition (Williams and Browne)**.** Starting with our two unit circles, at stage 0, at the nth stage we have $2^n + 1$ circles whose radii, reading across, we designate as $\left(\frac{1}{A_v^n}\right)^2$, where $v = 0, 1, ..., 2^n$ and where the A are succesively the positive integers:

$$\begin{array}{ccccccccc}
1 & & & & & & & & 1 \\
1 & & & & 2 & & & & 1 \\
1 & & 3 & & 2 & & 3 & & 1 \\
1 & 4 & 3 & 5 & 2 & 5 & 3 & 4 & 1 \\
. & . & . & . & . & . & . & . & .
\end{array}$$

$$\begin{array}{ccccccccc}
A_0^0 & & & & & & & & A_1^0 \\
A_0^1 & & & & A_1^1 & & & & A_2^1 \\
A_0^2 & & A_1^2 & & A_2^2 & & A_3^2 & & A_4^2 \\
A_0^3 & A_1^3 & A_2^3 & A_3^3 & A_4^3 & A_5^3 & A_6^3 & A_7^3 & A_8^3 \\
. & . & . & . & . & . & . & . & .
\end{array}$$

Figure 1.2. Stern-Brocot Array

From this point on we will refer to this configuration as the Stern-Brocot Array (SBA). The values in SBA satisfy relationships which are briefly explored by Williams and Browne. We wish to clarify the arguments given in the original paper for the reader.

By definition, the values of the SBA satisfy the property $A_{2v}^{n+1} = A_v^n$ since once an element appears in the n^{th} row, it appears in the $(n+1)^{th}$ row in the same column. Also by definition, $A_{2v+1}^{n+1} = A_v^n + A_{v+1}^n$ as this corresponds to the iterative process of the circles produced in the Ford arrangement. A symmetry relation, namely $A_v^n = A_{2^n-v}^n$, also occurs which is apparent from the circles.

Williams and Browne notice the following property at this point:

Theorem 1.2.1. *If S_n denotes $\sum_{v=0}^{2^n} A_v^n$, then $S_n = 3^n + 1$.*

Proof By Induction. Let $P(k)$ be the statement 'If $S_k = \sum_{v=0}^{2^k} A_v^k$, then $S_k = 3^k + 1$.'
Base Case $P(0)$ holds readily.
$S_0 = \sum_{v=0}^{1} A_v^0 = A_0^0 + A_1^0 = 1 + 1 = 2$ and $S_0 = 3^0 + 1 = 2$
Inductive Case Assume that $P(k)$ is true for some $k > 0$. We need to show that $P(k+1)$ is true. For the $(k+1)^{th}$ row we have all the values of the k^{th} row and new values from the sums of these terms. Each term in the k^{th} row is used exactly 2 times in a sum to make our new elements in the $(k+1)^{th}$ row, except for the first and last terms of the k^{th} row, which are used only once. Thus, we have

$$S_{k+1} = S_k + 2S_k - 2 = 3S_k - 2$$

$$= 3(3^k + 1) - 2 \text{ by Inductive Assumption}$$

$$= 3^{k+1} + 3 - 2 = 3^{k+1} + 1$$

Therefore $P(k+1)$ is true.
Thus $P(k)$ is true $\forall k \geq 0$. □

Next, Williams and Browne state two Theorems without proof. We do not prove either of these as they are tangential to the topic at hand.

Theorem 1.2.2. *A_v^n is a linear function of n for constant v.*

Theorem 1.2.3. *For fixed n, the maximum A_v^n is f_n, the nth Fibonacci number ($\{f_n\}_0^\infty = 1, 2, 3, 5, ...$), and is given by $v = \frac{2^n - (-1)^n}{3}$, and symmetrically.*

The next theorem from [1] on Figure 1.2 concerns divisibility.

Theorem 1.2.4.

$$A_v^n \big| \left(A_{v-1}^n + A_{v+1}^n\right) \qquad \forall n, v$$

Williams and Browne. The proof is inductive. By assuming the proposition for $n-1$ and all v, we deduce its validity for n and all v. When the subscript is odd, the theorem holds trivially. When it is even, we observe that

$$A_{2v-1}^n + A_{2v+1}^n = A_{v-1}^{n-1} + 2A_v^{n-1} + A_{v+1}^{n-1}$$

and by the inductive hypothesis,

$$A_v^{n-1} \big| \left(A_{v-1}^{n-1} + A_{v+1}^{n-1}\right)$$

But

$$A_v^{n-1} = A_{2v}^n$$

Hence it is established generally. □

We will attempt to make this proof a little more straightforward for the reader.

Proof. The Base Case can be verified by looking at the values in the SBA. For the inductive case, asssume that $A_v^{n-1} \big| \left(A_{v-1}^{n-1} + A_{v+1}^{n-1}\right)$ for $n > 2$ and for all v. We want to show that $A_v^n \big| \left(A_{v-1}^n + A_{v+1}^n\right)$. We have two cases:

(1) When v is odd, notice that $A_{2k-1}^n = \left(A_{k-1}^{n-1} + A_k^{n-1}\right)$ for $v = 2k-1$ for some $k \in \mathbb{Z}^+$. Since $A_{k-1}^{n-1} = A_{2k-2}^n$ and $A_k^{n-1} = A_{2k}^n$ from our properties of the A_k^n,

$$A_{2k-1}^n \big| \left(A_{2k-2}^n + A_{2k}^n\right) \Rightarrow A_v^n \big| \left(A_{v-1}^n + A_{v+1}^n\right)$$

(2) When v is even, notice that $A_k^{n-1} = A_{2k}^n$ for $v = 2k$ for some $k \in \mathbb{Z}^+$. Also,

$$\begin{aligned} A_{2k-1}^n + A_{2k+1}^n &= \left(A_{k-1}^{k-1} + A_k^{n-1}\right) + \left(A_k^{n-1} + A_{k+1}^{n-1}\right) \\ &= A_{k-1}^{n-1} + 2A_k^{n-1} + A_{k+1}^{n-1} \end{aligned}$$

By inductive hypothesis, we know that $A_k^{n-1}\big|\left(A_{k-1}^{n-1} + A_{k+1}^{n-1}\right)$ and $A_k^{n-1}\big|2A_k^{n-1}$ so $A_k^n\big|\left(A_{k-1} + A_{k+1}^n\right)$.

Therefore, $A_v^n\big|\left(A_{v-1}^n + A_{v+1}^n\right)$ for all n, v. $\square$

CHAPTER 2

THE STERN-BROCOT TREE

2.1 THE STERN-BROCOT TREE

Now we begin to take a look at the actual numbers found in the SBA. We will look at how these numbers, which are generated from the circles, help us to arrive to a conclusion for the area-sum of the Ford circles. In the Williams and Browne paper, the proofs of the theorems about these values leave out details which we will clarify below.

We modify the SB array into a tree, which we will call the Stern-Brocot Tree (SBT). We will pair up elements that are next to each other, starting on the left. The elements of SBT will pair each element of a row twice, except for the endpoints which will be paired only once. So our Figure 1.2 will now become:

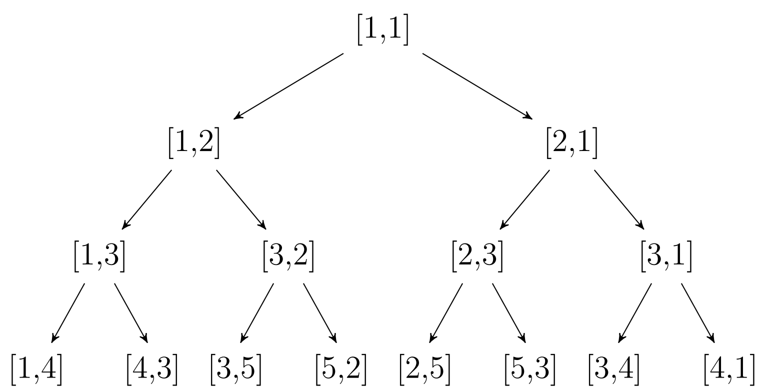

Figure 2.1. Stern-Brocot Tree

The SBT allows for us to easily describe how to move throughout the tree of numbers. If the entries of SBT are viewed as row-vectors, we can describe their children by multiplying on the right by the following matrices

$$L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The matrix L and R are used to find the left and right child, respectively, in SBT. For example, if we want to find the entry [5,2] we can begin with entry [1,1] and act on the right by LRR. We could also start with [3,2] and act on the right by RR and get the same result. If we want, the inverses,

$$L^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad R^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

can be used to walk up the tree as well.

2.2 PROPERTIES ON STERN-BROCOT TREE

Each element in SBT has a unique address, $[1,1]w$ where $w \in \{L, R\}^*$ (w is a word of L, R). We want to show this uniqueness.

Theorem 2.2.1. *Each pair $[a, b]$ of SBT has a unique address.*

Proof. Assume that the pair $[a, b]$ has both the address $[1, 1]w$ and $[1, 1]y$ where $w, y \in \{L, R\}^*$ and $w \neq y$. But, if $[1, 1]w = [a, b] = [1, 1]y$, then we have,

$$[1, 1]w = [1, 1]y$$

w^{-1} is well defined as if say $w = LRL...RR$, then $w^{-1} = R^{-1}R^{-1}...L^{-1}R^{-1}L^{-1}$. Also, notice that L and R are generators of the multiplicative group $SL(2, \mathbb{Z})$, so again w^{-1} is well defined. Thus, we can multiply on the right by w^{-1} and obtain,

$$[1, 1] = [1, 1]yw^{-1} \Leftrightarrow yw^{-1} = I_{2x2} \Leftrightarrow w = y$$

a contradiction. So each $[a, b]$ has a unique address in SBT. $\square$

Now that we have the uniqueness of addresses for each pair $[a, b]$, we can begin to prove other properties of the elements of SBT. Our next result shows that all of the pairs found in the SBT are mutually prime.

Theorem 2.2.2. *All pairs in the SBT are mutually prime.*

Proof. Assume that we have a pair $[a, b]$ in the SBT such that $(a, b) \neq 1$. Since $(a, b) \neq 1$, $(a, b - a) \neq 1$ and $(a - b, b) \neq 1$ where $(a, b - 1) = [a, b]L^{-1}$ and $(a - b, b) = [a, b]R^{-1}$ in the SBT. Continuing this process inductively, we have the pair $[0, k]$ or $[k, 0]$, where $(a, b) = k$ for $k \in \mathbb{Z}^+$ and $k \neq 1$. But, $[k, 0]$ and $[0, k]$ are not part of SBT as $[1, 1]w \neq [0, k]$ and $[1, 1]w \neq [k, 0]$ $\forall$ $w \in \{L, R\}^*$.
Therefore, all pairs of SBT are mutually prime. □

We also have the following theorem:

Theorem 2.2.3. *All pairs $[a, b]$ such that $(a, b) = 1$ are elements of SBT and each pair appears exactly once.*

Proof. First, we show that all relatively prime pairs $[a, b]$ exist in the SBT. Since every pair has a unique address, a pair $[x, y]$ not in the SBT, then has $[1, 1]w \neq [x, y]$ for any $w \in \{L, R\}^*$. Using the Euclidean Algorithm, we see that if $(x, y) = 1$ and $x > y$, then

$$[x, y] \rightarrow [x - ny, y] \rightarrow [x - ny, y - m(x - ny)] \rightarrow ... \rightarrow [1, 1]$$

which can be described in SBT by $y \in \{L^{-1}, R^{-1}\}^*$. If $x < y$, we can use symmetry to come to a similar argument. So, $[1, 1]w = [x, y]$ and this process is unique due to the uniqueness of the Euclidean Algorithm.

Therefore, all relatively prime pairs are in the SBT and appear exactly once. □

2.3 THEOREMS OF *A FAMILY OF INTEGERS AND A THEOREM OF CIRCLES*

The information presented in Chapter 2 so far is our attempt to reformulate the SBA to clarify the following theorem and proof from the original Williams and Browne paper:

Theorem 2.3.1. *Every coupled pair of relatively prime integers occurs in the table of A's.*

Proof. Williams and Browne Again the proof is inductive and depends on the two conditions,

1. If an integer m occurs next to each of the $\phi(m)$ integers $1, a_2, a_3, ..., m-1$ prime to and less than m,

2. then ultimately it must occur next to each of the integers

$$q = jm + 1 \qquad (i = 1, a_2, ..., m-1; j = 1, 2, ...)$$

as can be seen by simple inductive reasoning. Plainly these numbers comprise all q such that $q > m$, $(q, m) = 1$. Hence, *all* relatively prime pairs involving m occur if (i) holds for m. But if (i) is true for all integers less than m it is true for m, since the $\phi(m)$ integers prime to m are included among $1, 2, ..., m-1$ for which (i) and (ii) both hold. The induction is completed by noting the truth of (i) when $m = 2$. □

From our theorems from the SBT, we have shown an alternative way to prove this using Theorems 2.2.1-2.2.3. The next theorem is absolutely essential to our calculations for the area-sum in Figure 1.1.

Theorem 2.3.2. *Ultimately, every integer $m > 1$, appears precisely $\phi(m)$ times in a row of the table (SBA).*

The author believes that the two line proof provided in the original paper is not sufficient enough to show such a wonderful result. Here is our proof using both the SBA and SBT.

Proof. For some $m > 1$, notice the first appearances of m by looking at the columns of the SBA containing m. We will count the number of columns containing m.

The columns in SBA containing m begin when two values less than m appear next to one another in the row above and sum to m. From Theorem 2.2.2, these two values are relatively prime. So, each m column begins next to relatively prime factors less than m. There are $\phi(m)$ of these factors.

For the rest of the column, m only appears next to values greater than m, so no new columns of m can appear. Therefore, we have $\phi(m)$ columns containing m.

Thus, each integer $m > 1$ appears exactly $\phi(m)$ times in a row of the SBA. $\square$

CHAPTER 3

AREA-SUM OF THE FAMILY OF CIRCLES

3.1 AREA-SUM OF THE FAMILY OF CIRCLES

The remainder of this paper will concentrate on solving the area-sum of the circle arrangement. Now that we have the tools to describe the radii of the circle packing, we just need to determine the other tools necessary to prove the final theorem from [1].

Theorem 3.1.1. *The area-sum of the configuration is*

$$\pi + \frac{\pi\zeta(3)}{\zeta(4)} = 6.6307288,$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ *is the Riemann-zeta function.*

Using what we discovered in Theorem 2.3.2, we can describe the area-sum by:

$$\begin{aligned}\sum_{m=1}^{\infty} \pi \left(\frac{1}{m^2}\right)^2 &= \sum_{m=1}^{\infty} \pi \left(\frac{\phi(m)}{m^4}\right) \text{ as there are } \phi(m) \text{ circles with radius } \frac{1}{m^2} \\ &= \pi + \pi \sum_{m=1}^{\infty} \frac{\phi(m)}{m^4} \text{ as } \pi \text{ is the area of one of the unit circles}\end{aligned}$$

3.1.1 Dirichlet Series and Dirichlet convolution

In order to obtain the result in Theorem 3.1.1, we use Dirichlet series and convolution, Möbius inversion, and properties of arithmetic and multiplicative functions. We begin by defining a Dirichlet series.

Definition. A *Dirichlet series* is any series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $s \in \mathbb{C}$ and a_n is a complex defined sequence. [4]

The most famous of these Dirichlet series is the Riemann-zeta function, which is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This function, as discovered by Euler, has a nice connection with the prime numbers as it can be shown that [5]

$$\zeta(s) = \prod_{p\ prime} \frac{1}{1 - p^{-s}}$$

We also will need the concept of Dirichlet convolution for calculations.

Definition. Dirichlet convolution, denoted $*$, is a binary operation for arithmetic functions, such that if f, g are arithmetic [5]

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right) = \sum_{ab=n} f(a)g(b)$$

An *arithmetic function* is any mapping $f : \mathbb{N} \to \mathbb{C}$. An arithmetic function f is said to be *mutiplicative* if f $\forall\ m, n \in \mathbb{N}$ $f(mn) = f(m)f(n)$ where $f(1) = 1$ and $(m, n) = 1$. A multiplicative function f is said to be *complete*, if $\forall\ m, n \in \mathbb{N}$ $f(mn) = f(m)f(n)$.

Examples of arithmetic functions

1. $1(n)$, function which maps $n \in \mathbb{Z}^+ \mapsto 1$

2. $Id(n)$, function which maps $n \in \mathbb{N} \mapsto n$

3. $\mu(n)$, the Möbius function defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^r, & \text{if } n = p_1 p_2 \cdots p_r \text{ all } p_i \text{ diffferent} \\ 0, & \text{if } n \text{ has a squared factor} \end{cases}$$

4. $\phi(n)$, Euler's totient function, a function which gives back the number of natural numbers up to n which are relatively prime to n. $\phi(n)$ is multiplicative and evaluated by

$$\phi(n) = \begin{cases} 1, & \text{if } n = 1 \\ p-1, & \text{if } n = p \text{ a prime} \\ p^k - p^{k-1}, & \text{if } n = p^k \end{cases}$$

In Theorem 3.1.1 we were left with the function $\phi(m)$ in our summation. We propose the following:

Proposition 3.1.2. $\qquad \phi = \mu * Id \qquad \forall\ m \in \mathbb{Z}^+$

To show that this proposition is true we will act on $m \in \mathbb{Z}^+$ on both sides. On the right hand side, we can use Dirichlet convolution to get

$$(\mu * Id)(m) = \sum_{d|m} \mu(d) Id\left(\frac{m}{d}\right) = \sum_{d|m} \mu(d)\left(\frac{m}{d}\right)$$

On the left hand side, we will use Möbius inversion formula from number theory.

Möbius Inversion Formula [4]

If f, g are arithmetic functions such that

$$g(m) = \sum_{d|m} f(d)\ \forall m \in \mathbb{Z}^+,$$

then

$$f(m) = \sum_{d|m} \mu(d) g\left(\frac{m}{d}\right)\ \forall m \in \mathbb{Z}^+.$$

To write $\phi(m)$ as a summation, we will turn to a proposition of Gauss [7], which we will not prove, but rather show through an example.

Proposition 3.1.3.

$$\sum_{d|m} \phi(d) = m$$

Example 3.1.1. Let $m = 12$. List all of the fractions in the interval $(0, 1]$ with denominator $m = 12$ and reduce those fractions to lowest terms. Then collect all of the fractions with like denominators together as follows:

$$\left\{\frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}, \frac{7}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}, \frac{11}{12}, \frac{12}{12}\right\}$$

$$\left\{\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{11}{12}, \frac{1}{1}\right\}$$

$$\begin{aligned}
\left|\left\{\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}\right\}\right| &= 4 = \phi(12)\\
\left|\left\{\frac{1}{6}, \frac{5}{6}\right\}\right| &= 2 = \phi(6)\\
\left|\left\{\frac{1}{4}, \frac{3}{4}\right\}\right| &= 2 = \phi(4)\\
\left|\left\{\frac{1}{3}, \frac{2}{3}\right\}\right| &= 2 = \phi(3)\\
\left|\left\{\frac{1}{2}\right\}\right| &= 1 = \phi(2)\\
\left|\left\{\frac{1}{1}\right\}\right| &= 1 = \phi(1)
\end{aligned}$$

Taking the sum $\sum\limits_{d|m} \phi(d)$ we obtain m.

From Proposition 3.1.2, $\sum\limits_{d|m} \phi(d) = m = Id(m)$, so using the Möbius Inversion Formula, we let $g(m) = Id(m)$ and $f(m) = \phi(m)$ to get

$$\phi(m) = \sum_{d|m} \mu(d) Id\left(\frac{m}{d}\right) = \sum_{d|m} \mu(d)\left(\frac{m}{d}\right)$$

Therefore, $\phi = \mu * Id$ for any $m \in \mathbb{Z}^{+}$.

3.1.2 Dirichlet series of convolution products

Next we need to discuss the Dirichlet series of convolution products before applying the material in this section to the area-sum of our Ford circles.

Theorem 3.1.4. *Let f and g be arithmetic functions associated with Dirichlet series $F(s)$ and $G(s)$, respectively. Let $h = f * g$ be the Dirichlet convolution of f and g and let $H(s)$ be the associated Dirichlet series of h.*

If $F(s)$ and $G(s)$ converge absolutely at some point s, then so does $H(s)$ and we have $H(s) = F(s)G(s)$.

Proof.

$$\begin{aligned} F(s)G(s) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{f(k)g(m)}{(km)^s} &= \sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{km=n} f(k)g(m) \\ &= \sum_{n=1}^{\infty}\frac{(f*g)(n)}{n^s} \\ &= \sum_{n=1}^{\infty}\frac{h(n)}{n^s} \\ &= H(s) \end{aligned}$$

$H(s)$ converges absolutely as

$$\begin{aligned} \sum_{n=1}^{\infty}\left|\frac{h(n)}{n^s}\right| &\leq \sum_{n=1}^{\infty}\left|\frac{1}{n^s}\right|\sum_{km=n}|f(k)||g(m)| \\ &= \left(\sum_{k=1}^{\infty}\left|\frac{f(k)}{k^s}\right|\right)\cdot\left(\sum_{m=1}^{\infty}\left|\frac{g(m)}{m^s}\right|\right) \end{aligned}$$

which are both absolutely convergent by the hypothesis of the theorem. □

3.1.3 The Finale!

We ended our calculation for the area-sum of the configuration with the following

$$\pi + \pi\sum_{m=1}^{\infty}\frac{\phi(m)}{m^4}$$

We will look at $\sum_{m=1}^{\infty}\frac{\phi(m)}{m^s}$. From Proposition 3.1.2 and Theorem 3.1.4, we have

$$\sum_{m=1}^{\infty}\frac{\phi(m)}{m^4}=\sum_{m=1}^{\infty}\frac{(\mu*Id)(m)}{m^4}$$
$$=\left(\sum_{m=1}^{\infty}\frac{\mu(m)}{m^4}\right)\cdot\left(\sum_{m=1}^{\infty}\frac{Id(m)}{m^4}\right)$$

For the second sum we have,

$$\sum_{m=1}^{\infty}\frac{Id(m)}{m^4}=\sum_{m=1}^{\infty}\frac{1}{m^3}=\zeta(3)$$

For the first sum we will use tools from number theory to manipulate this into something which we can recognize. First, we propose a proposition [5]

Proposition 3.1.5. *Let f be a multiplicative function. Then*

$$D(f,s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}=\prod_{p}\sum_{l=0}^{\infty}\frac{f(p^l)}{p^{ls}}$$

Using Euler's product formula for the Riemann-Zeta function,

$$\prod_{p\ prime}\frac{1}{1-p^{-s}}=\zeta(s)$$

we can see clearly that

$$\frac{1}{\zeta(s)}=\prod_{p}\left(1-\frac{1}{p^s}\right)$$

Using Proposition 3.1.5 to determine the value of $D(\mu,s)$, we have

$$D(\mu,s)=\prod_{p}\sum_{l=0}^{\infty}\frac{\mu(p^l)}{p^{ls}}=\prod_{p}\left(1-\frac{1}{p^s}\right)$$

as μ is multiplicative and $\mu(p^l)$ is evaluated by

$$\mu(p^l) = \begin{cases} 1, & \text{if } l = 0 \\ -1, & \text{if } l = 1 \\ 0, & \text{if } l > 1 \end{cases}$$

Thus,

$$\frac{1}{\zeta(s)} = D(\mu, s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

So, $\sum\limits_{n=1}^{\infty} \frac{\mu(n)}{n^4} = \frac{1}{\zeta(4)}$ and we have

$$\sum_{m=1}^{\infty} \frac{\phi(m)}{m^4} = \frac{\zeta(3)}{\zeta(4)}.$$

as desired. Thus, we have shown that the area-sum of the Ford circle arrangement is

$$\pi + \frac{\pi\zeta(3)}{\zeta(4)}$$

What a beautiful result! We began with circles and obtain a result for the area-sum of the circles which uses the Riemann zeta function. While this result is great, in the next chapter we will explore a disk packing, commonly referred to as the Apollonian Window, and attempt to translate these results to its partial area-sum.

CHAPTER 4

COMPARING TO THE APOLLONIAN WINDOW

4.1 INVERSION OF THE APOLLONIAN WINDOW

In this chapter, we will consider the Apollonian window as shown in the figure below.

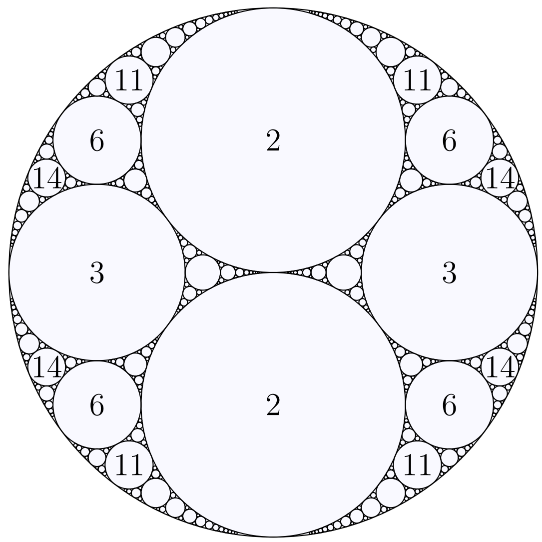

Figure 4.1. Apollonian Window

The goal of this chapter is to explore the area-sum of different parts of the Apollonian Window. For this window, we will use geometric inversion to invert the disks in the interior of the Apollonian Window, as described in [3] and [6], through the boundary disk of the packing. This inversion will give us the following image in Figure 4.2.

A few notational things. We will describe all disks by their curvature in the original Apollonian Window from this point on. After a circle, denoted X, is inverted through the boundary disk, denoted by $\mathcal{O}$, we will denote this new disk by X'. We will only be looking at the disks which are directly tangent to the boundary disk of the Apollonian window.

A *chain* of disks is defined to be a collection of succesively smaller, that is smaller in terms of radii, tangent disks which are all tangent to $\mathcal{O}$ in the Apollonian Window. For example, the disks $6, 11, 18, 27, 38, 51, ...$ can be considered a chain just as $2, 3, 6, 11, 54, ...$

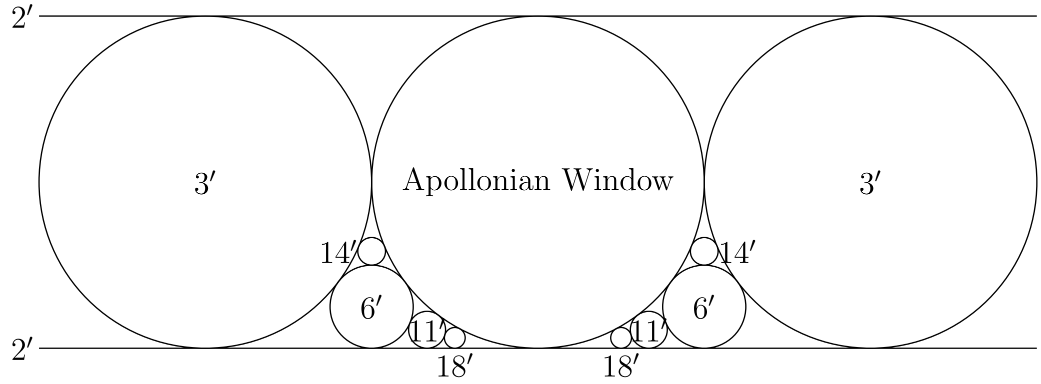

Figure 4.2. Inversion of Apollonian Window

can. We will denote the chain $6, 11, 18, 27, 38, ..., n^2 + 2$, for $n = 2, 3, 4, ...$, by C_1 and the chain $6, 14, 26, 42, 62, 86, ..., 2(n^2 - n + 1)$, for $n = 2, 3, 4, ...$, by C_2. We will look at C_1 and C_2 to generalize a pattern for the other disks and chains located between C_1,C_2, and $\mathcal{O}$.

After inversion, we see that we get a configuration similar to that of the Ford circles from before. We will use the results of [6] on this inversion to compare the area of this disk packing under inversion to the area of our Ford circles. First, we notice a stunning relationship between the radii of the disks in the Apollonian Window and their new radii after inversion. To find the radii of the disks after inversion, we will use the following formula found in [6]

$$r' = \frac{rR^2}{d^2 - r^2}$$

where r' is the radius of the new disk after inversion, r is the disk's original radius, d is the distance between the center of the disk we are inverting through to the center of the original disk in the packing, and R is the radius of the disk we are inverting through, which in this case is 1. We display our results for C_1 in Figure 4.1.

We notice a few interesting patterns being developed. For one, under inversion, the radius of a circle $\dfrac{1}{r}$ becomes $\dfrac{1}{r-2}$, much like Farey series addition [6] of $-\frac{1}{2}$, which can be found in numerous other fractal images. Another pattern is that the radii of the circles

Circle	Old Radius	Radius under Inversion
6	$\frac{1}{6}$	$\frac{1}{4}$
11	$\frac{1}{11}$	$\frac{1}{9}$
18	$\frac{1}{18}$	$\frac{1}{16}$
27	$\frac{1}{27}$	$\frac{1}{25}$
38	$\frac{1}{38}$	$\frac{1}{36}$
51	$\frac{1}{51}$	$\frac{1}{49}$
$\vdots$	$\vdots$	$\vdots$

Table 4.1. Radii of C_1 under Inversion

after inversion become the sequence of inverted perfect squares starting with $n = 2$. This pattern also continues for the chain C_2 as we see in Figure 4.2.

Circle	Old Radius	Radius under Inversion
6	$\frac{1}{6}$	$\frac{1}{4}$
14	$\frac{1}{14}$	$\frac{1}{12}$
26	$\frac{1}{26}$	$\frac{1}{24}$
42	$\frac{1}{42}$	$\frac{1}{40}$
62	$\frac{1}{62}$	$\frac{1}{60}$
86	$\frac{1}{86}$	$\frac{1}{84}$
$\vdots$	$\vdots$	$\vdots$

Table 4.2. Radii of C_2 under Inversion

which also has the same fractal addition by $-\frac{1}{2}$ property as C_1, but has the pattern $\dfrac{1}{2(n)(n+1)}$, for $n = 1, 2, ...$, for the radii of the circles after inversion. It can be shown that the fractal addition by $-\frac{1}{2}$ is true for the other chains of the Apollonian window bounded between C_1, C_2 and $\mathcal{O}$.

Next we look at results for finding the area of the circles in Figure 4.2. The next tables, Table 4.3 and Table 4.4, compare the area of the circles in chains C_1 and C_2 of the Apollonian Window to the area of the same circles after inversion. The scalar column is a value k such that the original area, A, and the area under inversion, A', satisfy $A \cdot k = A'$.

Circle	Original Area	Area under Inversion	Scalar
6	$\pi\left(\frac{1}{36}\right)$	$\pi\left(\frac{1}{16}\right)$	$\frac{9}{4}$
11	$\pi\left(\frac{1}{121}\right)$	$\pi\left(\frac{1}{81}\right)$	$\frac{121}{81}$
18	$\pi\left(\frac{1}{324}\right)$	$\pi\left(\frac{1}{256}\right)$	$\frac{81}{64}$
27	$\pi\left(\frac{1}{729}\right)$	$\pi\left(\frac{1}{625}\right)$	$\frac{729}{625}$
38	$\pi\left(\frac{1}{1444}\right)$	$\pi\left(\frac{1}{1296}\right)$	$\frac{361}{324}$
51	$\pi\left(\frac{1}{2601}\right)$	$\pi\left(\frac{1}{2401}\right)$	$\frac{2601}{2401}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4.3. Area of C_1 under Inversion

Notice that the scalar multiplier k, obeys a particular sequence for both of these chains. For C_1, the scalar multiplier is given by

$$k = \frac{(n^2+2)^2}{((n-2)^2+2)^2} = \frac{(n^2+2)^2}{(n^2-4n+6)^2}$$

where $n = 2, 3, ...$ as we know the sequence $\{n^2+2\}_{n=2}^{\infty}$ generates the values of C_1. For C_2, we have that the scalar multiplier is given by

$$k = \frac{(2(n^2-n+1))^2}{(2((n-2)^2-(n-2)+1))^2} = \frac{(n^2-n+1)^2}{(n^2-5n+7)^2}$$

Circle	Original Area	Area under Inversion	Scalar
6	$\pi\left(\frac{1}{36}\right)$	$\pi\left(\frac{1}{16}\right)$	$\frac{9}{4}=\frac{3^2}{2^2}$
14	$\pi\left(\frac{1}{196}\right)$	$\pi\left(\frac{1}{144}\right)$	$\frac{49}{36}=\frac{7^2}{6^2}$
26	$\pi\left(\frac{1}{676}\right)$	$\pi\left(\frac{1}{576}\right)$	$\frac{169}{144}=\frac{13^2}{12^2}$
42	$\pi\left(\frac{1}{1764}\right)$	$\pi\left(\frac{1}{1600}\right)$	$\frac{441}{400}=\frac{21^2}{20^2}$
62	$\pi\left(\frac{1}{3844}\right)$	$\pi\left(\frac{1}{3600}\right)$	$\frac{961}{900}=\frac{31^2}{30^2}$
86	$\pi\left(\frac{1}{7396}\right)$	$\pi\left(\frac{1}{7056}\right)$	$\frac{1849}{1764}=\frac{23^2}{21^2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 4.4. Area of C_2 under Inversion

where $n = 2, 3, ...$ for the sequence $\{2(n^2 - n + 1)\}_{n=2}^{\infty}$ which generates these values. If we defined the sequence of scalar products in C_2 by using the label for each circle, namely X, we can define the pattern

$$k = \frac{\left(\frac{X}{2}\right)^2}{\left(\frac{X-2}{2}\right)^2} = \frac{X^2}{(X-2)^2}$$

At this point, we notice that even though we have a pattern for determining the area of each individual circle of the chains C_1 and C_2, it is difficult to establish a pattern like the one from the Ford circles, for the area-sum of this disk packing after inversion. Even though we have a similar picture after inversion, the number theoretic structure of the disks is not as simple as the circles. Finding such a pattern for the area-sum is my goal moving forward.

REFERENCES

[1] G.T. Williams and D.H. Browne, A family of integers and a theorem on circles, Amer. Math Monthly 54 (1947) no. 9, 534–536.

[2] L. R. Ford, Fractions, Amer. Math Monthly 45 (1938) no. 9, 586-601.

[3] P. Sarnak, MAA Lecture on Apollonian circle packings (preprint 2009).

[4] G.H. Hardy and E.M. Wright, An introduction to the theory of numbers, Sixth edition, Oxford University Press, 2008.

[5] Dubravka Ban, Analytic Number Theory, Southern Illinois University-Carbondale, Course notes, Spring 2015.

[6] Jerzy Kocik, On a diophantine equation that generates all integral apollonian gaskets, ISRN Geometry Article ID 348618, Southern-Illinois University-Carbondale, 2012

[7] David S. Dummit and Richard M. Foote, Abstract Algebra, 3rd edition, Wiley & Sons.

VITA

Graduate School
Southern Illinois University

Ethan Lightfoot

email address: etlightfoot@gmail.com

Southeast Missouri State University
Bachelor of Science, Mathematics, May 2013

Southeast Missouri State University
Bachelor of Science in Education, Mathematics, May 2013

Thesis Title:
A Family of Integers in a Window

Major Professor: Dr. J. Kocik