ON THE FOUNDATION AND TECHNIC OF ARITHMETIC.*

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MENSURATION.

NEVER forget that no exact measurement is ever possible, that no theorem of arithmetic, algebra, or geometry could ever be proved by measurement, that measure could never have been the basis or foundation or origin of number.

But the approximate measurements of life are important, and the best current arithmetics give great space to mensuration.

Geometry.

Geometry is an ideal construct.

Of course the point and the straight are to be assumed as elements, without definition. They are equally immeasurable, the straight in Euclidean geometry being infinite. What we first measure and the standard with which we measure it are both *sects.* A sect is a piece of a straight between two points, the end points of the sect. The sides of a triangle are sects.

A *ray* is one of the parts into which a straight is divided by a point on it.

An *angle* is the figure consisting of two coinitial rays. Their common origin is its vertex. The rays are its sides.

When two straights cross so that the four angles made are congruent, each is called a *right angle.*

One ninetieth of a right angle is a *degree* (1°).

A *circle* is a line on a plane, equidistant from a point of the plane (the center). A sect from center to circle is its radius.

An *arc* is a piece of a circle. If less than a semi-circle it is a minor arc.

* Continuation of an article begun in the February *Open Court.*

One quarter of a circle is a *quadrant.*

One ninetieth of a quadrant is called a *degree of arc.*

A sect joining the end points of an arc is its *chord.*

A straight with one, and only one, point in common with the circle is a *tangent.*

Length of a Sect.

To measure a sect is to find the number L (its length) when the sect is conceived as $Lu+r$, where u is the standard sect and r a sect less than u. In science, u is the centimeter.

Thus the length, L, of the diagonal of a square centimeter, true to three places of decimals, is 1.414.

Since there are different standard sects in use, it is customary to name u with the L. Here 1.414 cm.

Knowing the length of a sect, from our knowledge of the number and the standard sect it multiplies we get knowledge of the measured sect, and can always approximately construct it.

Length of the Circle.

We assume that with every arc is connected one, and only one, sect not less than the chord, and, if the arc be minor, not greater than the sum of the sects on the tangents from the extremities of the arc to their intersection, and such that if the arc be cut into two arcs, this sect is the sum of their sects. The length of this sect we call the *length of the arc.*

If r be the length of its radius, the length of the semicircle is πr.

Archimedes expressed π approximately as $3+1/7$.

True to two places of decimals, $\pi = 3.14$ or 3.1416 true to four places.

The approximation $\pi = 3+1/7$ is true to three significant figures. But since $\pi = 3.1416 = 3+1/7-1/800$, a second approximation, true to five significant figures, can be obtained by a correction of the first.

Again $\pi = 3.1416 = (3+1/7)(1-.0004)$, which gives the advantage that in a product of factors including π, the value $3+1/7$ can be used and the product corrected by subtracting four thousandths of itself.

The circle with the standard sect for radius is called the *unit circle.* The length of the arc of unit circle intercepted by an angle with vertex at center is called the *size* of the angle.

The angle whose size is 1, the length of the standard sect, is called a *radian.*

A radian intercepts on any circle an arc whose length is the length of that circle's radius.

The number of radians in an angle at the center intercepting an arc of length L on circle of radius length r, is L/r. $180° = \pi\rho$.

An arc with the radii to its endpoints is called a *sector.*

Area.

The area of a *triangle* is half the product of the length of either of its sides (the base) by the length of the corresponding altitude, the perpendicular upon the straight of that side from the opposite vertex.

A figure which can be cut into triangles is a *polygon,* whose area is the sum of theirs. Its *perimeter* is the sum of its sides.

Area of Circle. In area, an inscribed regular polygon (one whose sides are equal chords) of $2n$ sides equals a triangle with altitude the circle's radius r and base the perimeter of an inscribed regular polygon of n sides.

A circumscribed regular polygon (one with sides on tangents) of n sides equals a triangle with altitude r and base the polygon's perimeter.

There is one, and only one, triangle intermediate between the series of inscribed regular polygons and the series of circumscribed regular polygons, namely that with altitude r and base equal in length to the circle. This triangle's area, $rc/2 = r^2\pi$, is the *area of the circle,* $r^2\pi$.

From analogous considerations, the *area of a sector* is the product of the length of its arc by the length of half its radius.

Volume.

A *tetrahedron* is the figure constituted by four noncoplanar points, their sects and triangles.

The four points are called its *summits,* the six sects its *edges,* the four triangles its *faces.*

Every summit is said to be *opposite* to the face made by the other three; every edge opposite to that made by the two remaining summits.

A *polyhedron* is the figure formed by n plane polygons such that each side is common to two. The polygons are called its faces; their sects its edges; their vertices its summits.

One-third the product of the area of a face by the length of the perpendicular to it from the opposite vertex is the *volume of the tetrahedron.*

The *volume of a polyhedron* is the sum of the volumes of any set of tetrahedra into which it is cut.

A *prismatoid* is a polyhedron with no summits other than the vertices of two parallel faces.

The altitude of a prismatoid is the perpendicular from top to base.

A number of different prismatoids thus have the same base, top, and altitude.

If both base and top of a prismatoid are sects, it is a tetrahedron.

A *section* of a prismatoid is the polygon determined by a plane perpendicular to the altitude.

To find the *volume of any prismatoid.* Rule: Multiply one-fourth its altitude by the sum of the base and three times a section at two-thirds the altitude from the base.

Halsted's Formula: $V = (a/4)(B+3S)$.

All the solids of ordinary mensuration, and very many others heretofore treated only by the higher mathematics, are nothing but prismatoids or covered by Halsted's Formula.

A *pyramid* is a prismatoid with a point as top. Hence its volume is $aB/3$.

A circular *cone* is a pyramid with circular base.

A *prism* is a prismatoid with all lateral faces parallelograms.

Hence the volume of any prism $= aB$.

A circular *cylinder* is a prism with circular base.

A *right* prism is one whose lateral edges are perpendicular to its base.

A *parallelopiped* is a prism whose base and top are parallelograms.

A *cuboid* is a parallelopiped whose six faces are rectangles.

A *cube* is a cuboid whose six faces are squares.

Hence the volume of any cuboid is the product of its length, breadth and thickness.

The cube whose edge is the standard sect has for volume 1.

Therefore the volume of any polyhedron tells how oft it contains the cube on the standard sect, called the unit cube.

Such units, like the unit square, though traditional, are unnecessary.

A *sphere* is a surface equidistant from a point (the center).

A sect from the center to sphere is its radius.

A *spherical segment* is the piece of a sphere between two parallel planes.

If a sphere be tangent to the parallel planes containing opposite edges of a tetrahedron, and sections made in the sphere and tetrahedron by one plane parallel to these are of equal area, so are sections made by any parallel plane. Hence the volume of a sphere is given by Halsted's Formula.

$$V = (a/4)(B+3S) = (3/4)aS.$$

But $a = 2r$ and $S = (2/3)r\pi(4/3)r$.

So Vol. sphere $= (4/3)\pi r^3$.

Hence also the volume of a spherical segment is given by Halsted's Formula.

Area of sphere $= 4\pi r^2$.

The area of a sphere is quadruple the area of its great circle.

As examples of solids which might now be introduced into elementary arithmetic, since they are covered by Halsted's Formula, may be mentioned: oblate spheroid, prolate spheroid, ellipsoid, paraboloid of revolution, hyperboloid of revolution, elliptic hyperboloid, and their segments or frustums made by planes perpendicular to their axes, all solids uniformly twisted, like the square-threaded screw, etc.

ORDER.

In the counting of a primitive group, any element is considered equivalent to any other. But in the use even of the primitive counting apparatus, the fingers, appeared another and extraordinarily important character, order.

The savage in counting systematically begins his count with the little finger of the left hand, thence proceeding toward the thumb, which is fifth in the count. When number-words come to serve as extended counting apparatus, order is a salient characteristic.

By one-to-one adjunction of these numerals the individuals of a collection are given a factitious order.

When the order is emphasized the number-names are modified, becoming first, second, third, fourth, etc., and are called ordinal numbers or ordinals, but this designation is now applied also to the ordinary forms, one, two, three, etc., when order is made their fundamental characteristic.

DEPICTION.

If we can so correlate each element of the set A with a definite element of the set B that two different elements of A are never correlated with the same element of B, the element of A is considered as depicted or pictured or imaged by the correlated element of B, its picture or image.

Such a correlation we call a *depiction* of the set A upon the set B. The elements of A are called the *originals.*

An assemblage contained entirely in another is called a component of the latter.

A *proper component* or *proper part* of an assemblage is an aggregate made by omitting some element of the assemblage.

INFINITE.

An assemblage is called *infinite* if it can be depicted upon some proper part of itself, or distinctly imaged, element for element, by a constituent portion, a proper component of itself. Otherwise it is *finite.*

Stand between two mirrors and face one of them. Your image in the one faced will be repeated by the other. If this replica could be separately reflected in the first, this reflection imaged by itself in the second, this image pictured as distinct in the first, this in turn depicted in the second, and so on forever, this set would be infinite, for it is depicted upon the proper part of it made by omitting you. It is *ordered.* You may be called 1, your image 2, its image 3, and so on.

Sense.

A relation has what mathematicians call *sense,* if, when A has it to B, then B has to A a relation different, but only in being correlatively opposite. Thus "greater than" is a sensed relation. "Greater than" and "less than" are different relations, but differ only in sense.

Any number of numbers, all individually given, form a finite set. If numbers be potentially given through a given operand and a given operation, law, of successive education, they are still said to form a set. If the law educes the numbers one by one in definite succession, they have an *order,* taking on the order inherent in time or in logical or causal succession.

A set in order is a *series.*

Analysis of Order.

Intrinsic order depends fundamentally upon relations having sense, and, for three terms, upon a relation and its opposite in sense attaching to a given term.

The unsymmetrical sensed relation which determines the fixed order of sequence may be thought of as a logic-relation, that an element shall involve a logically sequential element creatively or as representative. An individual or element 1 has its shadow 2, which in turn has its shadow 3, and so on.

Linear order is established by an unsymmetrical relation for one sense of which we may use the word "precede," for the opposite sense "follow."

The ordering relation may be envisaged as an operation, a transformation, which performed upon a preceding gives the one next succeeding it; turns 1 into 2, and 2 into 3, and so on.

If we have applicable to a given individual an operation which turns it into a new individual to which in turn the operation is applicable with like result, and so on without cease, we have a recurrent operation which recreates the condition for its ongoing. If in such a set we have one and only one term not so created from any other, a first term, and if every term is different from all others, we have a commencing but unending ordered series. The number series, 1, 2, 3, and so on, may be thought of as the outcome of a recurrent operation, that of the ever repeated adjunction of one more unit. It is a system such that for every element of it there is always one and only one next following. This successor may be thought of as the depiction of its predecessor. Every element is different from all others. Every element is imaged. There is an element which though imaged is itself no image.

Thus the series is depicted without diminution upon a proper part of itself; is infinite, and by constitution endless. It has a first element, but no element following all others, no "last" element.

Any set which can be brought into one-to-one correspondence with some or all of the natural numbers is said to be *countable,* and, if not finite, is called *countably infinite.*

Ordered Set.

A set of elements is said to be in simple order if it has two characteristics:

1°. Every two distinct arbitrarily selected elements, A and B, are always connected by the same unsymmetrical relation, in which relation we know what rôle one plays, so that always one, and only one, say A, comes before B, is source of B, precedes B, is less than B; while B comes after A, is derived from A, follows A, is greater than A.

2°. Of three elements ABC, if A precedes B, and B precedes C, then A precedes C. Thus the moments of time between twelve and one o'clock, and the points on the sect AB as passed in going from A to B are simply ordered sets.

Finite Ordinal Types.

An arranged finite set of, say, n elements can be brought into one-to-one correspondence with the first n integers.

Such an ordered set has a first and a last element; so has each ordered component.

Inversely an ordered set with a first and a last element, whose every component has a first and a last element, is finite. For let a_1 be the first element. The remaining elements form an ordered component; let a_2 be the first of these elements. In the same way determine a_3. We must thus reach the last, else were there an ordered component without last element, contrary to hypothesis. These then are the characteristics of the finite ordinal types.

THE NATURAL NUMBER SERIES AS A TYPE OF ORDER.

The characteristic property of a countably infinite set, when arranged in countable order, is that we know of any element a whether, or no, it corresponds to a smaller integer than does the element b. Should a and b correspond to the same integer they would be identical. Thus when arranged in countable order, the order of any countably infinite set is that of the natural numbers. The defining characteristics of this order are that it, as well as each of its ordered components, has a first element, and that every element, except the first, has another immediately preceding it; while there is no last element.

Any simply-ordered set between any pair of whose elements there is always another element is said to be in *close order.*

Well-ordered Sets.

A simply ordered set is said to be "well-ordered" if the set itself, as well as every one of its components, has a first element.

In a well-ordered set its elements so follow one another according to a given law that every element is immediately followed by a completely determined element, if by any. As typical of well-ordered sets we may take first the finite sets of the ordinal numbers: 1st; 1st, 2d; 1st, 2d, 3d; and so on.

As typical of the first transfinite well-ordered set we may take the set of all the ordinal numbers, the ascending order of the natural numbers.

The thousandth even number is immediately followed by the number 2001.

But if a point B is taken on a sect AC, there is no next consecutive point to B determinable.

The way in which an iterative operation develops from an individual operand not only infinity but endless variety unthought of and so waiting to be thought of, lights up the fact that mathematics though deductive is not troubled with the syllogism's tautology but offers ever green fields and pastures new. Thus in the number series is the series of even numbers, in this the set of even even numbers, 4, 8, 12, 16, 20, etc., each a system in which every element of every preceding system of this series of systems can have its own uniquely determined picture, the first term depicting any first term, the second any second, etc.

ORDINAL NUMBER.

Ordinal Number.

Numbers are ordinal as individuals in a well-ordered set or series, and used ordinally when taken to give to any one object its position in an arrangement and thus to individually identify and place it. So its number identifies the automobile.

The ordinal process has also as outcome knowledge of the cardinal; when we have in order ticketed the ninth, we have ticketed nine. Thus the last ordinal used tells the result of the count.

Children's Counting.

The assignment of order to a collection and ascertainment of place in the series made by this putting in order is shown by that use of *count* which occurs in children's games, in their *counting out* or counting to fix who shall be *it*. This counting is the use of a set of words not ever investigated as to multiplicity, but characterized by order. Such is the actually-used set: ana, mana, mona, mike; bahsa, lona, bona, strike; hare, ware, frounce, nack; halico, baliko, we, wo, wy, wak. Applied to an assemblage, it gives order to the assemblage until exhausted, and the last one of the ordered but unnumbered group is *out* or else *it*. How many individual words the ordering group contains is never once thought of. There is successive enumeration without simultaneous apprehension.

Every element has an ordinal significance. No element has any cardinal significance.

E nee, me nee, my nee, mo;
Crack ah, fee nee, fy nee, fo;
Amo neu ger, po po tu ger;
Rick stick, jan jo.

Such a group but indefinitely extensible, having a first but no last term, is the ordinal number series.

Uses of the Ordinal System.

But in our ordinary system of numeral words, with fixed and rote-learned order, each word is used to convey also an exact notion of the multiplicity of individuals in the group whose tagging has used up that and all preceding numerals. Thus each one characterizes a specific group, and so has a cardinal content.

Yet it is upon the ordered system itself that we chiefly rely to get a working hold of the number when beyond the point where we try to have any complete appreciation, as simultaneous, of the collection of natural units involved. Thus it is to the ordinal system that we look for succor and aid in getting grasp and understanding particularly of numbers too great for their component individual units to be at once and together separately picturable. Thus the ideas we get of large numbers come not from any attempt to realize the multiplicity of the discrete manifold, but rather from place in the number-set.

Number in its genesis is independent of quantity, and number-science consists chiefly, perhaps essentially, in relations of one number in the number series to another and to the series.

That a concept is dependent for its existence upon a word or language-symbol is a blunder. The savage has number-concepts beyond words. On the other hand, the modern child gets the words of the ordinal series before the cardinal concepts we attach to them. If a little child says, "Yes, I can count a hundred," it simply means it can repeat the series of number-names in order. Its slips would be skips or repetitions. The ordinal idea has been formed. It is used by the child who recognizes its errors in this ordinal counting. The ordinal idea has been made, has been embodied perhaps in rythmical movements. The child's rudimentary counting set is a sing-song ditty. The number series when learned is perhaps chanted. Just so there is a pleasurable swing in the count by fives.

The use of the terms of the number series as instruments for individual identification appears in the primitive child's game as in the identification of the automobile. Before making or using number, children delight in making series. Succession is one of the earliest made thoughts.

We think in substituted symbols. It is folly to attempt to hold back the child in this substitution. The abstractest number becomes a thing, an objective reality.

Number has not originated in comparison of quantity nor in quantity at all. Number and quantity are wholly independent categories, and the application of number to quantity, as it occurs in measurement, has no deeper motive than one of convenience.

It has often been stressed, that children knowing the number-names, if asked to count objects, pay out the series far faster than the objects; the names far outstrip the things they should mate.

The so-called passion of children for counting is a delight in ordinal tagging, in ordinal depiction with names, with no attempt to carry the luggage of cardinals.

The "which one" is often more primitive and more important than the "how many." The hour of the day is an ordinal in an ordered set. Its interest for us is wholly ordinal. It identifies one element in an ordered set. The strike of the clock is a word. The striking clock has a vocabulary of 12 words. These words are distinguished by the cardinal number of their syllables. But even when recognized by the cardinal number of syllables in its clock-spoken name, the hour is in essence an ordinal.

So the number series as a word-song may well in our children precede any application to objects. Objects are easily over-estimated by those who have never come to the higher consciousness that objects are mind-made, that every perception must partake of the subjective.

Children often apply the number-names to natural individuals as animals might, that is without making any artificial or man-made individual, and so without any cardinal number. Each name depicts a natural individual, but not as component of a unity composed of units. What passes for knowledge of number among animals is only recognition of an individual or an individual form.

Serial depiction under the form of tallying or beats or strokes may precede all thought of cardinal number. Nine out of ten children learn number names merely as words, not from objects or groups.

The typical case is given of the girl who could "count" 100 long before she could recognize a group of six or even of four objects.

The names of the natural numbers are an unending child's ditty, primarily ordinal, but a ditty to whose terms cardinal meanings have also been attached. Ordinally the number name "one" is simply the initial term of this series; any number name is simply a term of this series. The ordinal property it designates is the positional property of an element in a well-ordered set.

The natural scale is the standard for civilized counting. Its symbols in sequence are mated with the elements of an aggregate and the last symbol used gives the outcome of the count, tells the cardinal number of the counted aggregate. The cardinal, *n*, of a set is that attribute by which when the set's elements are coupled with ordinals, the ordinal *n* and all ordinals preceding *n* are used.

The very first step in the teaching of arithmetic should be the child's chanting of the number names in order. Then the first application should be ordinal. Use the numbers as specific tags, conveying at first only order and individual identification. Afterward connect with each group, as *its* name, the last numeral it uses, which thus takes on a cardinal significance.