ON THE FOUNDATION AND TECHNIC OF ARITHMETIC.*

BY GEORGE BRUCE HALSTED.

Decimals.

IT is the characteristic of our positional notation for number that shifting a digit one place to the left multiplies it by the base of the system. The zero enables us to indicate such shifting. Thus since our base is ten, 1 shifted one place to the left, 10, becomes ten; two shifted two places to the left, 200, becomes two hundred.

Inversely, shifting a digit one place to the right divides it by the base of the system. Thus 3 in the thousands place, 3000, shifted one place to the right becomes 300.

We now create that this shifting to the right may go on beyond the units place with no change of meaning or effect.

In order to write this, we use a device, a notation to mark or point out the units place, a point immediately to its right called the decimal point or unital point. The decimal point appears first in 1617 on page 21 of Napier's *Rabdologiae*. Thus 4 shifted one place to the right becomes 0.4 and of course means a number which multiplied by the base gives 4. Such numbers have been called decimals. Their theory is independent of the base, which might be say 12 or 2, in which case the word decimals would be a distinct misnomer.

If however the base be ten, then shifting a digit one place to the left multiplies it by ten. But this is accomplished for every digit in the number simply by shifting the point one place to the right. Thus .05 is tenfold .005. If our unit is a dollar, $1, then the first place to the right will be dimes. Thus $0.6 means six dimes. The next place to the right of dimes means cents. Thus $.07 means seven cents. The next place to the right of cents means mills. Thus $.008 means eight mills.

* Continuation of an article appearing in the February and March numbers.

Ten mills make a cent. Ten cents make a dime. Ten dimes make a dollar.

In general we name these basal subunitals so as to indicate by symmetry their place with reference to the units column. As the first column to the left of units is tens, so the first column to the right of units is called tenths. As the second column to the left of the units column is called hundreds, so the second column to the right of the units' column is called hundredths. As the third column to the left of the units column is called thousands, so the third column to the right of the units column is called thousandths.

But these names need not be used in reading a subunital. Thus 0.987 may be read: Point, nine, eight, seven.

One-tenth is a number ten of which are together equal to a unit.

The word "and" connecting the different parts of a number is generally dropped; in English, however, it is retained after the hundreds (Homersham Cox, *Arithmetic,* p. 9).

If an integer be read by merely pronouncing in succession the names of its digits, as in reading 7689 as seven, six, eight, nine, we do not know the rank and so all the value of any figure read until after all have been read.

Hence the advantage of reading 7689 seven thousand six hundred and eighty-nine. But in reading the decimal .7689 as "point, seven, six, eight, nine" we know every thing about each figure as it is read, which on the contrary we do not know if it be read seven thousand six hundred and eighty-nine ten-thousandths.

Moreover such a habit of reading decimals detracts from our confident certainty of understanding integers step by step as read. There may be coming at the end a wretched subunital designation like this "ten-thousandths" to metamorphose everything read.

So always read decimals by pronouncing the word *point* and the names of the separate single digits.

Read 7000.008 seven thousand, point, nought, nought, eight. Read .708 point, seven, nought, eight.

Sum and Difference.

To add decimals, write the terms so that the decimal points fall precisely under one another, in a vertical column. Then proceed just as with integers, the point in the sum falling under those of the terms.

Just so it is with subtraction.

Product.

In multiplying decimals remember we are dealing simply with a symmetrical completion, extension of positional notation to the *right* from units' place. Realize the perfect balance resting on the units' column. 4321.234.

A shift of the decimal point changes the rank of each of the digits. So to multiply or divide by any power of ten is accomplished by a simple shift of the point.

Thus 98.76×10 is 987.6. Just so $98.76/10$ is 9.876, and is identical with 98.76×0.1. Twice this is $98.76\times0.1\times2$ or $98.76\times0.2 = 19.752$.

So to multiply by a decimal is to multiply by an integer and shift the point.

Hence the rule, useful for check, that the number of decimal places in the product is the sum of the places in the factors. There is no need for thinking of tenths as fractions to realize that two-tenths of a number is twice one-tenth of it.

In multiplying decimals, write the multiplier so that its point comes precisely under the point in the multiplicand, and in vertical column with these put the point in each partial product. The figure obtained from multiplying the *units* figure of the multiplicand must come precisely under the figure by which we are multiplying.

$$\begin{array}{r@{.}l}
1293 & 015 \\
132 & 02 \\
\hline
129301 & 5 \\
38790 & 45 \\
2586 & 030 \\
25 & 86030 \\
\hline
170703 & 8403
\end{array}$$

Here, beginning to multiply by the 1, think five while writing it two places to the left of the figure multiplied because the 1 is two places to the left of the units column. Proceed to multiply by the 3, thinking fif*teen*; 3, four; nought; nine; *twenty*-seven; etc.

Rule: Multiplying shifts as many places right or left as the multiplier is from the units column.

$$\begin{array}{r@{.}l}
41 & 27 \\
 & 03 \\
\hline
1 & 2381
\end{array}$$

Here think *twenty*-one while writing the 1 two places more to the right than the 7 because the 3 is two places to the right of the units column.

Quotient.

In division of decimals place the decimal point of the quotient precisely over the decimal point of the dividend and, when the divisor is an integer, the first figure of the quotient over the last figure of the first partial dividend.

Rule: The first figure of the quotient stands as many places to the left of the last figure of the first partial dividend as there are decimal places in the divisor.

```
     638
.021)13.4
       8
       17
         2
```

Here the quotient 638 is an integer.

The sign + at the end of a number means there is a remainder, or that the number to which it is attached falls short of completely, exactly expressing all it represents, though increasing the last figure by unity would overpass exactitude and so should be followed by the sign − (minus).

```
      6+
2.1).0134
       8
```

Thus $\pi = 3.14+$ and

$\pi = 3.1416-$

When there is a remainder we may get additional places in the quotient by annexing ciphers to the dividend and continuing the division.

```
    63
.21)13.4
      8
      17
```

The phrase "true to 2 (or 3, etc.) places of decimals" means that a closer approximation can not be written without using more places.

Thus as a value for π, 3.14 is true or "correct" to two places of decimals, since $\pi = 3.14159+$; while 3.1416 is true to four places.

As an approximation to 1.235 we may say either 1.23 or 1.24 is true to two places of decimals.

FRACTIONS.

Principle of Permanence:

For the new numbers hold the old laws.

1st. Every number combination which gives no already existing number, is to be given such an interpretation that the combination can be handled according to the same rules as the previously existing numbers.

2d. Such combination is to be defined as a number, thus enlarging the number idea.

3d. Then the usual laws (freedoms) are to be proved to hold for it.

4th. Equal, greater, less are to be defined in the enlarged domain.

This was first given by Hankel as generalization of a principle given by G. Peacock, British Association, III, London, 1834, p. 195. *Symbolic Algebra*, Cambridge, 1830, p. 105; 2d ed., 1845, p. 59.

Fractions.

If unity in pure number be considered as indivisible, fractions may be introduced by conventions. Take two integers in a given order and regard them as forming a couple; create that this couple shall be a number of a new kind, and define the equality, addition, and multiplication of such numbers by the conventions,

$$a/b = c/d \text{ if } ad = bc;$$
$$a/b + c/d = (ad+bc)/bd;$$
$$(a/b)(c/d = (ac)/(bd).$$

The preceding number is called the *numerator* of the fraction: the succeeding number, the *denominator.*

Fractions have application only to objects capable of partition into portions equal in number to the denominator. No fraction is applicable to a person.

In accordance with the principle of permanence, we create that the compound symbol of the form a/b, two natural numbers separated by the slant, shall designate a number. Either the symbol or the number may be called a *fraction.* The slant is to stand for the division of a by b, of the preceding by the succeeding number, where this is possible. When a is exactly divisible by b, that is, without remainder, the fraction designates a natural number.

When a is a multiple of b, and a' of b', the equality $ab' = a'b$ is the necessary and sufficient condition for the symbols a/b, a'/b' to represent the same number. By this same condition we define the equality of the new numbers, the fractions.

A fraction is *irreducible* when its numerator and denominator contain no common factor other than 1.

To compare two fractions, reduce them to a common denominator, then that which has the greater numerator is called the greater.

A *proper* fraction is a fraction with numerator less than denominator. It is less than 1.

Subtraction is given by the equality $a/b - a'/b' = (ab' - a'b')/bb'$.

The *multiplication* of fractions is covered by the statement: A product is the number related to the multiplicand as unity to the multiplier. $(a/b)\ (a'/b') = aa'/bb'$.

Thus $(5/7)\times(2/3)$ means trisect, then double, giving $10/21$.

So $(a/b)\times(b/a) = 1$. Two numbers whose product is unity are called *reciprocal.*

Division is taken as the inverse of multiplication, hence (c/d) $/(a/b)$ means to find a number whose product with (a/b) is (c/d). Such is $(c/d)(b/a)$.

So $(c/d)/(a/b) = (c/d)(b/a) = bc/ad$.

1°. This last expression may be considered simply a more compact form of the first, obtained by reducing to a common denominator and cancelling this denominator. This compact form can be obtained by a procedure sometimes called the rule for division by a fraction: *Invert the divisor and multiply.*

2°. If we interchange numerator and denominator of a fraction we get its *inverse* or *reciprocal*. So the inverse of a is $1/a$.

$(a/b)\ (b/a) = 1$.

Now $(x/y)/(a/b)$ means to find a number which multiplied by a/b gives x/y and so the answer is $(x/y)\ (b/a)$. Hence: *To divide by a fraction, multiply by its reciprocal.*

3°. Again to find $(a/b)/(c/d)$, note that c/d is contained in 1 d/c times, and hence in a/b it is contained $(a/b)\ (d/c)$ times.

Fractions Ordered.

A *reduced* fraction is one whose numerator and denominator contain no common factor.

The fractions arranged according to size are an ordered set, but not well ordered; for no fraction has a determinate next greater fraction, since between any two numbers, however near in size, lie always innumerable others.

But all reduced fractions can be arranged in a well-ordered set arranged according to groups in which the sum of numerator and denominator is the same:

1/1, 1/2, 2/1, 1/3, 3/1, 1/4, 2/3, 3/2, 4/1, 1/5, 5/1, 1/6, 2/5, 3/4, 4/3, 5/2, 6/1,. . . .

Thus they make a simply infinite series equivalent to the number series.

Proper fractions can be arranged by denominators:

$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \ldots$

To turn a fraction a/b into a decimal $c/10^k$ must give $a10^k = bc$, where c is a whole number. Since a/b is in reduced form, therefore a and b have no common factor. So 10^k must be exactly divisible by b. Thus only fractions with denominator of the form $2^n 5^m$ can be turned exactly into decimals.

Fractions may be thought of as like decimals in being also subunitals. The unit operated with in a fraction, the fractional unit, is a subunit, and the *denominator* is to tell us just what subunit, just what certain part of the whole or original or primal unit is taken as this subunit; while the *numerator* is the number of these subunits. Thus 3/10 is a three of subunits ten of which make a unit. Thus, like an integer, a fraction is a unity of units (or one unit), but these are subunits. Different subunits may be very simply related, as are 1/2, 1/4, 1/8.

To add 3/4 and 1/2 we first make their subunits the same by bisecting the subunit of 1/2, which thus becomes 2/4. Then 3/4 and 2/4 may be counted together to give 5/4.

Fractions having the same subunit are added by adding their numerators, the same denominator being retained since the subunit is unchanged. The like is true of subtraction.

To add unlike fractions change to one same subunit. The technical expression for this is "reduce to a common denominator."

Since we already know that to be counted together the things must be taken as indistinguishably equivalent, the procedure of changing to one same subunit is crystal clear.

To change a half to twelfths is simply to split up the one-half, the first subunit, into subunits twelve of which make the whole or original unit.

Thus, operatively, to express a fraction in terms of some other subunit, the procedure is simply to multiply (or divide) numerator and denominator by the same number.

Thus $1/2=(1\times6)/(2\times6)=6/12$. So $6/12=(6/3)/(12/3)=2/4$. This principle in the form: "The value of a fraction is unaltered by dividing both numerator and denominator by the same number," is freely applied in what is technically called "reducing fractions to their lowest terms."

It should be applied just as freely and directly in the form: "The value of a fraction is unaltered by multiplying both numerator and denominator by the same number." Thus the complex fraction $(2+2/3)/(3+2/9)$, multiplying both terms by 9, gives at once 24/29. Again (3 feet 5 inches)/(2 feet 7 inches), multiplying both terms by 12, gives 41/31.

$13\frac{1}{4}$	To subtract $7+3/4$ from $13+1/4$, that is to evaluate
$7\frac{3}{4}$	$13\frac{1}{4}-7\frac{3}{4}$, think 3/4 and two-fourths make 5/4, carry 1;
$5\frac{2}{4}$	8 and five make thirteen.

Division of a Fraction by an Integer.

The 1 in $1/n$ is the subunit, the n specifying what particular subunit. In division of a fraction by an integer we meet the same limitation which theoretically led to the creation of fractions; namely

2/5 is no more divisible by three than any other two. But here we can easily transform our fraction into an equivalent divisible by 3. Just trisect the subunit. Thus 2/5 becomes 6/15, which is divisible by 3 giving 2/15.

Such result is always at once attained simply by multiplying the given denominator by the given integral divisor. Hence the rule: To divide a fraction by an integer, multiply its denominator by the integer.

RELATION OF DECIMALS TO FRACTIONS.

6.214	Fractions may be freely combined with decimals.
$3\frac{1}{3}$	Thus $1/24 = .04\frac{1}{6}$.
18.642	1 meter = 39.37 inches = 3 feet $3\frac{3}{8}$ inches.
$2.071\frac{1}{3}$	In finding the product of a decimal and a fraction
$20.713\frac{1}{3}$	use the fraction as multiplier.

1st. *Conversion of Decimals Into Fractions.*

By our positional notation, 0.1 means one subunit such that ten of them make the unit. But just this same thing is meant by 1/10. Therefore any decimal may be instantly written as a fraction; e. g., $0.234 = 2/10+3/100+4/1000 = 234/1000$.

2d. *Conversion of Fractions Into Decimals.*

First Method.

Any fraction equals the quotient of its numerator divided by its denominator. Consider the fraction, then, simply as indicating an example in division of decimals, and proceed to find the quotient.

Thus for 1/2 we have:

$$\begin{array}{r} .5 \\ 2)\overline{1.0} \end{array} \quad \text{So } 1/2 = 0.5.$$

For 3/4 we have:

$$\begin{array}{r} .75 \\ 4)\overline{3.00} \end{array} \quad \text{So } 3/4 = 0.75.$$

For 7/8 we have:

$$\begin{array}{r} .875 \\ 8)\overline{7.000} \end{array} \quad \text{So } 7/8 = 0.875.$$

Second Method.

Apply the principle: The value of a fraction is unaltered by multiplying both numerator and denominator by the same number.

$$\begin{aligned} \text{Thus } 7/8 &= 7/(2\times2\times2) \\ &= (7\times5\times5\times5)/(2\times5\times2\times5\times2\times5) \\ &= 875/1000 = 0.875. \end{aligned}$$

Considering the application of this second method to 1/3, we see there is no multiplier which will convert 3 into a power of 10, since 10 contains no factors but 2 and 5. Ten does not contain 3 as a factor, so we cannot convert 1/3 into an ordinary decimal. We cannot, as an example in division of decimals, divide 1 by 3 *without remainder.* But we can freely apply remainder-division, at any length. Thus

$$\begin{array}{r@{}l} & .333 \\ 3)\,1. & \\ & .1 \\ & .01 \\ & .001 \end{array}$$

The procedure is recurrent, and if continued the 3 would simply recur.

$$\begin{array}{l} \;\;\;.142857 \\ 7)\,1. \\ \;\;\;\;.3 \\ \;\;\;\;\;\;2 \\ \;\;\;\;\;\;\;6 \\ \;\;\;\;\;\;\;\;4 \\ \;\;\;\;\;\;\;\;\;5 \\ \;\;\;\;\;\;\;\;\;\;1 \end{array}$$

In division by n, not more then n–1 different remainders can occur. But as soon as a preceding dividend thus recurs, the procedure begins to repeat itself. Here then this division by 7 must begin to repeat, and the figures in the quotient must begin to recur.

If the recurring cycle begins at once, immediately after the decimal point, the decimal is called a pure recurring decimal. As notation for a pure recurring decimal, we write the recurring period, the repetend, dotting its first and last figures; thus $1/11 = .\dot{0}\dot{9}$; $1/9 = \cdot\dot{1}$.

Every fraction is a product of a decimal by a pure recurring decimal. Thus $1/6 = (1/2)(1/3) = 0.5 \times \cdot\dot{3}$.

To convert recurring decimals into fractions:

$$\begin{array}{l} \cdot\dot{1}\dot{2}\times\;100 = 12\cdot\dot{1}\dot{2} \\ \underline{\cdot\dot{1}\dot{2}\times\;\;\;\;\;1 = \;\;\;\;\cdot\dot{1}\dot{2}} \\ \cdot\dot{1}\dot{2}\times\;\;\;99 = 12 \\ \cdot\dot{1}\dot{2} = 12/99 = 4/33 \end{array}$$

Therefore subtracting,

Rule: Any pure recurring decimal equals the fraction with the repeating period for a numerator, and that many nines for denominator.

Base.

The base of a number system is the number which indicates how many units are to be taken together into a composite unit, to be named, and then to be used in the count instead of the units composing it, this first composite unit to be counted until, upon reaching as many of them as units in the base, this set of composite units is taken together to make a complex unit, to be named, and in turn to be used in the count, and enumerated until again the basal number of these complex units be reached, which manifold is again to be made a new unit, named, etc.

Thus twenty-five, twain ten + five, uses ten as base. Using twelve as base, it would be two dozen and one. Using twenty, it would be a score and five. In positional notation for number, a digit in the units' place means so many units, but in the first place to the left of units' place it means so many times the base, while in the first place to the right of the units' place it means so many subunits each of which multiplied by the base gives the unit. And so on, for the second, etc., place to the left of the units' column, and for the second, etc., place to the right of the units' column.

It is the systematic use of a base in connection with the significant use of position, which constitutes the formal perfection of our Hindu notation for number. The actual base itself, ten, is a concession to our fingers.

Compare these subunital expressions for the fundamental fractions, to base ten, to base twelve, to base two.

DECIMALLY.	DUODECIMALLY.	DUALLY. [IN THE BINARY NOTATION.]
$1/2=0\cdot5$	$1/2=0\cdot6$	$1/2=1/10\ =0\cdot1$
$1/3=\ \cdot\dot{3}$	$1/3=0\cdot4$	$1/3=1/11\ =\ \cdot\dot{0}\dot{1}$
$1/4=0\cdot25$	$1/4=0\cdot3$	$1/4=1/100\ =\ \cdot01$
$1/6=0\cdot1\dot{6}$	$1/6=0\cdot2$	$1/6=1/110\ =\ \cdot0\dot{0}\dot{1}$
$1/8=0\cdot125$	$1/8=0\cdot16$	$1/8=1/1000=\ \cdot001$
$1/9=\ \cdot\dot{1}$	$1/9=0\cdot14$	$1/9=1/1001=\ \cdot\dot{0}0011\dot{1}$

MEASUREMENT.

Says Dr. E. W. Hobson: "It is a very significant fact that the operation of counting, in connection with which numbers, integral and fractional, have their origin, is the one and only absolutely exact operation of a mathematical character which we are able to undertake upon the objects which we perceive. On the other hand, all operations of the nature of measurement which we can perform in connection with the objects of perception contain an essential element of inexactness. The theory of exact measurement in the domain of the ideal objects of abstract geometry is not immediately derivable from intuition."

Arithmetic is a fundamental engine for our creative construction of the world in the interests of our dominance over it. The world so conceived bends to our will and purpose most completely. No rival construct now exists. There is no rival way of looking at the world's discrete constituents. One of the most far-reaching achievements of constructive human thinking is the arithmetization of that world handed down to us by the thinking of our animal predecessors.

Why Count?

In regard to an aggregate of things, why do we care to inquire "how many"? Why do we count an assemblage of things? Why not be satisfied to look upon it as an animal would? How does the cardinal number of it help?

First of all it serves the various uses of identification. Then the inexhaustible wealth of properties individual and conjoined of exact science is through number assimilated and attached to the studied set, and its numeric potential revealed. Mathematical knowledge is made applicable and its transmission possible.

Thus the number is basal for effective domination of the world social as well as natural.

Number arises from a creative act whose aim and purpose is to differentiate and dominate more perfectly than do animals the perceived material, primarily when perceived as made of individuals. Not merely must the material be made of individuals, but primarily it must be made of individuals in a way amenable to treatment of this particular kind by our finite powers. Powers which suffice to make specific a clutch of eggs, say a dozen, may be transcended by the stars in the sky.

Number is the outcome of an aggressive operation of mind in making and distinguishing certain multiplex objects, certain manifolds. We substitute for the things of nature the things born of man's mind and more obedient, more docile. They, responsive to our needs, give us the result we are after, while economising our output of effort, our life. The number series, the ordered denumerable discrete infinity is the prolific source of arithmetic progress. Who attempts to visualize 90 as a group of objects? It is nine tens. Then the fingers tell us what it is, no graphic group visualization. First comes the creation of artificial individuals having numeric quality. The cardinal number of a group is a selective representation of it which takes or pictures only one quality of the group but takes that all at once. This selective picture process only applies primarily to those particular artificial wholes which may be called discrete aggregates. But these are of inestimable importance for human life.

The Measure Device.

The overwhelming advantages of the number picture led, after centuries, to a human invention as clearly a device of man for himself as the telephone. This was a device for making a primitive individual thinkable as a recognizable and recoverable artificial individual of the kind having the numeric quality, having a number picture. This is the recondite device called measurement.

Measurement is an artifice for making a primitive individual conceivable as an artificial individual of the group kind with previously known elements, conventionally fixed elements, and so having a significant number-picture by which knowledge of it may be transmitted, to any one knowing the conventionally chosen standard unit, in terms of this previously known standard unit and an ascertained number.

From the number and the standard unit for measure the measured thing can be approximately reproduced and so known and recovered. No knowledge of the thing measured must be requisite for knowledge of the standard unit for the measurement. This standard unit of measure must have been familiar from previous direct perception. So the picturing of an individual as three-thirds of itself is not measurement.

All measurement is essentially inexact. No exact measurement is ever possible.

Counting Prior to Measuring.

Counting is essentially prior to measuring. The savage, making the first faltering steps, furnished number, an indispensable prerequisite for measurement, long ages before measurement was ever thought of. The primitive function of number was to serve the purposes of identification. Counting, consisting in associating with each primitive individual in an artificial individual a distinct primitive individual in a familiar artificial individual, is thus itself essentially the identification, by a one-to-one correspondence, of an unfamiliar with a familiar thing. Thus primitive counting decides which of the familiar groups of fingers is to have its numeric quality attached to the group counted. To attempt to found the notion of number upon measurement is a complete blunder. No measurement can be made exact, while number is perfectly exact.

Counting implies first a known ordinal series or a known series of groups; secondly an unfamiliar group; thirdly the identification of the unfamiliar group by its one-to-one correspondence with a familiar group of the known series. Absolutely no idea of measurement, of standard unit of measure, of value is necessarily involved or indeed ordinarily used in counting. We count when we wish to find out whether the same group of horses has been driven back at night that was taken out in the morning. Here counting is a process of identification, not connected fundamentally with any idea of a standard measurement-unit-of-reference, or any idea of some value to be ascertained. We may say with perfect certainty that there is no implicit presence of the measurement idea in primitive number. The number system is not in any way based upon geometric congruence or measurement of any sort or kind.

The numerical measurement of an extensive quantity consists in approximately making of it, by use of a well-known extensive quantity used as a standard unit, a collection of approximately equal, quantitatively equal, quantities, and then counting these approximately equal quantities. The single extensive quantity is said to be numerically measured in terms of the convened standard quantitative extensive unit.

New Assumptions.

For measurement, assumptions are necessary which are not needed for counting or number. Spatial measurement depends upon the assumption that there is available a standard body which may be transferred from place to place without undergoing any other change. Therein lies not only an assumption about the nature of space but also about the nature of space-occupying bodies. Kindred assumptions are necessary for the measuring of time and of mass.

Now in reality none of these assumptions requisite for measurement are exactly fulfilled. How fortunate then that number involves no measurement idea!

But still other assumptions are made in measurement. After this device for making counting apply to something all in one piece has marked off the parts which are to be assumed as each equal to the standard, their order is unessential to their cardinal number. But it is also assumed that such pieces may be marked out beginning anywhere, then again anywhere in what remains, without affecting the final remainder or the whole count. Moreover measurement, even the very simplest, must face at once incommensurability. Whatever you take as standard for length, neither it nor any part of it is exactly contained in the diagonal of the square on it. This is proven. But the great probabilities are that your standard is not exactly contained in anything you may wish to measure. There is a remainder large or small, perceptible or imperceptible. Measurement then can only be a way of pretending that a thing is a discrete aggregate of parts equal to the standard, or an aliquot part of it. We must neglect the remainder. If we do it unconsciously, so much the worse for us.

No way has been discovered of describing an object exactly by counting and words and a standard. Any man can count exactly. No man can measure exactly.

Arithmetic applies to our representation of the world, to the constructed phenomena the mind has created to help, to explain, its own perceptions. This representation of things lends itself to the application of arithmetic. Arithmetic is a most powerful instrument for that ordering and simplification of perception which is fundamental for dominance over so-called nature.

Measurement may be analysed into three primary procedures: 1°. The conventional acceptance or determination of a standard object, the unit of measure. 2°. The breaking up of the object to be measured into pieces each congruent to the standard object. 3° The counting of these pieces.

(TO BE CONTINUED.)