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The Open Court

A MONTHLY MAGAZINE

Devoted to the Science of Religion, the Religion of Science, and the Extension of the Religious Parliament Idea

Founded by Edward C. Hegeler

A PAGAN "NOAH'S ARK" OF PREHISTORIC ORIGIN, DISCOVERED IN A TOMB OF ANCIENT VETULONIA, ETRURIA.

(See page 88.)

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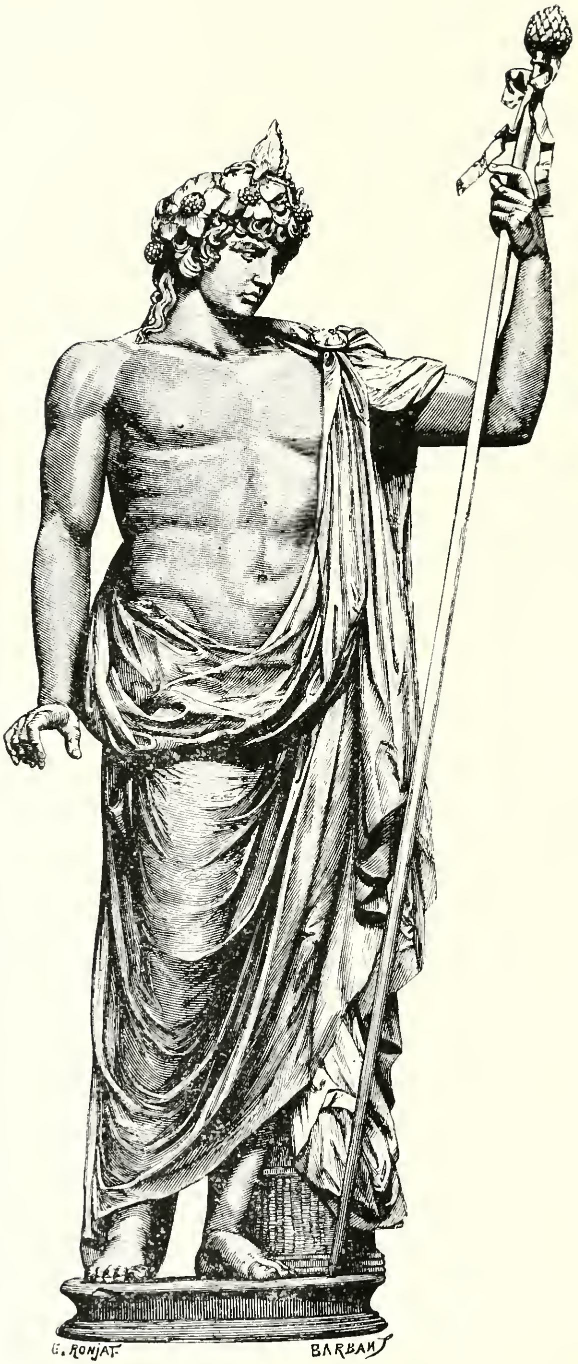

ANTINOUS, WHO DIED AS A VICARIOUS SACRIFICE FOR EMPEROR HADRIAN.

Frontispiece to The Open Court.

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ON THE FOUNDATION AND TECHNIC OF ARITHMETIC.

BY GEORGE BRUCE HALSTED.

INTRODUCTION.

IN the French Revolution, when called before the tribunal and asked what useful thing he could do to deserve life, Lagrange answered: "I will teach arithmetic."

Almost invariably now arithmetic is taught by those whose knowledge of mathematics is most meager. No wonder it and the children suffer. In this day of the arithmetization of mathematics and later its logicization, are the beauty, the elegance of arithmetical procedures to still remain unexplained? Is the singular, the lonely precision of this science and art to remain unheralded, unexpounded?

In arithmetic a child may taste the joy of the genius, the joy of creative activity.

Arithmetic is for man an integrant part of his world construction. Thus do his fellows make their world, and so must he. Now this is not by passive apprehension of something presenting itself, but by permeating vitalization spreading life and its substance through what the ignorant teacher would present as the dead mechanism of mechanical computation.

More than in any other science, there has been in mathematics an outburst of most unexpected, most deep-reaching progress. Its results, if made available for the teacher, will revivify this first, most precious of educational organisms; the more so since mathematics is seen to possess of all things the most essential, most fundamental objective reality.

THE PREHUMAN CONTRIBUTIONS TO ARITHMETIC.

Properly to understand or to teach arithmetic, one should have a glimpse of its origin, foundation, meaning, aim.

Arithmetic is the science of number, but for the ordinary school-teacher it is to be chiefly the doctrine of primary natural number, the decimal and later the fraction, and the art of reckoning with them.

Numbers are of human make, creations of man's mind; but they are first created upon and influenced by a basis which comes from the prehuman.

The Natural Individual.

Before our ancestors were men, they represented to themselves, as do some animals now, the world as consisting of or containing individuals, definite objects of thought, things. They exercised an individuating creative power. In now understanding by *thing* a definite object of thought, conceived as individual, we are using a method of world presentation which served animals before there were any men to serve.

The child's consciousness certainly begins with a sense-blur into which specification is only gradually introduced. At what stage of animal development the vague and fluctuating fusion, which was the world, begins to be broken up into persistently separate entities would be an interesting comparative biologico-psychologic investigation. However that might turn out, yet things, separate objective things, are a gift to man from the prehuman. Yet simple multiplicity of objects present to perception or even to consciousness does not give number. The duck does not count its young. The crow, wise old bird, has no real counting power to help its cunning. The animals' senses may be keener than ours, yet they never give number.

A babe sees nothing numeric. Even an older child may attend to diverse objects with no suggestion from them of number. Sense-perception may be said to have to do with natural individuals, but never, unaided by other mind-act, does it give number.

The Artificial Individual.

To the animal habit of postulating entities as separate must be added, before cardinal number comes, the human unification of certain of them into one whole, one totality, one assemblage or group or set, one discrete aggregate or artificial individual man-made.

This artificial whole, this discrete aggregate it is to which cardinal number pertains. Thus number rests upon a prehuman basis, yet is not number itself prehuman. Cardinal number involves more than the animal or natural individuals or things. It comes only with a human creation, the creation of artificial individuals, discrete aggregates taken each as an individual, an individual of human make, fleeting perhaps as our thought, transient, yet the necessary substratum for cardinal number. Unification is necessary. The mind must make of the distinct things a whole, a totality. Else no cardinal number.

Now to an educated man a number concept is suggested when a specific simple aggregate of objects is attended to. Not so to any animal, though just the same individual objects be recognized and attended to. The animal has the unity of the natural object or individual, but that unity is not enough. There is needed the new, the artificial, the man-made individuality of the total aggregate. To this artificial individual it is that cardinal number pertains. There is thus a unity, man-made, of the aggregate of natural individuals, of the set of constituent units. To this unity made of units cardinal number belongs.

Going for quite different articles, or to accomplish entirely different things, may we not help and check memory by fixing in our mind that we are to get *three* things, or that we are to do three things? How man-made, arbitrary, and artificial, this conjoining of acts most diverse into a fleeting unified whole!

Each finger of the left hand is different. A dog might be taught to recognize each as a separate and distinct individual. Only a man can make of all at once an individual which, conceived as a whole is yet multiple, multiplex, a manifold, fivefold, a five of fingers, a product of rational creation beyond the dog.

Primary Number.

A primary cardinal number is a character or attribute of an artificial unit made of natural units. It needs this single individuality and this multiplicity of individuals. The fingered hand has fiveness only if taken as an individual made of individuals.

Number is a quality of a construct. If three things are completely amalgamated, emulsified, like the components of bronze or the ingredients of a cake, there remains no threeness. If some things are in no way taken together the number concept is still inapplicable, we do not see them as a trio.

The animally originated primitive individuals, however complete in their distinctness, have no numeric suggestion. The creative synthesis of a manifold must precede the conscious perception of its numeric quality. It is only to man-made conceptual unities that the numeric quality pertains. This "number of natural individuals" in an artificial individual is called its *cardinal number* or *cardinal.*

Primary number would seem in some sense a normal creation of man's mind. No primitive language has ever been investigated without therein finding records of the number idea, unmistakable though perhaps slight, limited, meager, it may be not going beyond our baby stage, one, two, many.

There is a baby stage when no *many* is specialized but *two.* One, two, many, then baby waits how long before that many called *three* is specialized? Numeric *one* as cardinal only comes into existence in contrast with *many.* Number comes when we make a vague *many* specific.

The number of a particular totality represents the particular multiplicity of its individual elements and nothing more. So far as represented in a number, each natural individual loses everything but its distinctness; all are alike, indistinguishably equivalent. The idea of unity is doubly involved in number, which applies to a unity of a plurality of units. The units are arithmetically identical; not so the complex unities man-made out of collections of the units. To these pertain the differing cardinal numbers.

Our Base Ten.

In our developed number systems certain *manys* take on a peculiar prominence, are of basal character. Of these ten has now permanently the upper hand.

What is the origin of this preeminence?

Its origin is prehuman. Our system is decimal, not because ten is scientifically, arithmetically a good base, a superior number, but solely because our prehuman ancestors gave us five fingers on each of two hands.

THE GENESIS OF NUMBER.

Cardinals.

In nature, distinct things are made and perceived as individual. Each distinct thing is a whole by itself, a qualitative whole. The individual thing is the only whole or distinct object in nature. But the human mind takes individuals together and makes of them a single whole of a new kind, and names it. Thus we have made the concept a flock, a herd, a bevy, a covey, a genus, a species, a bunch, a gang, a host, a class, a family, a group, an array, a crowd, a party, an assemblage, an aggregate, a throw, a set, etc. These are artificial units, discrete magnitudes; the unity is wholly in the concept, not in nature; it is artificial. We constitute of certain things an artificial individual when we distinguish them collectively from the rest of the world, making out of subsidiary individuals a single thing. From the contemplation of the natural individual or element in relation to the artificial individual, the group, spring the related ideas "many" and "one." We must have numeric many before we can have cardinal one. A natural qualitative unit thought of in contrast to "many" as *not-many* gives the idea "one" as cardinal. A unity, a "many" composed of "one" and another "one" is characterized as *two*.

The unity, the "many" composed of "one" and the special many "two" is characterized as *three*.

Among the primitive ideas of cardinal number, the idea of "two" is the first to be formed definitely. There are ever present things which can be grasped in pairs. This two is the very simplest many. It is incalculably simpler than three, as witness whole savage tribes whose spoken number system is "one, two, many"; as witness the mind-wasting primitive stupidities of the dual number in Greek grammar.

The special many, a one made of three, a trinity, a trio, triplets, here is an advance. When to the grasp of the pair, the dominance over the trio is added, when the three is created, then after-progress is rapid.

With a couple of pairs goes four; with a couple of threes, six. A hand represents five coming in between four and six. A pair of hands says ten. A pair of tens is twenty, a score. A pair of fours is eight. A trio of threes is nine. A pair of sixes or a trio of fours is twelve, a dozen.

Arithmetic flowers like a rocket. That seven is left out, is missed, makes it the sacred, the mystic number of superstition. To numbers, however complicated their genesis, is finally ascribed a certain objective reality. In our mind the number concepts finally become simple things, objectively real.

COUNTING AND NUMERALS.

Correlation.

The ability of mind to relate things to things, to correlate, to represent something by something else, to make or perceive a correspondence between things or thought creations is fundamental, essential, necessary.

The operation of establishing such a correspondence between two sets that every thing or element of each set is mated with, paired with, just one particular thing or element of the other, is called establishing a one-to-one correspondence between the sets. Two sets which can be so mated are said to be *equivalent,* or to have the same *potency.* Two sets equivalent to the same are equivalent to each other, their elements correlated to the same element being thereby mated.

A set's cardinal number is what is common to the set and every equivalent set. Thus a set's cardinal is independent of every characteristic or quality of any element beyond its distinctness. To find the cardinal of a set, we count the set.

Counting is the establishing of a one-to-one correspondence of two aggregates, one of which belongs to a well-known series of aggregates. If a group of things have this correspondence with this standard group, then those properties of this standard group which are carried over by the correspondence will belong to the new group. They are the properties of the group's cardinal number.

To Count.

To count an aggregate, an artificial individual, is to identify it as to numeric quality with a familiar assemblage by setting up a one-to-one correspondence between the elements of the two groups. Thus counting consists in assigning to each natural individual of an aggregate one distinct individual in a familiar set, originally a group of fingers, now usually a set of words or marks. So counting is essentially the numeric identification, by setting up a one-to-one correspondence, of an unfamiliar with a familar group. Thus it ascertains, it fixes the nature of the less familiar through the preceding knowledge of the more familiar.

The Primitive Standard Sets.

Primitively the known groups were the groups of fingers. The fingers gave the first set of standard groups and formed the original apparatus for counting, and served for the symbolic transmission of the concepts, the number ideas generated. More than that, this finger counting gave the names of the numbers, the numeric words so helpful in the further development of numeric creation. The name of a number, when referring to an artificial unit, as of sheep, denoted that a certain group of fingers would touch successively the natural units in the discrete magnitude indicated, or a certain finger would stand as a symbol for the numerical characteristic of that group of natural units.

Our word "five" is cognate with the Latin *quinque*, Greek *pente*, Sanskrit *pankan*, Persian *pendji*; now in Persian *penjeh* or *pentcha* means an outspread hand.

In Eskimo "hand me" is *tamuche*; "shake hands" is *tallalue*; "bracelet" is *talegowruk*; "five" is *talema*.

In the language of the Tamanocs of the Orinoco, five means "whole hand"; six is "one of the other hand"; and so on up to ten or "both hands."

Philology confirms that the original counting series or outfit was the series of sets of fingers, and this primitive method preceded the formation of numeral words. In very many languages the counting words come directly and recognizably from the finger procedure.

But of the fingers there are only a few distinct aggregates, only ten. Developing man needs more, needs to enlarge and extend his standards.

The Abacus.

The Chinese, even at the present day, extend the series of primary groups, the finger-groups, by substituting groups of counters movably strung on rods fixed in an oblong frame. With this *abacus* they count and perform their arithmetical calculations.

The Word-Numeral System.

In many languages there are not even words for the first ten groups. Higher races have not only named these groups, but have extended indefinitely this system of names. They no longer count directly with their fingers, but use a series of names, so that the operation of counting an assemblage of things consists in assigning to each of them one of these numeral words, the words being always taken in order, and none skipped, each word being thus capable of representing not merely the individual with which it is associated, but the entire named group of which this individual is the last named.

In making this series of word numerals, there is evidently need for a system of periodic repetition. The prehuman fixes five, ten, or twenty as the number after which repetition begins. Of these, ten has become predominant. Thus come our word-numerals, each applicable to just one of a counted set and to the aggregate ending with this one. This dekadic word-system makes easy, with a simple, a light notational equipment, the perfectly definite expression of any number, however advanced.

So for us to count is to assign the numerals one, two, three, etc., successively and in order, to all the individual objects of a collection, one to each. The collection is said to be given in number, the number of things in it, by the cardinal number signified by the numeral assigned to the last natural unit or component of the collection in the operation of counting it. Numerals are also called numbers. The numeral and a word specifying the kind of objects counted make what is called a concrete number. In distinction from this, a number is called an abstract number.

When children are to count, the things should be sufficiently distinct to be clearly and easily recognizable as individual, yet not so disparate as to hinder the human power to make from them an artificial individual. The objects should not be such as to individually distract the attention from the assemblage of them.

A Partitioned Unit.

In counting, an artificial individual may take the place of a natural individual. Children enjoy counting by fives. Inversely, a unit may be thought of as an artificial individual, composed of subsidiary individuals, as a dollar of 100 cents.

Recognition of Number Without Counting.

An interesting exercise is the instantaneous recognition of the cardinal, the particular numeric quality of the collection, its specification without counting. But this power to picture all the separate individuals and to recognize the specific given picture is very limited. If it be attempted to facilitate this recognition by arrangement, the recognition may easily become that of form instead of number. It is then simply recognizing a shape which we know should have just so many elements.

Decimal Word-Numerals.

In the making of numeral words it is necessary to fix upon one after which repetition is to begin. Otherwise there would be no end to the number of different words required. We have noted that the prehuman has narrowed the choice, by the fiveness of the extremities of mammalian limbs, to five, ten, or twenty. The majority of races, especially the higher, in prehistoric time chose ten, the number of our fingers. Then was developed a system to express by a few number-names a vast series of numbers. If we interpret eleven as "one and ten" and twelve as "two and ten," *teen* as "and ten," *ty* as "tens," then English, until it took "million," ("great thousand," Latin *mille*, a thousand), bodily from the Italian, used only a dozen words in naming numbers, in making a series of word-numerals with fixed order.

The systematic formation of numerical words is called *numeration.*

Invariance of the Cardinal Number.

The cardinal number of any finite set of things is the same in whatever order we count them.

This is so fundamental a theorem of arithmetic, it may be well to make its realization more intuitive.

That the number of any finite group of distinct things is independent of the order in which they are taken, that beginning with the little finger of the left hand and going from left to right, a group of distinct things comes ultimately to the same finger in whatever order they are counted, follows simply from the hypothesis that they are distinct things. If a group of distinct things comes to, say, five when counted in a certain order, it will come to five when counted in any other order.

For a general proof of this, take as objects the letters in the word "triangle," and assign to each a finger, beginning with the little finger of the left hand and ending with the middle finger of the right hand. Each of these fingers has its own letter, and the group of fingers thus exactly adequate is always necessary and sufficient for counting this group of letters in this order.

That the same fingers are exactly adequate to touch this same group of letters in any other order, say the alphabetical, follows because, being distinct, any pair attached to two of my fingers in a certain order can also be attached to the same two fingers in the other order.

In the new order I want *a* to be first. Now the letters *t* and *a* are by hypothesis distinct. I can therefore interchange the fingers to which they are assigned, so that each finger goes to the object previously touched by the other, without using any new fingers or setting free any previously employed. The same is true of *r* and *e*, of *i* and *g*, etc.

As I go to each one, I can substitute by this process the new one which is wanted in its stead in such a way that the required new order shall hold good behind me, and since the group is finite, I can go on in this way until I come to the end, without changing the group of fingers used in counting, that is without altering the cardinal number, in this case 8.

The group of fingers exactly adequate to touch a group of objects in any one definite order is thus exactly adequate for every order. But when touching in one definite order each finger has its own particular object and each object its own particular finger, so that the group of fingers exactly adequate for one peculiar order is always necessary and sufficient for that one order. But we have shown it then exactly adequate for every order; therefore it is necessary and sufficient for every order.

GENESIS OF OUR NUMBER NOTATION.

Positional System of Counting.

The systematic decimal system in accordance with which, even in the times of our prehistoric ancestors, a few number names were used to build all numeral words, is paralleled by the procedure, even at the present day, of those Africans who in counting use a row of men as follows: The first begins with the little finger of the left hand, and indicates, by raising it and pointing or touching, the assignment of this finger as representative of a certain individual from the group to be counted; his next finger he assigns to another individual; and so on until all his fingers are raised. And now the second man raises the little finger of his left hand as representative of this whole ten, and the first man, thus relieved, closes his fingers and begins over again. When this has been repeated ten times, the second man has all his fingers up, and is then relieved by one finger of the third man, which finger therefore represents a hundred; and so on to a finger of the fourth man, which represents a thousand, and to a finger of the fifth man, which represents a myriad (ten thousand).

The Abacus.

An advance on this actual use of fingers with a positional value depending only on the man's place in the row, is seen in the widely occurring *abacus,* a rough instance of which is just a row of grooves in which pebbles can slide. With most races, as with the Egyptians, Greeks, Japanese, the grooves or columns are vertical, like a row of men. The counters in the right-most column correspond to the fingers of the man who actually touches or checks off the individuals counted; it is the units column.

But in the abacus a simplification occurs. One finger of the second man is raised to picture the whole ten fingers of the first man, so that he may lower them and begin again to use them in representing individuals. Thus there are two designations for ten, either all the fingers of the first man or one finger of the second man. The abacus omits the first of these equivalents, and so each column contains only nine counters.

Recorded Symbols.

For purposes of counting, a group of objects can be represented by a graphic picture so simple that it can be produced whenever wanted by just making a mark for each distinct object. Thus the marks I, II, III, IIII, picture the simplest groups with a permanence beyond gesture or word; and for many important purposes, one of these stroke-diagrams, though composed of individuals all alike, is an absolutely perfect picture, as accurate as the latest photograph, of any group of real things no matter how unlike.

The ancient Egyptians denoted all numbers under ten by the corresponding number of strokes; but with ten a new symbol was introduced. The Romans regularly used strokes for numbers under five, using V for five. The ancient Greeks and Romans both however indicated numbers by simple strokes as high as ten. The Aztecs carried this system as high as twenty, but they used a small circle in place of the straight stroke. I have seen the same thing done in Japan.

Each stroke of such a picture-group may be called a unit. Each group of such units will correspond always to the same group of fingers, to the same numeral word.

The Hindu Numerals.

Though to this primitive graphic system of number-pictures there is no limit, yet it soon becomes cumbrous. Abbreviations naturally arise. Those the world now uses, the Hindu numerals, have been traced back to inscriptions in India probably dating from the early part of the third century B. C.

The Zero.

But a whole millennium was yet to pass before the creation of the most useful symbol in the world, the naught, the zero, a sign for nothing, a mark for the absence of quantity, the cypher, whose first known use in a document is in 738 A. D. This little ellipse, picture for airy nothing, is an indispensable corner-stone of modern civilization. It is an Ariel lending magic powers of computation, promoting our kindergarten babies at once to an equality with Cæsar, Plato or Paul in matters arithmetical.

The user of an abacus might instead rule columns on paper and write in them the number of pebbles or counters. But zero, 0, shows an empty column and so at once relieves us of the need of ruling the columns, or using the abacus. Modern arithmetic comes from ancient counting on the columns of the abacus, immeasurably improved by the creation of a symbol for an empty column.

The importance of the creation of the zero mark can never be exaggerated. This giving to airy nothing not merely a local habitation and a name, but a picture, a symbol, is characteristic of the Hindu race whence it sprang. It is like coining the Nirvana into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power. From the second half of the eighth century Hindu writings were current at Bagdad. After that the Arabs knew positional numeration. They called the zero *çifr*. The Arab word, a substantive use of the adjective *çifr* ("empty"), was simply a translation of the Sanskrit name *śūnya*, literally "empty." It gave birth to the low-Latin *zefirum* (used by Leonard of Pisa), whence the Italian form *zefiro*, contracted to *zefro* then *zero*, whose introduction goes back to the 15th century.

In the oldest known French treatise on algorithm (author unknown, of the thirteenth century) we read, "iusca le darraine ki est appellee *cifre* 0." In the thirteenth century in Latin the word *cifra* for "naught" is met in Jordan Nemorarius and in Sacrobosco who wrote at Paris about 1240. Maximus Planudes (14th century) uses *tziphra*. Euler used (1783) in Latin the word *cyphra*. We still say "cipher" or "cypher." In German *Ziffer* has taken a more general meaning, as has the equivalent French word *chiffre*, the most important numeral coming to mean any. The oldest coin positionally dated is of 1458.

Zero may be looked upon as indicating that a class is void, containing no object whatever. But though it is thus one of the answers to the question, "How many?", it is not given a place in the series of natural numbers, though chief in the series of algebraic numbers. Only after the seventeenth century does naught appear as common symbol for all differences in which minuend and subtrahend are equal.

So to-day we use nine digits and have no digit corresponding to the Roman X, for X is all the fingers of the first man, while we, like the abacus, use 10, which is one finger of the second man. Thus the ten, hundred, thousand are only expressed by the position of the number which multiplies them.

In the written numeral IIII, we still see in the symbol the units of which the fourfold unit four is composed. Later abbreviation veils the constituent units, but their independence and all-alike-ness remain fundamental, giving to cardinal number its independence of the order in which the things are enumerated.

Our Present Notation.

The use of the digits (Latin, *digitus,* "a finger"), the substitution of a single symbol for each of the first nine picture-groups, and that splendid creation of the Hindus, the zero, 0, nought, cypher, made possible our present notation for number. This still has a bad base, ten, in which the sins of our fathers, the mammals, are visited on their children. Its perfection is in its use of position, a positional notation for number, which the decimal point (or unital point) empowers to run down below the units, giving the indispensable *decimals.*

Calculus, (Latin, "a pebble"), cyphering, which thus by the aid of zero attains an ease and facility which would have astounded the antique world, consists in combining given numbers according to fixed laws to find certain resulting numbers.

Teaching is to enable the ordinary child to do what the genius has done untaught.

A Hindu genius created the zero. The common, even the stupid, child is now to be taught to understand and use this wonderful creation just as it is taught to use the telephone. So the teacher incites, provokes the self-activity of the child's mind and guides it and confirms it, stopping this kaleidoscope at a certain turn, when the evershifting picture is near enough for life to the picture in the teacher's mind.

Without theory, no practice, yet need not the theory be conscious. There is a logic of it, yet the child need not necessarily know, had perhaps better not know, that logic. The teacher should know, the child practise.

Though language so long precedes writing, nevertheless it is striking to realize the centuries that passed after the present system of number-naming, numeration, had been developed, before it had analogous, adequate symbolization, adequate written notation.

As compared with their number-names, how bungling the Greek and Roman numerals, how arithmetically helpless the men of classic antiquity for lack of just one written symbol, the Hindu naught, giving us a written system which, except for its bad base ten, seems to be final and for all time. That prehuman parasite, the ten, is fixed on us like an Old Man of the Sea, else we could take the easily superior system with base twelve.

In each case the prebasal figures, by help of the zero, always express as written in succession to left or to right of the units place (fixed by the unital point) ascending and descending powers of the base. But while the two and six of twelve are like the two and five of ten, yet twelve has three and four besides as divisors, as submultiples, for which tremendous advantage ten offers no equivalent whatsoever. The prehuman imposition of ten as base, disbarring twelve, is thus a permanent clog on human arithmetic.

The mere numerals, 1, 2, 3,—or the numeral words "one," "two," "three,"—are signs for what are called "natural numbers," or positive integers. Integer with us shall always mean positive integer. If pure numbers, integers, have an intrinsic order, so do these, their symbols.

The unending series, 1, 2, 3, 4, 5,.... or one, two, three, four, five,.... is called the "natural scale," or the scale of the natural numbers, or the number series. Each symbol in it, besides its ordinal, positional significance in the sequence of symbols, is used also to indicate the cardinal number of the symbols in the piece of the scale it ends, and so of any group correlated to that piece.

[TO BE CONTINUED.]