VOL. XXIV. (No. 2.) FEBRUARY, 1910. NO. 645.

CONTENTS:

THE GHOST DANCE: ECSTASY. (After Mooney.)

Frontispiece to The Open Court.

THE OPEN COURT

A MONTHLY MAGAZINE

Devoted to the Science of Religion, the Religion of Science, and the Extension of the Religious Parliament Idea.

VOL. XXIV. (No. 2.) FEBRUARY, 1910. NO. 645.

THE REPETEND.

BY WALTER R. EGBERT.

IN the process of division, the quotient sometimes reminds us of Tennyson's "Brook,"—it goes on forever,—the perpetual motion of numbers. Our first impulse is to blame the divisor, and often the fault lies there; but it is possible, with a different dividend, for that same divisor to furnish a perfect, finished quotient.

Custom seems to sanction the continuance of the operation of division as far as the third decimal place. In long continuous and successive operations of multiplication and division, this lopping of the quotient may seriously impair the accuracy of the final result. Whenever possible, it is best to defer the division until the final operation.

In the case of dollars and cents, the third decimal place is sufficiently accurate for all practical purposes, yet in such operations as are involved in partial payments and compound interest, the final result is considerably affected by retaining or rejecting all decimal figures beyond the third place.

In denominate numbers, the third decimal place—or the fourth or even the fifth,—should be carefully considered with reference to the *name* it bears. A difference of half a thousandth of a mile is a difference of nearly one yard; while a difference of half a thousandth of a square mile would be about equivalent to a square lot 116 feet on a side,—a piece of ground sufficient for raising 175 bushels of potatoes. Who could tell the value of such a lot if situated along Broadway? Portia's "twentieth part of one poor scruple" seems very attenuated, like the ghost of departed quantity, yet in the Philadelphia Mint is a weighing apparatus so delicately and accurately adjusted that the weight of the lead recording your name on a visiting card can be told. In chemical and physical experimentation, the attenuated thousandth part may defeat a valuable purpose, and produce results far-reaching and sometimes fatal to science.

The question of the stopping-place in very accurate and important calculations becomes a question of no little moment. In a scale of sizes, let a coconut represent the first, or tenths decimal unit. Then would an English walnut represent the hundredths unit, a cherry seed the thousandths decimal unit, and a cabbage seed the next down the line. Yet if the first decimal unit were a globe the size of the sun, the earth upon which we live would probably take its size further down the line than the fourth. Nothing is great or small save by comparison; so the practice of stopping the division at the third decimal point ought to be carefully considered as to the *character* of the quotient.

In a division which does not terminate, the result can be brought as near as desired to accuracy, yet it can not be brought to the perfect finish. Something must remain incomplete, hence unsatisfactory.

If, as a matter of experiment, the division be continued into the realm of attenuated decimal values and a careful inspection of the quotient figures be made, almost invariably there will be found a succession of groups of the same figures in the same order. The discovery of repetition in the work is soon discovered when the divisor is a small number. With large numbers, the experimenter will rarely have the patience to work on until the quotient figures and the work are duplicates of the earlier processes. With the divisor 19, the duplication of work is not revealed until the eighteenth quotient decimal figure has been passed; while with the divisor seven, the repetition may be noticed after the sixth quotient figure.

Such series of repeating quotient figures are called *repetends* or *circulates*. Repetends are the result of division when the divisors are prime numbers. Some composite numbers also produce repetends; but these repetends will be identical with the circulates of the prime factor found in the divisor, if there be but *one* prime factor there. With a divisor 14 and a dividend 1, the circulate 714285 is found. This is the circulate of five-sevenths.

It is possible that with another basis of notation, fewer unending divisions could be found. Our decimal system of notation seems to be responsible for the circulates. A duodecimal basis of notation would yield more satisfactory results in the operations of division; but none of us would yield our decimal system of notation for a complicated system which would require constant reductions from one denomination to another. When we compare our system of writing money—the decimal notation—with the English system—not decimal—, and mentally perform those constant reductions in the English currency, we appreciate the difference, greatly to the advantage of the decimal system. Were we only familiar with the metric system of denominate numbers, we should deplore our duped condition of servitude to reductions in denominate numbers. But spare us the transition stage! There is a mental repugnance against the supposed possibilities of the concomitant "reductions" necessary. The anticipated condition seems parallel to the physical discomfort of breaking in new shoes,—only more prolonged.

The repetend is neither formidable nor mysterious. Does not our antipathy frequently originate in our real lack of knowledge of things? The very dots which mark the innocent repetend seem like sentinels of mathematical woe;—yet the device is excellent. There is no other possible way of indicating this continuous array of numbers save by some device or system of marking. The ± does too much indefinite service at the "tired" part of the quotient, and the "etc." imparts a careless mysteriousness very misleading in the field of repetends. A dot over the first and another dot over the last of the figures in the series which repeats, is an excellent device. A single dot answers if but one figure is repeated.

A few hours' close study should yield a good return in knowledge of the repetend. There may lie in the circulate possibilities yet unheard of in the domain of numbers.

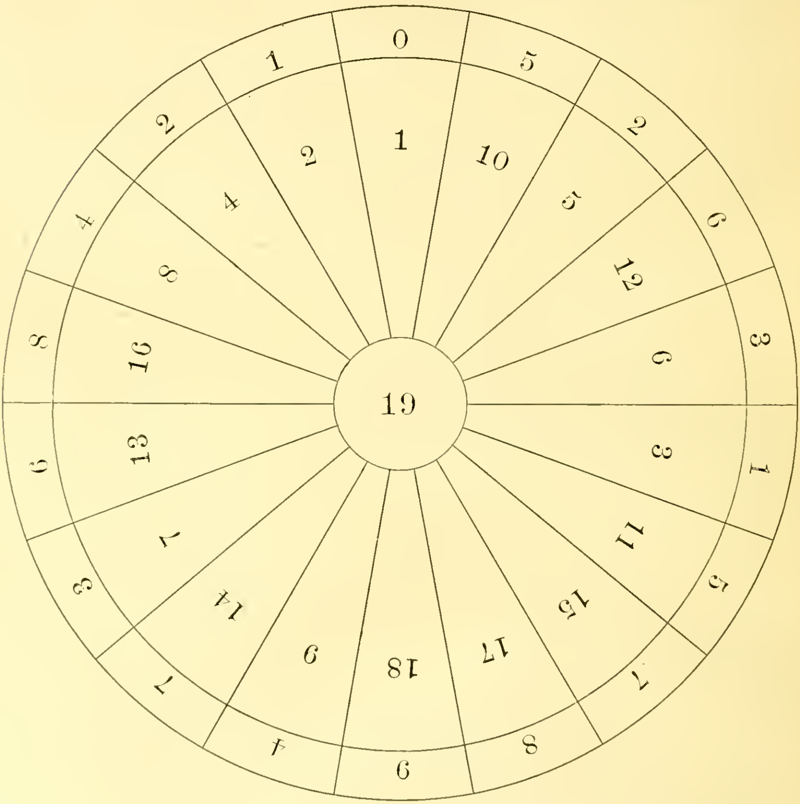
If 1 be divided by 7, it will soon be discovered that there can be but six possible remainders,—1, 2, 3, 4, 5, and 6,—consequently if all these six remainders appear before the original dividend again occurs (regarding the first dividend as a remainder), there will be a series of numbers repeating in the quotient. This is exactly what happens with the divisor 7.

Again with the divisor 17, since there are sixteen numbers below it, there are sixteen figures in the repetend. It frequently happens—though not always—that the number of figures in the repetend is one fewer than the number of units in the divisor. In all such cases, there is a single cycle of the repetend.

With 13 as a divisor, using all the possible numbers below it as dividends, two different repetends will be found, each cycle consisting of six figures. The two cycles conjointly contain one fewer figure than the divisor has units. Likewise 71 has two cycles of 35 figures each.

If 1 be divided by 19 until the remainder 1 appears again as a dividend, a repetend of eighteen figures is found. This furnishes an excellent example of the *single cycle.* A very convenient arrangement of the quotient figures and the remainders may be made to conform to the circular arrangement by placing the divisor at the center of the circle, the dividends (or remainders) next, and the quotient figures, which are the repetend, farthest out. A little study of the attached diagram, or cycle for 19, will soon reveal the advantage of this arrangement.

Nineteen is not contained in one, so there can be no "whole number" in the quotient. With the annexed cypher, 19 is not con-

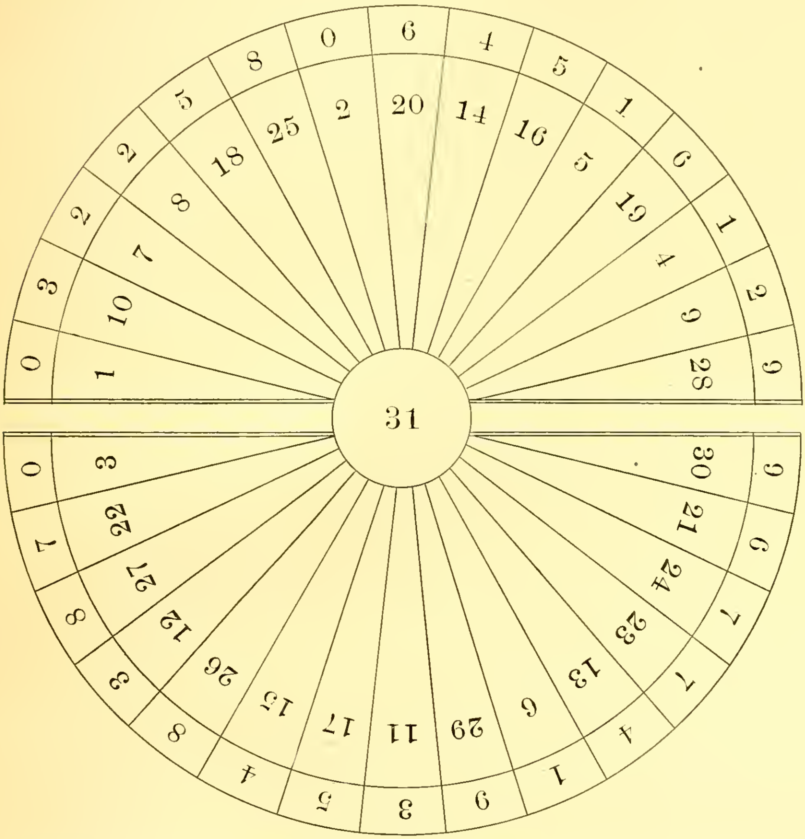


tained, so the decimal portion begins with 0; and ten, placed in the next sector, is treated as a remainder. With an annexed cypher, (100) as a dividend, 19 is contained 5 times with a remainder of 5. The quotient is placed above the dividend and the remainder in the next sector to the right. The work advances clockwise.

The first decimal figure of any repetend of 19 lies just outside the number which is the dividend, and in the same sector. Thus the repetend for $^{14}/_{19}$ is $.\dot{7}3684210526315789\dot{4}$.

Some peculiarities of the cyclic arrangement may here be noticed. The sum of any two opposite repetend figures will always be 9. The sum of any two opposite remainders will always be equal to the divisor.

When in the process of division a remainder one unit below the divisor is found, the middle point of the division has been struck,



and the completion of the work may be done by taking complements of 9—starting with the first quotient figure—for the other half of the repetend, and complements of the divisor for the opposite remainders. The cyclic arrangement relieves the divider of half the work.

When two different repetends produced by the same divisor are taken conjointly and conform to all the essentials of the single cycle, they may be termed complementary cycles. Thus 31 as a divisor produces complementary cycles which may be arranged as in the diagram on the preceding page.

This arrangement of the complementary cycles gives all the characteristics of the single cycle. A ruler edge will assist to find the opposite complementary figures. The reading of the complementary repetend must be confined to that semicircle in which the dividend is found. Thus the reading for $^{20}/_{31}$ is $.\dot{6}4516129032258\dot{0}$. Do you appreciate that with the exception of a few figures as a whole number, here are all the possible problems in division with a divisor of 31? Here are thirty problems in division, from which a teacher can test thirty problems of busy work,—both as to quotient figures and remainders.

The divisor 79 gives three sets of complementary circulates, of thirteen figures each. In the cyclic arrangement, each repetend will occupy a sector of sixty degrees.

It sometimes happens that a divisor will produce repetends that of themselves conform to all the characteristics of the single cycle. Such series may be called independent cycles. With the divisor 13, two independent circulates are found, each of which conforms to all the conditions of the single cycle. They may be arranged in two different circles. These two circles will contain all the possible conclusions of all divisions by 13, which do not "come out even." The divisor 73 gives nine complete independent cycles of eight figures each.

The prime numbers in their order as divisors, with 1 as the dividend, give the following results:

3, produces two complementary cycles of one figure each.
7, produces one cycle of six figures.
11, produces five sets of complementary cycles of two figures each.
13, produces two independent cycles.
17, produces one cycle of sixteen figures.
19, produces one cycle of eighteen figures.
23, produces one cycle of twenty-two figures.
29, produces one cycle of twenty-eight figures.
31, produces two complementary cycles of fifteen figures each.
37, produces six sets of complementary cycles of three figures each.
41, produces four sets of complementary cycles of five figures.
43, produces two complementary cycles of twenty-one figures.
47, produces one cycle of forty-six figures.
53, produces two sets of complementary cycles of thirteen figures.
59, produces one cycle of fifty-eight figures.
61, produces one cycle of sixty figures.

67, produces two complementary cycles of thirty-three figures.
71, produces two complementary cycles of thirty-five figures.
73, produces nine independent cycles of eight figures each.
83, produces two complementary cycles of forty-one figures each.
89, produces two complementary cycles of forty-four figures.
107, produces two complementary cycles of fifty-three figures.
113, produces one cycle of one hundred and twelve figures.

If in arranging the remainders within the circle, the circumference be divided into equal arcs, and lines be drawn from the middle points of these arcs starting with 1 and drawing right on in the consecutive order, a beautiful design of parallel chords will appear. There will be two exceptions to the parallel arrangement, and these two chords will intersect at the center of the circle, and of course be equal. The parallel chords will also be equal. The long-sought-for method of trisecting an arc may lie in this field.

In working with repetends, two operations are found. The division which produces the repetend may be called reduction descending, and the operation which brings the repetend back to its original dividend and divisor may be called reduction ascending. The process of bringing back the countless smaller and smaller values to an expression of the highest numerical value, is not easy; and because not always understood, it elicits the student coinage "horrid" and "dreadful." The rule of giving the repetend a denominator of nines, should be understood before it is allowed to be used. A simple device in subtraction clears the repetend of its comet-like tail; but, at the same time, compensates for that by giving the denominator of 9's. The process will be explained under the caption, "Teaching the Repetend."

TEACHING THE REPETEND.

The class in recitation may be divided into six sections, and work in division assigned as follows:

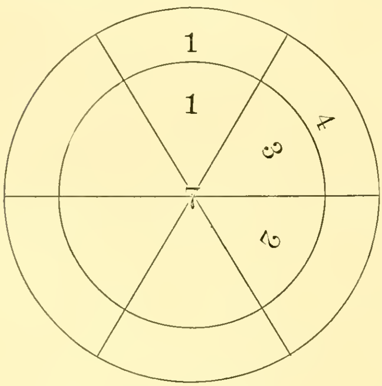
First section, reduce one-seventh to a decimal.
Second section, reduce two-sevenths to a decimal.
Third section, reduce three-sevenths to a decimal.
Fourth section, reduce four-sevenths to a decimal.
Fifth section, reduce five-sevenths to a decimal.
Sixth section, reduce six-sevenths to a decimal.

Soon some alert pupil will report, "It goes back to the same thing all the time." The sections report in order while the teacher writes the results on the wall-slate.

First section, $\frac{1}{7} = .14285714285714$
Second section, $\frac{2}{7} = .285714285714$
Third section, $\frac{3}{7} = .4285714$
Fourth section, $\frac{4}{7} = .57142857142857$
Fifth section, $\frac{5}{7} = .714285714285$
Sixth section, $\frac{6}{7} = .8571428$

These results should be carefully inspected and compared. Develop the point that in each quotient six figures repeat. Erase from each quotient all beyond the sixth figure and place the dots over the first and the last of each quotient. Explain their use. Note that these six figures are always in each repetend, and always in the same order.

These results may be concisely tabulated by means of a circle. Each pupil draws a circle about two inches in diameter. Divide the circumference into six parts, using the radius as a measure. Join each point of division with its opposite point. Let an inner circle be drawn. Let 7 be placed at the center and 1 in the sector just above the 7. Mental division is now in order. Since 7 is not contained in one, we proceed into decimals. 7 in 10, 1 and three remaining. Place the quotient 1 in the outer space above the one, and the 3 in the adjoining sector. 7 in 30, 4 and 2 remaining. Place the quotient figure in the outer space above its dividend 3, and the remainder in the next sector.



Continue the division and record until the cycle is finished.

The definition of repetend may now be called for. Each pupil finds his particular problem in the diagram. Pupils discover a kind of division table. Each pupil reads from the "wheel,"—the reading being always clockwise. Thus $\frac{6}{7} = .\dot{8}5714\dot{2}$. Pupils will appreciate the economy. Here are six problems in one.

In the assignment of work for the next lesson, ask each pupil to prepare a "wheel" of repetends, using prime numbers not higher than 23 as the divisors. Not all the numbers within this range will produce a single cycle. This discovery may annoy some pupils, but the teacher assists to make the "wheel" in this case.

The reduction to the dividend and divisor which produced a given repetend is as follows:

$$
\begin{array}{rl}
? = & .\dot{4}2857\dot{1} \\
1000,000 \times ? = & 428571.\dot{4}2857\dot{1} \\
1 \times ? = & .\dot{4}2857\dot{1} \\
\hline
999,999 \times ? = & 428571.
\end{array}
$$

$$? = \frac{428571}{999999} = \frac{47619}{111111} = \frac{15873}{37037} = \frac{429}{1001} = \frac{39}{91} = \frac{3}{7}.$$

$$\tfrac{3}{7} = .\dot{4}2857\dot{1}.$$

In this process of reduction, multiply the given decimal repetend by such multiple of ten as will cast the entire repetend to the left of the decimal point. Then subtract once the repetend, and the attenuated decimal parts will disappear, leaving an entire quantity which is one unit below the multiple times the number. Hence the denominator of nines.

It seems improper to take several numbers at random, mark them as a repetend, and ask for the corresponding fraction. It seems like inverting the process of creation. Let the teacher who thinks otherwise find the fraction which would give $.\dot{9}9\dot{9}$.

Not a little educational value attaches to work with the repetend. With what persistence men seek the North Pole! — an impulse which urges them to know the unknown, the ultimate. To what extent is education responsible for this continuity of persistence? He who has been taught to terminate all division at the third decimal point could never be a man of great research. He would want to stop at the third cache on the northwest coast of Baffin's Bay. The dogged quality of sticking to a thing is valuable, and everything in the course of education that develops this quality should be utilized. A degree of satisfaction always attaches to a *finished* thing, though it be a jack-o-lantern.

The discovery of a repetend satisfies; for, though not perfectly finished, it is possible to see down the line of infinitesimal values into the realm of non-assignable values. We appreciate the end. The otherwise inconclusive becomes satisfactory; and we place the repetend dots over the discovered repetition of figures and, in imagination, see them dance off down into the infinite space of incalculable small values. A result to the third decimal place +, draws the curtain before the last act is over.

The cycles which we discover in mathematics seem to be the numerical counterparts of the cycles in nature. The life story of the moth furnishes a physical cycle in the insect world; while a grain of corn discloses a valuable repetend whose scale of values is permanent.