2019

Applications of Graph Theory for Controlling City Infrastructure

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APPLICATIONS OF GRAPH THEORY FOR CONTROLLING CITY INFRASTRUCTURE

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B.S., Taibah University, 2010

A Research Paper
Submitted in Partial Fulfillment of the Requirements for the
Master of Science

Department of Mathematics
in the Graduate School
Southern Illinois University Carbondale
December, 2019

RESEARCH PAPER APPROVAL

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November 12, 2019

AN ABSTRACT OF THE RESEARCH PAPER OF

Ibahim Alnassar, for the Master of Science degree in Mathematics, presented on November 13, 2019 , at Southern Illinois University Carbondale.

TITLE: APPLICATIONS OF GRAPH THEORY FOR CONTROLLING CITY INFRASTRUCTURE

MAJOR PROFESSOR: Dr. John McSorley

Graph Theory is a relatively new field of mathematics, and it has grown into a very critical component in many applications in real-life problems such as transportation networks, computers, sciences, social networks, etc. A transportation network provides a means for goods and citizens to travel from a starting location to a destination. Each of these networks plays an essential role in civilization. Bus networks and bus stations are essential components of our national infrastructure, and intricate systems connect them. Any situation that has linked items can be symbolized by graph theory. Algorithms and theorems of graph theory have been used as a tool to analyze and improve potential vulnerabilities in infrastructure networks. This paper presents several applications of graph theory in different topics that can be used for optimizing and analyzing locations for bus systems.

Keywords: Graph Theory, Spanning Subgraphs, Eulerian Graph, Spanning Tree, Minimum Spanning Tree, Graph Center, Kruskal's Algorithm, Dominating Set, Domination Number.

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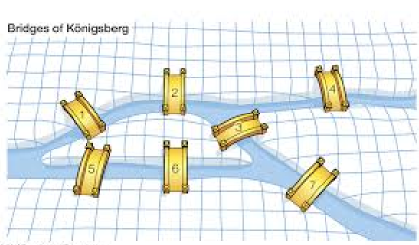
CHAPTER 1

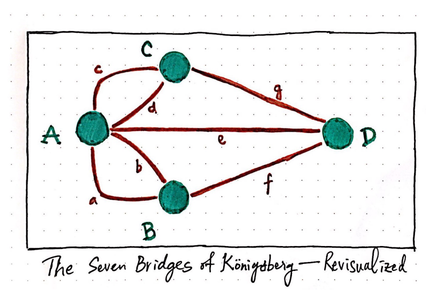
INTRODUCTION

A Graph is a simple concept with extremely powerful uses. It's a new branch of modern mathematics which is a graphical picture of a set of objects which are connected by paths, mathematically the objects and paths are called vertices and edges.

A Graph is an ordered pair $G = (V, E)$ where $V(G)$ represents the set of vertices and $E(G)$ represents the set of all edges in the graph. Graphs can be used to model many processes in computer science, biological, physical, and information systems. Graph Theory is an important branch of Mathematics. The history of graph theory began with the Swiss mathematician Leonhard Euler when he tried to solve the seven bridges of Königsberg problem in 1735. It was an old problem. The question (see Figure 1.1a) Is it possible to choose a starting point in the city and find a walking route which would take you over each bridge exactly once? Euler simplified this map with each of the four landmasses represented as single points: he called them vertices, and with lines which he called edges that link



(a) A real picture



(b) A Simplified graph

Figure 1.1: Seven bridges of Königsberg problem

them to represent bridges. And this simplified graph (Figure 1.1b) allows us to easily count the degrees of each vertex. Euler proved in his first paper in graph theory that no such paths exists in this problem. Then he established a way to determine wherein a graph is an Eulerian graph.

CHAPTER 2

EULERIAN GRAPH AND SEVEN BRIDGES OF KONIGSBERG PROBLEM

An Euler path is a simple path in a graph, by which we can visit every edge just once. We can use the same vertices several times. The Euler Circuit is a special style of Euler path. When the beginning vertex of the Euler path is also connected with the ending vertex of that path, then it is named **an Euler Circuit**. A graph that contains either an Euler Path or an Euler Circuit is named an **Eulerian graph**. The **degree** of a vertex is the number of edges that are connected to it. The degree of a vertex v is denoted $deg(v)$.

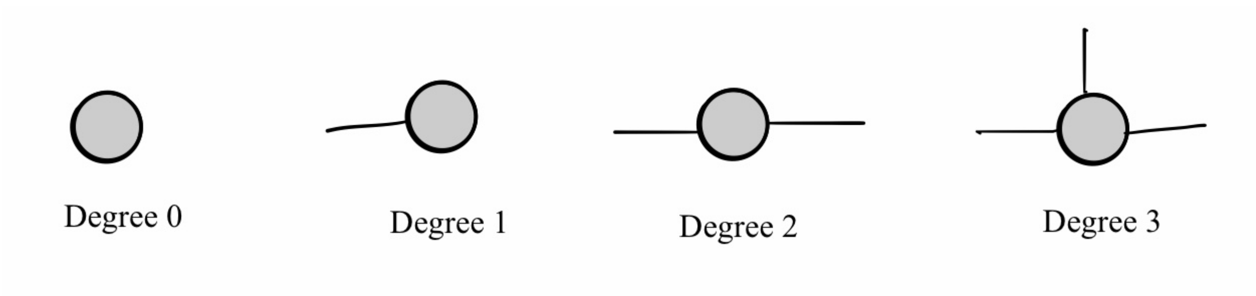


Figure 2.1: Vertex degree

Theorem 1. *A graph is Eulerian if and only if it is connected and every vertex has even degree.*

Theorem 2. *If a graph is connected and has just two odd vertices, then it has an Euler path. Any such path must start at one of the odd vertices and end at the other one.*

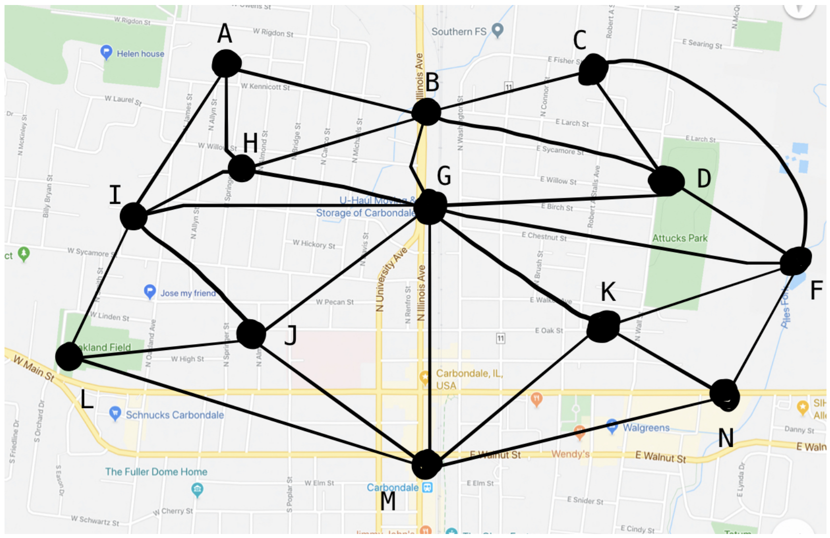


Figure 2.2: The paths that link the stations

Using theorem 1 and 2 in the seven bridges of Königsberg problem (Figure 1.1), all the four vertices have odd degree, so it is impossible that seven bridges of Königsberg problem has an Eulerian circuit. The graph has more than 2 odd vertices, so it also does not contain an Euler path.

We can use an Eulerian graph for organizing transpiration networks that can use particular paths just only one time. For example, as in Figure 2.2, a postal carrier wants to use bus stations (vertices) to traverse each street (edges) exactly once. Besides, he has to start and end his route at the same station. To let him do that, the graph must be an Euler circuit. In this case, we need to use Theorem 1. We start checking the degrees of the vertices one by one. As soon as we caught an odd vertex, we know that the graph will be not an Euler circuit. The graph (Figure 2.3) has eight vertices A, B, C, F, N, M, L, and I; their degrees are odds. So it is not an Euler circuit. Let's try to make the graph an Eulerian circuit. There are actually many several Euler circuits he could have taken. One of them could be by deleting four edges AI, BC, FN, and LM (Figure 2.3). Then, all vertices will have even degrees. Now, the mail carrier can traverses each street exactly once, and he can start and end from one station.

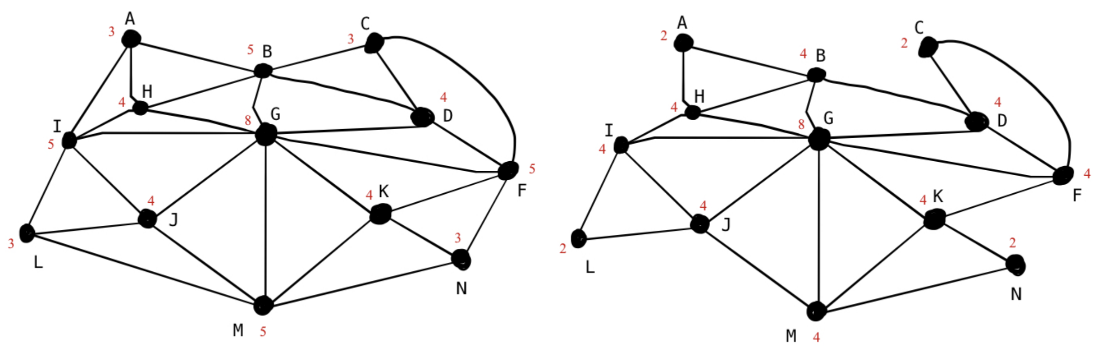


Figure 2.3: After deleting the edges that cause odd degrees

CHAPTER 3

USING SPANNING SUBGRAPHS FOR BUS PATH PROBLEMS

Each city has several diverse bus companies; these companies want to have service in every station in the city which can help the customers using that company to move from station to another station. But the city government does not let different companies to run on the same road. Here the city government wants to know the minimum number of bus companies that can run the city. By using graph theory, we need to use a graph that has vertices to represent stations and edges represent the road between the stations.

A subgraph of a graph G is a graph whose set of vertices and set of edges are all subsets of G. **A spanning subgraph** of G is a subgraph that contains all vertices of G.

Theorem 3. *Let G be a connected graph with n vertices. Then $s(G)$ is the largest integer for which the the following statement is true: for each positive integer $k \leq n$, at least $(k-1) \times s(G)$ edges must be removed in order to disconnected G into k components.*

Figure 2.2 represents a city that has 13 stations linked by 28 paths. Each bus company needs a network that connects with all stations, and each company is not allowed to have the same path. We need to divide the graph into the maximum number of connected subgraphs and we must use all the vertices. $s(G)$ is the required number of subgraphs. Let's applied Theorem 3 to this problem.

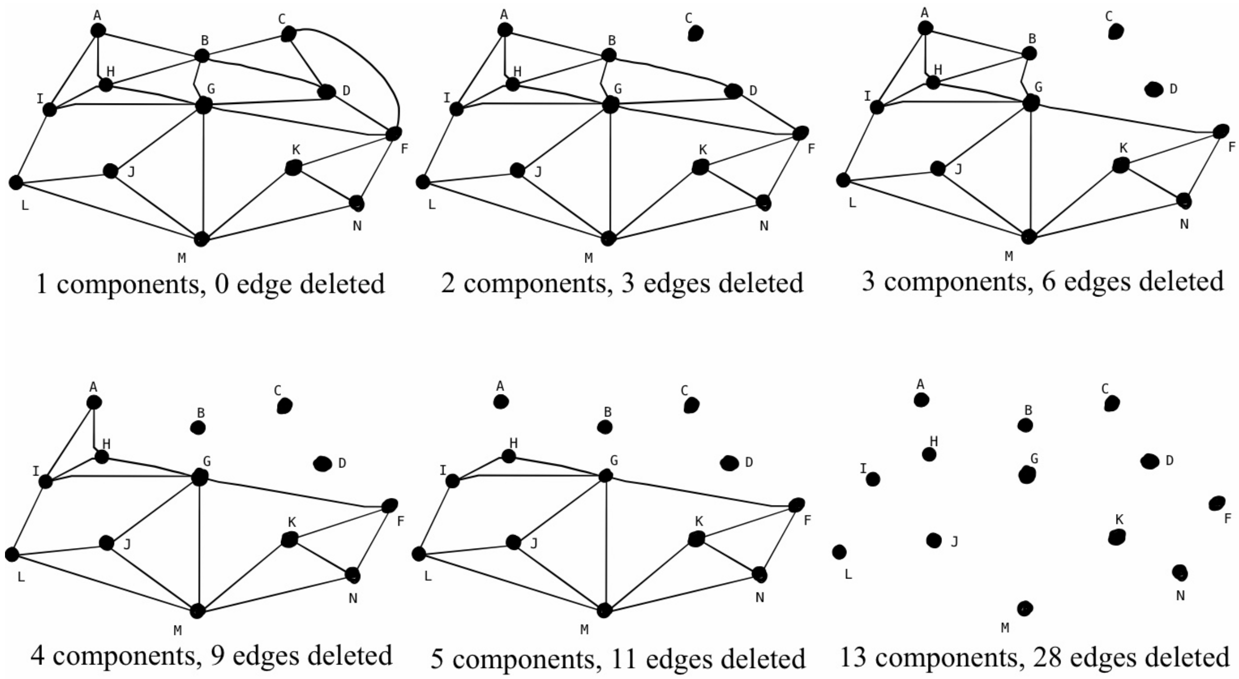


Figure 3.1: Deleting edges

by seeing Figure 3.1, to disconnect the graph into

2 components, we need to eliminate at least 3 edges, so

$$s(G) \leq 3/(2-1) = 3 \tag{3.1}$$

3 components, we need to eliminate at least 6 edges, so

$$s(G) \leq 6/(3-1) = 3 \tag{3.2}$$

4 components, we need to eliminate at least 9 edges, so

$$s(G) \leq 9/(4-1) = 3 \tag{3.3}$$

5 components, we need to eliminate at least 11 edges, so

$$s(G) \leq 11/(5-1) = 11/4 = 2.75 \tag{3.4}$$

$$\vdots$$

13 components, we need to eliminate at least 28 edges, so

$$s(G) \leq 28/(13-1) = 28/12 = 2.33 \tag{3.5}$$

The largest integer $s(G)$, which satisfies all these inequalities, is 2. Hence, the maximum number of company that can run the city is 2. The red company and blue company in Figure 3.2 and 3,3 show a correct allocation to two companies.

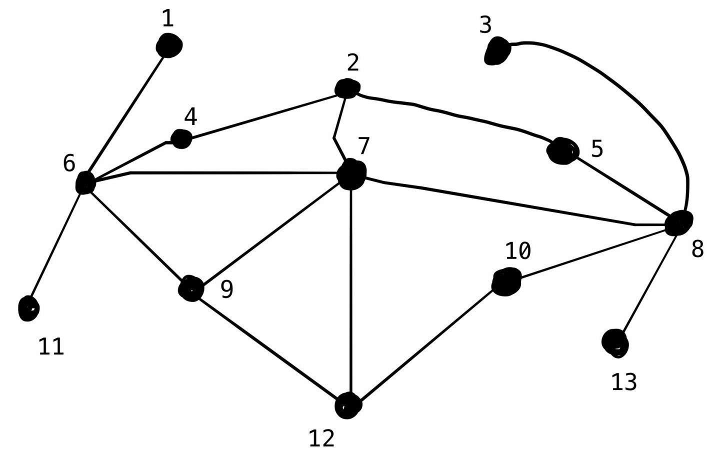


Figure 3.2: Blue bus company

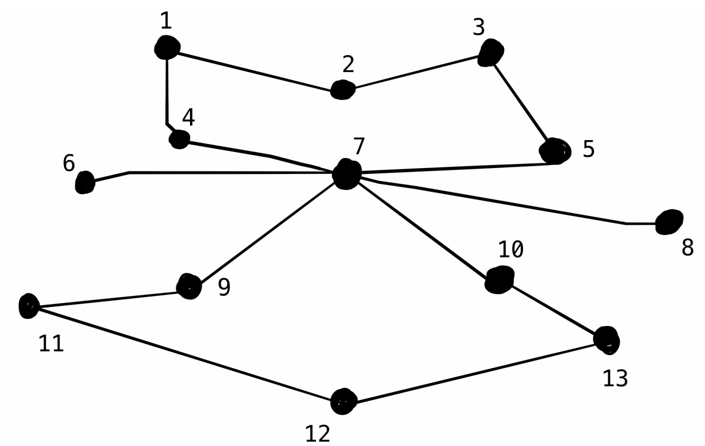


Figure 3.3: Red bus company

CHAPTER 4

USING THE GRAPH CENTER TO CHOOSE OPTIMUM LOCATIONS

Graph center helps the problem of selecting one or several areas of a graph of a city to optimize a function that depends on distance with respect to provided points of the network. For example, when the number of citizens grows in the city, the city government decides to build a new hospital at a central point that reduces distances from each station. Where in the city should they build it? Consequently, they want to determine an appropriate location to one of the bus stations, which can be easy for all citizens to approach the hospital in a short time by using the buses. The hospital should be next to bus station. Finding the center, by using graph theory, of the bus stations is beneficial for hospital location problems where the goal is to eliminate the worst-case distance.

Eccentricity: Let G be a graph and v be a vertex of G then the eccentricity of the vertex v is the maximum distance from v to any vertex in the graph. It is denoted by $e(v) = max\{d(v, u) : u \in V(G)\}$.

The **diameter** of G is the maximum eccentricity among the vertices of G. Thus, $diameter(G) = \max\{e(v) : v \in V(G)\}$ and it is denoted by $diam(G)$.

The **radius** of G is the minimum eccentricity among the vertices of G.

Therefore, $radius(G) = \min\{e(v) : v \in V(G)\}$ and it is denoted by $red(G)$. From all these eccentricities of the vertices in a graph, the radius of the connected graph is the smallest

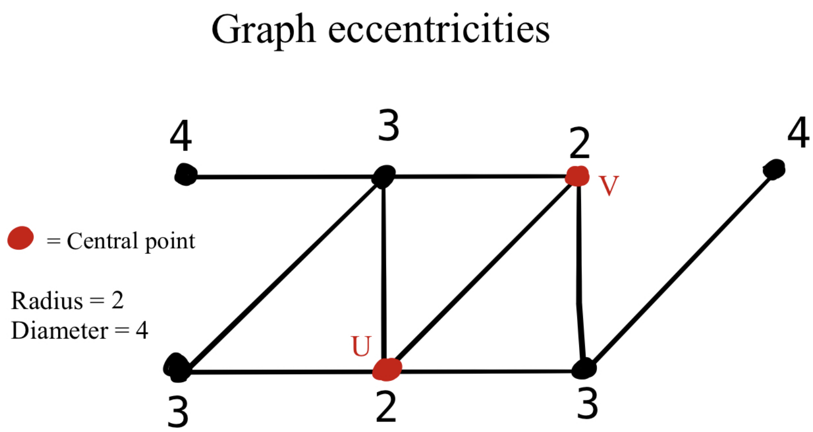


Figure 4.1: The diameter, the radius, and the center point

of all those eccentricities. If $e(V) = red(G)$, then V is the **central point** of Graph G. Now we can define the **center** as set of vertices of eccentricity equal to the radius(cental point). Hence, $cen(G) = \{v \in V(G) : e(v) = rad(G)\}$.

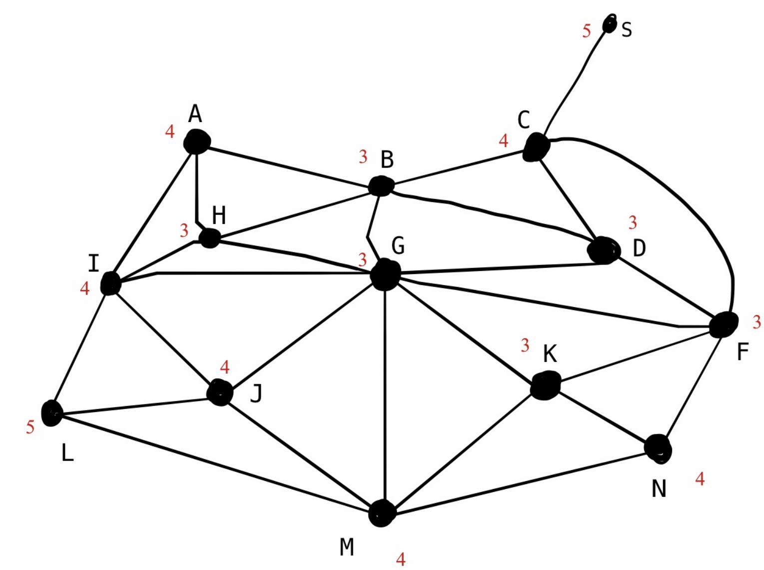


Figure 4.2: The eccentricities of the vertices

Figure 4.2 represents the graph of the city with its bus stations, and every vertex is labeled with its eccentricity. The new hospital should be located in an optimum location. Now see that an appropriate answer is for the hospital to be built at any of the six stations that correspond to center vertices that having eccentricity 3 in the figure which is $\{H, B, G, D, K, F\}$.

CHAPTER 5

FINDING A MINIMUM WEIGHT SPANNING TREE

(KRUSKAL'S ALGORITHM)

A graph is a **tree** if and only if the graph is connected and does not contain any cycle. **A minimum spanning tree** is a particular kind of tree that has minimum weight. A minimum spanning tree has $|V|-1$ edges where $|V|$ is the number of vertices in the original graph.

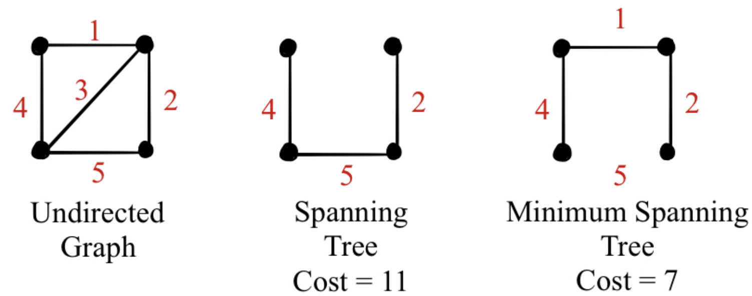
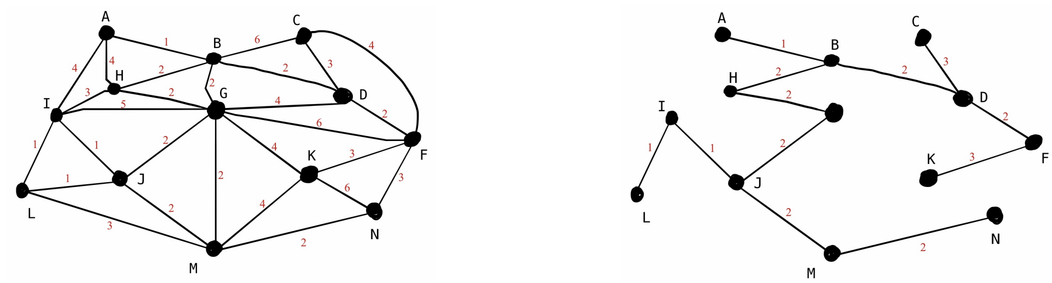


Figure 5.1: Minimum spanning tree

In the city, somehow, all the roads are destroyed simultaneously, so the city government has to repair the roads that can connect the bus's stations. There is a fixed cost to repair a particular path. They need to find out the minimum cost to connect all stations by repairing roads. The cost of the spanning tree is the total of the weights of all the edges in the tree. There can be several spanning trees. The minimum spanning tree is the spanning tree where the cost is minimum among all other spanning trees. The minimum spanning tree is the best solution for solving this problem.



(a) The original graph with the costs (b) Applying Kruskal's algorithm

Figure 5.2: Kruskal's algorithm

Kruskal's algorithm is a greedy algorithm that builds a minimum spanning tree by adding edges with minimum weight as long as doing so does not form a cycle. To do Kruskal's algorithm we first need to order all the edges in the non-decreasing order of their weight.

Second, choose the smallest edge. Verify if it makes a cycle with the spanning tree formed so far. If the cycle is not formed, include this edge. Else, delete it. Then, repeat that until there are $(|V| - 1)$ edges in the spanning tree. Back to the graph, we label all the cost of repairing the road and the cost in multiples of a million dollars. In the Table 5.1, we can see all possible repairing with the cost and * indicates the road content be fixed.

Table 5.1: The cost between stations

	A	B	C	D	F	G	H	I	J	K	L	M	N
A	-	1	*	*	*	*	4	4	*	*	*	*	*
B		-	6	2	*	2	2	*	*	*	*	*	*
C			-	3	4	*	*	*	*	*	*	*	*
D				-	2	4	*	*	*	*	*	*	*
F					-	6	*	*	*	3	*	*	3
G						-	*	2	2	4	*	2	*
H							-	3	*	*	*	*	*
I								-	1	*	1	*	*
J									-	*	1	2	*
K										-	*	4	6
L											-	*	*
M												-	2
N													-

The minimum cost is $1 + 1 + 1 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 3 + 3 = 23$ million dollars which is the minimum cost for the graph. Figure 5.2 shows the graph when we used Kruskal's algorithm.

CHAPTER 6

USING DOMINATION IN GRAPHS TO FIND THE BEST LOCATION OFPOLICE STATIONS

City hall decided to build small branches of police stations around the city, which should be nearby to bus stations. They want to make it easy for all citizens to approach these small police stations when they are in an emergency situation in its place of going to the main police department. City hall requires a minimum way to construct all these police stations. The question is: what is the minimum number of police stations needed to assist all citizens in all 13 bus stations or at least the citizens can approach the police stations just by using a bus for one station. Here, by using domination number, we can answer their question.

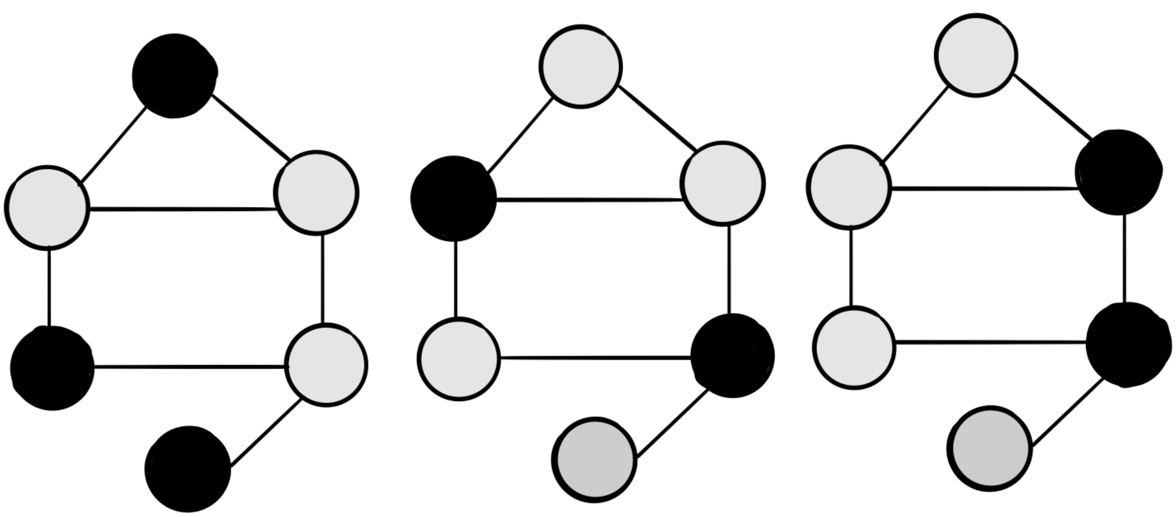
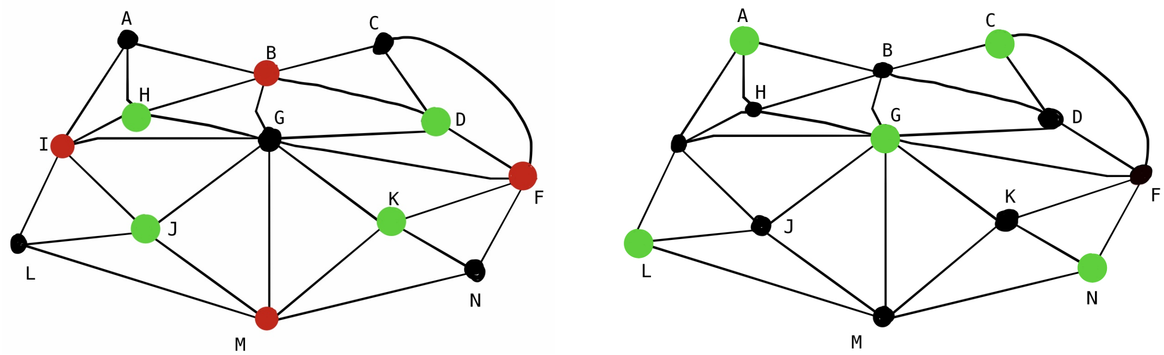


Figure 6.1: Domination number



(a) The smallest dominating sets
$S1 = \{H, D, J, K, \}$ and $S2 = \{B, L, F, M, \}$

(b) Dominating set
$S = \{A, C, G, L, N\}$

Figure 6.2: Domination sets of bus stations

As in Figure 6.1, all the black vertices in each graph are dominating sets. **Dominating set** for a graph $G = (V, E)$ is a subset S of V such that every vertex not in S is adjacent to at least one member of S. The **domination number** $\gamma(G)$ is the number of vertices in a smallest dominating set for G. As in Figure 6.1, the black vertices in two graphs on the right show domination number. Back to the problem by seeing Figure 6.2 we can see the graph of the bus stations with the difference between minimum dominating set and dominating set. In Figure 6.2 (b), the set $S = \{A, C, G, L, N\}$ is one of the dominating set but is not the minimum one. However, in Figure 6.2 (a), the sets $\{H, D, J, K, \}$ and $\{B, L, F, M, \}$ are minimum dominating sets. Hence, the dominating number is four $\gamma(G) = 4$. Now, by using minimum domination number, the minimum number of police stations can be fixed and any citizen can reach the police stations in any bus stations or at less they need to use the bus for only one station to reach the police station.

CHAPTER 7

USING CHROMATIC NUMBER TO CONSTRUCT DIFFERENT BUS STATION DESIGNS

A city government negotiates about changing all old bus-stations design in the town to new unique designs. However, stations do not have the same design if two bus stations are neighboring. In other words, when people move one station from any station, they will see different designs of bus stations. The city hall wants to know how many bus station designs needed to arrange their bus networks and inform their architects about the number of different designs. The chromatic number is the best way to solve the problem. Before we start applying the chromatic number in this problem, we need to understand the basics of graph coloring. Graph coloring is a particular case of graph labeling. By coloring , it means that a graph G is an assignment of the colors to the vertices of G in such a way that neighbors obtain different colors. In graph theory, often we use $S = \{1, 2, ...k\}$ as the colors. The number represent the number of colors. There are different kinds of graph coloring such as edge coloring, vertex coloring, face coloring, map coloring etc. Vertex coloring is one of the most popular graph coloring problems (Figure 7.1). Now, we can define the chromatic number of a graph G as the smallest number of colors required in coloring, it is denoted by $\chi(G)$. Back to the city government problem, we could use colors to represent the new design. Here, how to find a way of coloring the vertices of a bus network graph

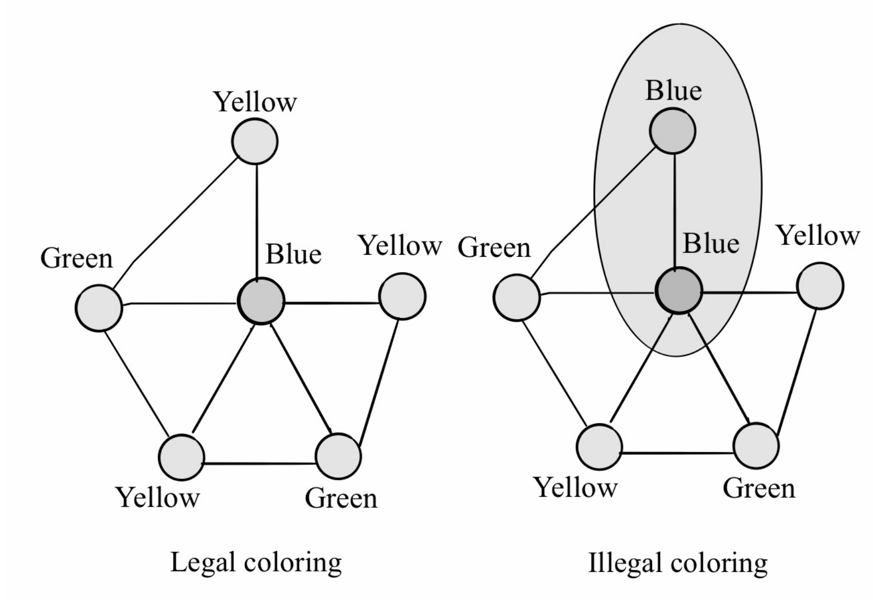


Figure 7.1: Vertex coloring

such that no two neighboring vertices are colored using the same color? Additionally, we need to use as few colors as possible to color the graph. By seeing Figure 7.2, we label the graph with three colors yellow, green, and blue. We can see that three colors are the minimum needed to color the graph so the chromatic number is three $\chi(G) = 3$. Now, each station that has the same color can have the same design.

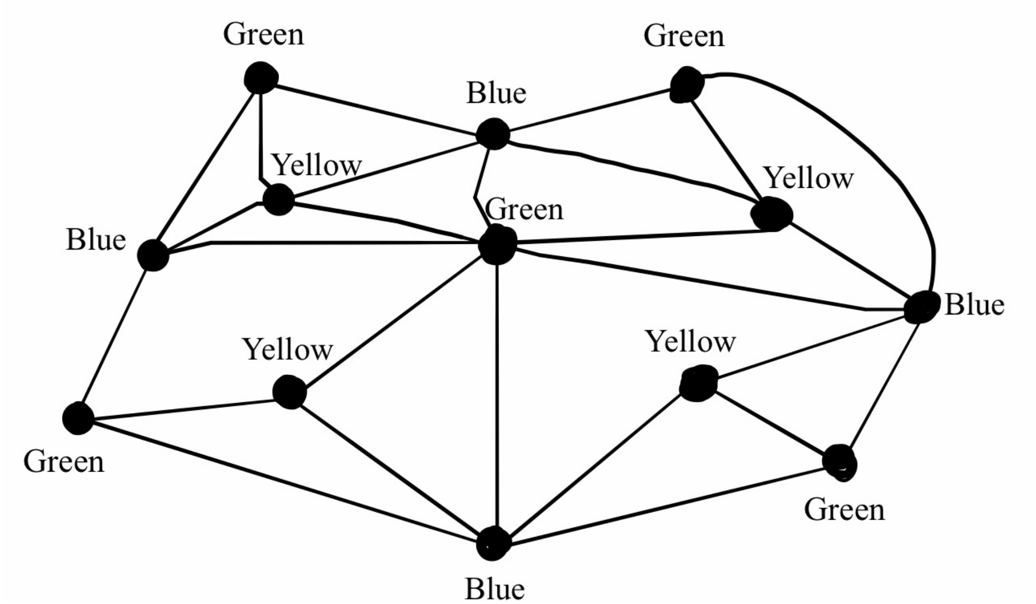


Figure 7.2: Coloring bus stations

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