GEOMETRIC PUZZLES.

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In the April number of *The Open Court,* in the article with the above title, there is a well-known puzzle in which a square containing 64 squares is apparently equal to a rectangle containing 65 squares. There are a few points about this puzzle which are not mentioned in the article referred to, which are interesting.

In Fig. 1, it is shown how we can arrange the same pieces so as to form the three figures, A, B, and C. If we take $x = 5$, $y = 3$, we shall have $A = 63$, $B = 64$, $C = 65$.

Let us investigate the three figures by algebra.

$$A = 2xy + 2xy + y(2y - x) = 3xy + 2y^2$$
$$B = (x + y)^2 = x^2 + 2xy + y^2$$
$$C = x(2x + y) = 2x^2 + xy$$
$$C - B = x^2 - xy - y^2$$
$$B - A = x^2 - xy - y^2.$$

These three figures would be equal if $x^2 - xy - y^2 = 0$, i. e., if

$$\frac{x}{y} = \frac{1 + \sqrt{5}}{2}$$

so the three figures cannot be made equal if x and y are expressed in rational numbers.

We will try to find rational values of x and y which will make the difference between A and B or between B and C unity.

Solving the equation

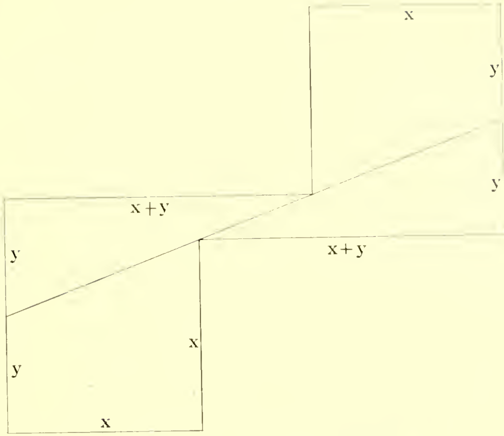
$$x^2 - xy - y^2 = \pm 1$$

we find by the Theory of Numbers that the y and x may be taken as any two consecutive numbers in the series

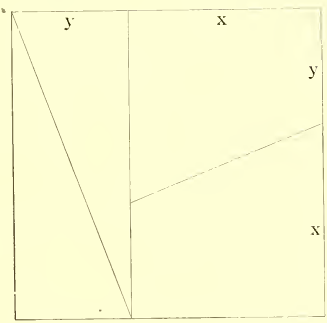
$$1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\ldots$$

where each number is the sum of the two preceding numbers.

A



B



C

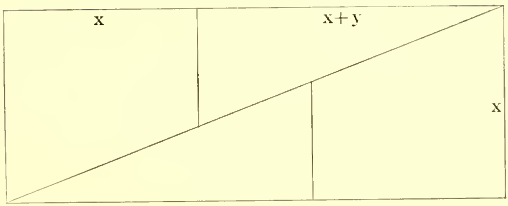


Fig. 1.

The values $y = 3$ and $x = 5$ are the ones commonly given. For these we have, as stated above, $A < B < C$.

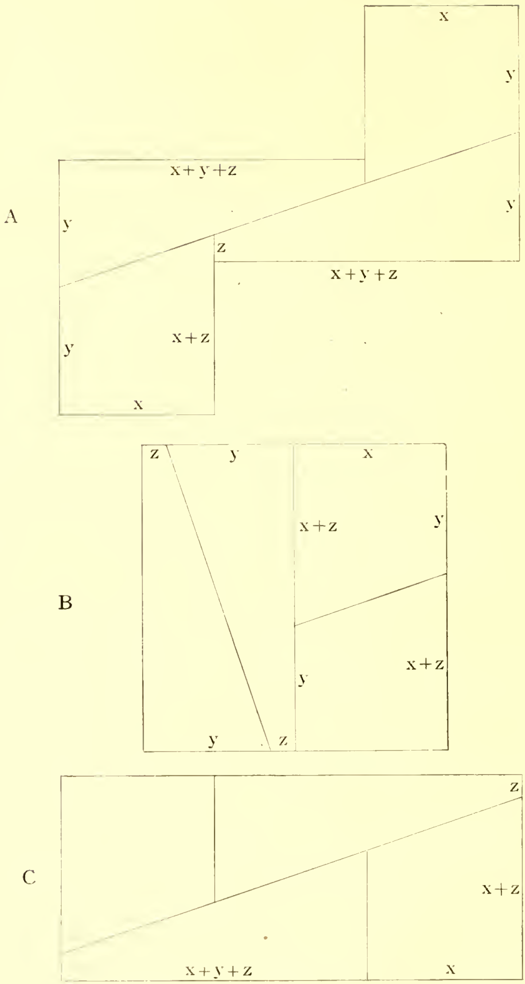


Fig. 2.

The next pair, $x=8$, $y=5$ give $A > B > C$, i. e., $A=170$, $B=169$, $C=168$.

This geometrical paradox is probably considerably older than is stated by Professor White. It seems to have been well known in 1868, as it was published that year in Schlömilch's *Zeitschrift für Mathematik und Physik,* Vol. 13, p. 162.

Fig. 2 shows an interesting modification of the puzzle.

$A = 4xy + (y + z)(2y - x) = 2y^2 + 2yz + 3xy - xz$

$B = (x + y + z)^2 = x^2 + y^2 + z^2 + 2yz + 2zx + 2xy$

$C = (x + 2z)(2x + y + z) = 2x^2 + 2z^2 + 2yz + 5zx + xy$

When $x = 6$, $y = 5$, $z = 1$ we have $A = B = C = 144$.

When $x = 10$, $y = 10$, $z = 3$ we have $A > B > C$, viz.,

$$A = 530, \quad B = 529, \quad C = 528.$$