

IN THE MAZES OF MATHEMATICS.

A SERIES OF PERPLEXING QUESTIONS.

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VIII. CHECKING THE SOLUTION OF AN EQUATION.

THE habit which many high-school pupils have of checking their solution of an equation by first substituting for x in both members of the given equation, performing like operations upon both members until a numerical identity is obtained, and then declaring their work "proved," may be illustrated by the following "proof," in which the absurdity is apparent:

	$1 + \sqrt{x+2} = 1 - \sqrt{12-x}$
Solution	$\sqrt{x+2} = -\sqrt{12-x}$
	$x+2 = 12-x$
	$2x = 10$
	$x = 5$
"Proof"	$1 + \sqrt{5+2} = 1 - \sqrt{12-5}$
	$\sqrt{5+2} = -\sqrt{12-5}$
	$5+2 = 12-5$
	$7 = 7$

Checking in the legitimate manner—by substituting in one member of the given equation and reducing the resulting number to its simplest form, then substituting in the other member and reducing to simplest form—we have $1 + \sqrt{7}$ for the first member, and $1 - \sqrt{7}$ for the second. As these are not equal numbers, 5 is not a root of the equation. There is no root.

In a popular algebra may be found the equation

$$x + 5 - \sqrt{x+5} = 6$$

and in the answer list printed in the book, "4, or -1" is given for this equation. 4 is a solution, but -1 is not. Unfortunately this instance is not unique.

As the fallacy in the erroneous method shown above is in assuming that all operations are reversible, that method may be caricatured by the old absurdity,

$$\begin{array}{ll} \text{To prove that} & 5 = 1 \\ \text{Subtracting 3 from each,} & 2 = -2 \\ \text{Squaring,} & 4 = 4 \\ & \therefore 5 = 1! \end{array}$$

IX. ALGEBRAIC FALLACIES.

A humorist maintained that in all literature there are really only a few jokes with many variations, and proceeded to give a classification into which all jests could be placed—a limited list of type jokes. A fellow humorist proceeded to reduce this number (to three, if the writer's memory is correct). Whereupon a third representative of the profession took the remaining step and declared that there are none. Whether these gentlemen succeeded in eliminating jokes altogether or in adding another to an already enormous number, depends perhaps on the point of view.

The writer purposes to classify and illustrate some of the commoner algebraic fallacies, in the hope, not of adding a striking original specimen, but rather of standardizing certain types, at the risk of blighting them. Fallacies, like ghosts, are not fond of light. Analysis is perilous to all species of the genus.

Of the classes, or subclasses, into which Aristotle divided the fallacies of logic, only a few merit special notice here. Prominent among these is that variety of paralogism known as undistributed middle. In mathematics it masks as the fallacy of converse, or employing a process that is not uniquely reversible as if it were. For example the following:*

Let c be the arithmetic mean between two *unequal* numbers a and b ; that is, let

$$a + b = 2c \quad (1)$$

Then $(a + b)(a - b) = 2c(a - b)$

$$a^2 - b^2 = 2ac - 2bc$$

Transposing,

$$a^2 - 2ac = b^2 - 2bc$$

Adding c^2 to each,

$$a^2 - 2ac + c^2 = b^2 - 2bc + c^2 \quad (2)$$

$$\therefore a - c = b - c \quad (3)$$

$$\text{and } a = b$$

But a and b were taken unequal.

Of course the two members of (3) are arithmetically equal but

* Taken, with several of the other illustrations, from the fallacies compiled by W. W. R. Ball. See his *Mathematical Recreations and Essays* (Macmillan, 1905), a book well deserving its popularity.

of opposite quality; their squares, the two members of (2), are equal. The fallacy here is so apparent that it would seem superfluous to expose it, were it not so common in one form or another.

For another example take the absurdity used in the preceding section to caricature an erroneous method of checking a solution of an equation. Let us resort to a parallel column arrangement:

A bird is an animal;	Two equal numbers have equal squares;
A horse is an animal;	These two numbers have equal squares;
∴ A horse is a bird.	∴ These two numbers are equal.

The untutored man pooh-poohs at this, because the *conclusion* is absurd, but fails to notice a like fallacy on the lips of the political speaker of his party.

The first-year high-school pupil duly derides this whenever the *conclusion* is absurd, but would allow to pass unchallenged the fallacious method of checking shown in the preceding section.

In case of indicated square roots the fallacy may be much less apparent. By the common convention as to sign, + is understood before $\sqrt{\quad}$. Considering, then, only the positive even root or the real odd root, it is true that "like roots of equals are equal," and

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

But if a and b are negative, and n even, the identity no longer holds, and by assuming it we have the absurdity

$$\begin{aligned} 1 \sqrt[2]{(-1)(-1)} &= 1 \sqrt[2]{-1} \cdot 1 \sqrt[2]{-1} \\ 1 \sqrt[2]{1} &= (1 \sqrt[2]{-1})^2 \\ 1 &= -1 \end{aligned}$$

Or take for granted that $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ for all values of the letters.

The following is an identity, since each member $= \sqrt{-1}$:

$$\begin{aligned} \sqrt{\frac{1}{-1}} &= \sqrt{\frac{-1}{1}} \\ \frac{1 \sqrt{1}}{1 \sqrt{-1}} &= \frac{1 \sqrt{-1}}{1 \sqrt{1}} \end{aligned}$$

Hence!

Clearing of fractions, $(1 \sqrt{1})^2 = (1 \sqrt{-1})^2$

Or $1 = -1$

The "fallacy of accident," by which one argues from a general rule to a special case where some circumstance renders the rule inapplicable, and its converse fallacy, and De Morgan's suggested third variety of the fallacy, from one special case to another, all find exemplification in pseudo-algebra. As a general rule, if equals be divided by equals, the quotients are equal; but not if the equal divisors are any form of zero. The application of the general rule

to this special case is the method underlying the largest number of the common algebraic fallacies.

$$x^2 - x^2 = x^2 - x^2$$

Factoring the first member as the difference of squares, and the second by taking out a common factor,

$$(x + x)(x - x) = x(x - x) \tag{1}$$

Canceling $x - x$, $x + x = x$ (2)

$$2x = x$$

$$2 = 1 \tag{3}$$

Dividing by 0 changes identity (1) into equation (2), which is true for only one value of x , namely 0. Dividing (2) by x leaves the absurdity (3).

Take another old illustration:*

Let $x = 1$

Then $x^2 = x$

And $x^2 - 1 = x - 1$

Dividing both by $x - 1$, $x + 1 = 1$

But $x = 1$

Whence, by substituting, $2 = 1$

The use of a divergent series furnishes another type of fallacy, in which one assumes something to be true of all series which in fact is true only of the convergent. For this purpose the harmonic series is perhaps oftenest employed.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

Group the terms thus:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots \text{ to 8 terms}\right) +$$

$$\left(\frac{1}{17} + \dots \text{ to 16 terms}\right) + \dots$$

Every term (after the second) in the series as now written $> \frac{1}{2}$. Therefore the sum of the first n terms increases without limit as n increases indefinitely.† The series has no finite sum; it is divergent.

But if the signs in this series are alternately $+$ and $-$, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

is convergent. With this in mind, the following fallacy is transparent enough:

* Referred to by De Morgan as "old" in a number of the *Athenæum* of forty years ago.

† The sum of the first 2^n terms $> 1 + \frac{1}{2}n$.

$$\begin{aligned}
 \log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\
 &= \left(1 + \frac{1}{3} + \frac{1}{5} + \dots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots\right) \\
 &= \left[\left(1 + \frac{1}{3} + \frac{1}{5} + \dots\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots\right)\right] - \\
 &\quad 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots\right) \\
 &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right) \\
 &= 0
 \end{aligned}$$

But $\log 1 = 0$

Suppose ∞ written in place of each parenthesis.

∞ and 0 are both convenient "quantities" for the fallacy maker.

By tacitly assuming that all real numbers have logarithms and that they are amenable to the same laws as the logarithms of arithmetic numbers, another type of fallacy emerges:

$$(-1)^2 = 1$$

Since the logarithms of equals are equal,

$$2 \log (-1) = \log 1 = 0$$

$$\therefore \log (-1) = 0$$

$$\therefore \log (-1) = \log 1$$

$$\text{and } -1 = 1$$

The idea of this type is credited to John Bernoulli. Some great minds have turned out conceits like these as by-products, and many amateurs have found delight in the same occupation. To those who enjoy weaving a mathematical tangle for their friends to unravel, the diversion may be recommended as harmless. And the following may be suggested as promising points around which to weave a snarl: the tangent of an angle becoming a discontinuous function for those particular values of the angle which are represented by $(n + \frac{1}{2})\pi$; discontinuous algebraic functions; the fact that when h, j and k are rectangular unit vectors the commutative law does not hold, but $hjk = -kjh$; the well-known theorems of plane geometry that are not true in solid geometry without qualification; etc.

Let us use one of these to make a fallacy to order. In the fraction $1/x$, if the denominator be diminished, the fraction is increased.

When $x = 5, 3, 1, -1, -3, -5$, a decreasing series;
 then $1/x = 1/5, 1/3, 1, -1, -1/3, -1/5$, an increasing series,
 as, by rule, each term of the second series is greater than the term

before it: $1/3 > 1/5$, $1 > 1/3$, $-1/5 > -1/3$. Then the fourth term is greater than the third; that is,

$$-1 > +1.$$

Neither the fallacies of formal logic nor those of algebra invalidate sound reasoning. From the counterfeit coin one does not infer that the genuine is valueless. Scrutiny of the counterfeit may enable us to avoid being deceived later by some particularly clever specimen. Counterfeit coins also, if so stamped, make good playthings.