IN THE MAZES OF MATHEMATICS.

A SERIES OF PERPLEXING QUESTIONS.

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IV. A QUESTION OF FOURTH DIMENSION BY ANALOGY.

AFTER class one day a normal-school pupil asked the writer the following question, and received the following reply:

Q. If the path of a moving point (no dimension) is a line (one dimension), and the path of a moving line is a surface (two dimensions), and the path of a moving surface is a solid (three dimensions), why isn't the path of a moving solid a four-dimensional magnitude?

A. If your hypotheses were correct, your conclusion should follow by analogy. The path of a moving point is, indeed, always a line. The path of a moving line is a surface *except* when the line moves in its own dimension, "slides in its trace." The path of a moving surface is a solid only when the motion is in a third dimension. The generation of a four-dimensional magnitude by the motion of a solid presupposes that the solid is to be moved in a fourth dimension.

V. LAW OF COMMUTATION.

This law, emphasized for arithmetic in McLellan and Dewey's *Psychology of Number,* and explicitly employed in all algebras that give attention to the logical side of the subject, is one whose importance is often overlooked. So long as it is used implicitly and regarded as of universal application, its import is neglected. An antidote: to remember that there are regions in which this law does not apply, e. g.:

In the "geometric multiplication" of rectangular vectors used in quaternions, the commutative property of factors does not hold, but a change in the order of factors reverses the sign of the product.

Even in elementary algebra or arithmetic, the commutative principle is not valid in the operation of involution. Professor Schubert, in his *Mathematical Essays and Recreations*, has called attention to the fact that this limitation—the impossibility of interchanging base and exponent—renders useless any high operation of continued involution.

VI. A FEW CATCH QUESTIONS.

What number can be divided by every other number without a remainder?

"Four-fourths exceeds three-fourths by what fractional part?" This question will usually divide a company.

Can a fraction whose numerator is less than its denominator be equal to a fraction whose numerator is greater than its denominator? If not, how can $\frac{-3}{+6} = \frac{+5}{-10}$?

In the proportion

$$+6 : -3 :: -10 : +5$$

is not either extreme greater than either mean? What has become of the old rule, "greater is to less as greater is to less"?

Where is the fallacy here?

1 mile square = 1 square mile,

$\therefore$ 2 miles square = 2 square miles. (Axiom: If equals be multiplied by equals, etc.)

VII. THE THREE FAMOUS PROBLEMS OF ANTIQUITY.

1. To trisect an angle or arc.
2. To "duplicate the cube" (Delian problem).
3. To "square the circle" (said to have been first tried by Anaxagoras).

Hippias of Elis invented the quadratrix for the trisection of an angle, and it was later used for the quadrature of the circle. Other Greeks devised other curves to effect the construction required in (1) and (2). Eratosthenes and Nicomedes invented mechanical instruments to draw such curves. But none of these curves can be constructed with ruler and compass alone. And this was the limitation imposed on the solution of the problems.

Antiquity bequeathed to modern times all three of the problems unsolved. Modern mathematics, with its greatly improved methods, has proved them all impossible of construction with ruler and compass alone—a result which the shrewdest investigator in antiquity could have only conjectured—has shown new ways of solving them if the limitation of ruler and compass be removed, and has devised and applied methods of approximation. It has *dissolved* the problems, if that term may be permitted.

It was not until 1882 that the transcendental nature of the number π was established (by Lindemann). The final results in all three of the problems, with mathematical demonstrations, are given in Klein's *Famous Problems of Elementary Geometry,* translated by Beman and Smith (Ginn, 1897).

It should be noted that the number π, which the student first meets as the ratio of the circumference to the diameter of a circle, is a number that appears often in analysis in connections remote from elementary geometry, e. g., in formulas in the calculus of probability.

The value of π was computed to 707 places of decimals by William Shanks. His result (communicated in 1873) with a discussion of the formula he used (Machin's) may be found in the *Proceedings of the Royal Society of London,* Vol. 21. No other problem has been worked out to such a degree of accuracy—"an accuracy exceeding the ratio of microscopic to telescopic distances." An illustration calculated to give some conception of the degree of accuracy attained may be found in Professor Schubert's *Mathematical Essays and Recreations* (translation by T. J. McCormack), p. 140. Most of this computation serves, apparently, no useful purpose. But it should be a deterrent to those who—immune to the demonstration of Lindemann and others—still hope to find an exact ratio.

The quadrature of the circle has been the most fascinating of mathematical problems. The "army of circle-squarers" has been recruited in each generation. "Their efforts remained as futile as though they had attempted to jump into a rainbow" (Cajori); yet they were undismayed. In some minds, the proof that no solution can be found seems only to have lent zest to the search.

That these problems are of perennial interest, is attested by the fact that contributions to them still appear. In 1905 a little book was published in Los Angeles entitled *The Secret of the Circle and the Square,* in which also the division of "any angle into any number of equal angles" is considered. The author, J. C. Willmon, gives original methods of approximation. *School Science and Mathematics* for May 1906 contains a "solution" of the trisection problem by a high-school boy in Missouri, printed, apparently, to show that the problem still has fascination for the youthful mind. In a later number of that magazine the problem is discussed by another from the vantage ground of higher mathematics.

While the three problems have all been proved to be insolvable under the conditions imposed, still the attempts made through many centuries to find a solution have led to much more valuable results, not only by quickening interest in mathematical questions, but especially by the many and important discoveries that have been made in the effort. The voyagers were unable to find the northwest passage, and one can easily see now that the search was *necessarily* futile; but in the attempt they discovered continents whose resources, when developed, make the wealth of the Indies seem poor indeed.