IN THE MAZES OF MATHEMATICS.

A SERIES OF PERPLEXING QUESTIONS.

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I. AXIOMS IN ELEMENTARY ALGEBRA.

MANY text-books on the subject introduce equations with a list of axioms such as the following:

1. Things equal to the same thing or equal things are equal to each other.

2. If equals be added to equals, the sums are equal.

3. If equals be subtracted from equals, the remainders are equal.

4. If equals be multiplied by equals, the products are equal.

5. If equals be divided by equals, the quotients are equal.

6. The whole is greater than any of its parts.

The fifth, or "division axiom," should receive the important qualification given it by the best of the books, "divided by equals, *except zero*." Without such limitation the statement is far from axiomatic.

A writer of the sixth "axiom" may also have on another page something like this: "+3 is the whole, or *sum*." Seeing that one of its parts is +7, one wonders how the author, in a text-book on algebra, could ever have written the "axiom," "The whole is greater than any of its parts."

$$\begin{array}{r} +7 \\ -5 \\ +2 \\ -1 \\ \hline +3 \end{array}$$

When we use the word "equal" in the axioms, do we mean anything else than "same"—If two numbers are the same as a third number, they are the same as each other, etc.?

II. DO THE AXIOMS APPLY TO EQUATIONS?

Most text-books in elementary algebra use them as if they applied. Most of the algebras have, somewhere in the first fifty or sixty pages, something like this:

$$3x+4=19$$

Subtracting 4 from each member,

$$3x=15 \qquad \text{Ax. 3}$$

Dividing by 3,

$$x=5 \qquad \text{Ax. 5}$$

This shows how common some very loose thinking on this subject is. As a matter of fact, the axioms do not apply directly to equations: for (A) one can follow the axioms, make no other mistake, and arrive at a result which is incorrect; (B) he can violate the axioms and come out right; (C) the axioms, from their very nature, can not apply directly to equations.

(A) *To follow axioms and come out wrong*:

$$x-1=2 \qquad (1)$$

Multiplying each member by $x-5$,

$$x^2-6x+5=2x-10 \qquad \text{Ax. 4}$$

Subtracting $x-7$ from each member,

$$x^2-7x+12=x-3 \qquad \text{Ax. 3}$$

Dividing each member by $x-3$,

$$x-4=1 \qquad \text{Ax. 5}$$

Adding 4 to each member,

$$x=5 \qquad \text{Ax. 2}$$

But $x=5$ does not satisfy (1). The only value of x that satisfies (1) is 3.

(B) *To violate the axioms and come out right*:

In order to avoid the objection that the errors made by violating two axioms may just balance each other, only *one* axiom will be violated.

$$x-1=2 \qquad (1)$$

Add 10 to one member *and not to the other*. This will doubtless be deemed a sufficiently flagrant transgression of the "addition axiom":

$$x+9=2 \qquad (2)$$

Multiplying each member by $x-3$,

$$x^2+6x-27=2x-6 \qquad (3) \qquad \text{Ax. 4}$$

Subtracting $2x-6$ from each member,

$$x^2+4x-21=0 \qquad (4) \qquad \text{Ax. 3}$$

Dividing each member by $x+7$,

$$x-3=0 \qquad (5) \qquad \text{Ax. 5}$$

Adding 3 to each member,

$$x=3 \qquad \text{Ax. 2}$$

Inasmuch as 3 is *the correct root* of equation (1), the error in the first step must have been balanced by another or several. It was done in obtaining (3) and (5), though at both steps the axioms were applied.

(C) *The axioms, from their very nature, can not have any direct application to equations.*

The axioms say that—if equals be added to equals etc.—the results are equal. But the question in solving equations is, For what value of x are they equal? Of course they are equal for *some* value of x. So when something was added to one member and not to the other, the results were equal *for some value of* x. Arithmetic, dealing with numbers, needs to know that certain resulting numbers are equal to certain others; but algebra, dealing with the equation, the conditional equality of expressions, needs to know on *what condition* the expressions represent the same number—in other words, for what values of the unknown the equation is true. In (B) above, the objection to equation (2) is not that its two members are not equal (they are "equal" as much as are the two members of the first equation) but that they are not equal *for the same value of* x as in the first equation; that is (2) is not *equivalent* to (1).

The principles of equivalency of equations as given in a few of the best of the texts are not too difficult for the beginner. The *proof* of them may well be deferred till later. Even if never proved, they would be, for the present purpose, vastly superior to axioms that do not apply. To give *no* reasons would be preferable to the practice of quoting axioms that do not apply.

The axioms have their place in connection with equations; namely, in the proof of the principles of equivalency. To apply the axioms directly in the solution of equations is an error.

Pupils can hardly be expected to think clearly about the nature of the equation when they are so misled. How the authors of the great majority of the elementary texts can have made so palpable a mistake in so elementary a matter, is one of the seven wonders of algebra.