ALICE IN THE WONDERLAND OF MATHEMATICS.

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YEARS after Alice had her "Adventures in Wonderland" and "Through the Looking-glass," described by "Lewis Carroll," she went to college. She was a young woman of strong religious convictions. As she studied science and philosophy, she was often perplexed to reduce her conclusions in different lines to a system, or at least to find some analogy which would make the coexistence of the fundamental conceptions of faith and of science more thinkable. These questions have puzzled many a more learned mind than hers, but never one more earnest.

Alice developed a fondness for mathematics and elected courses in it. The professor in that department had lectured on n-dimensional space, and Alice had read E. A. Abbott's charming little book, *Flatland; a Romance of Many Dimensions, by a Square,* which had been recommended to her by an instructor.

The big daisy-chain which was to be a feature of the approaching class-day exercises was a frequent topic of conversation among the students. It was uppermost in her mind one warm day as she went to her room after a hearty luncheon and settled down in an easy chair to rest and think.

"Why!" she said, half aloud, "I was about to make a daisy-chain that hot day when I fell asleep on the bank of the brook and went to Wonderland—so long ago. That was when I was a little girl. Wouldn't it be fun to have such a dream now? If I were a child again, I'd curl up in this big chair and go to sleep this minute. 'Let's pretend.'"

So saying, and with the magic of this favorite phrase upon her, she fell into a pleasant revery. Present surroundings faded out of consciousness, and Alice was in Wonderland.

"What a long daisy-chain this is!" thought Alice. "I wonder if I'll ever come to the end of it. Maybe it hasn't any end. Circles haven't ends, you know. Perhaps it's like finding the end of a rainbow. Maybe I'm going off along one of the infinite branches of a curve."

Just then she saw an arbor-covered path leading off to one side. She turned into it; and it led her into a room—a throne-room, for there a fairy or goddess sat in state. Alice thought this being must be one of the divinities of classical mythology, but did not know which one. Approaching the throne she bowed very low and simply said, "Goddess"; whereat that personage turned graciously and said, "Welcome, Alice." It did not seem strange to Alice that such a being should know her name.

"Would you like to go through Wonderland?"

"Oh! yes," answered Alice eagerly.

"You should go with an attendant. I will send the court jester, who will act as guide," said the fairy, at the same time waving a wand.

Immediately there appeared—Alice could not tell how—a courtier dressed in the fashion of the courts of the old English kings. He dropped on one knee before the fairy; then, rising quickly, bowed to Alice, addressing her as, "Your Majesty."

It seemed pleasant to be treated with such deference, but she promptly answered, "You mistake; I am only Miss —"

Here the fairy interruped: "Call her 'Alice.' The name means 'princess.'"

"And you may call me 'Phool,'" said the courtier; "only you will please spell it with a *ph*."

"How can I spell it when I am only speaking it?" she asked.

"*Think* the *ph*."

"Very well," answered Alice rather doubtfully, "but who ever heard of spelling 'fool' with a *ph*?"

Then he smiled broadly as he replied: "I am an anti-spelling-reformer. I desire to preserve the *ph* in words in place of *f* so that one may recognize their foreign origin and derivation."

"Y-e-s," said Alice, "but what does *phool* come from?"

Again the fairy interruped. Though always gracious, she seemed to prefer brevity and directness. "You will need the magic wand."

So saying, she handed it to the jester. The moment he had the wand, the fairy vanished. And the girl and the courtier were alone in the wonderful world, and they were not strangers. They

were calling each other "Alice" and "Phool." And he held the magic wand.

One flourish of that wand, and they seemed to be in a wholly different country. There were many beings, having length, but no breadth or thickness; or, rather, they were very thin in these two dimensions, and uniformly so. They were moving only in one line.

"Oh! I know!" exclaimed Alice, "This is Lineland. I read about it."

"Yes," said Phool; "if you hadn't read about it or thought about it, I couldn't have shown it to you."

Alice looked questioningly at the wand in his hand.

"It has marvelous power, indeed," he said. "To show you in this way what you have thought about, that is magic; to show you what you had never thought of, would be—"

Alice could not catch the last word. A little twitch of the wand set them down at a different point in the line, where they could get a better view of lineland. Alice thrust her hand across the line in front of one of the inhabitants. He stopped short. She withdrew it. He was amazed at the apparition: a body (or point) had suddenly appeared in his world and as suddenly vanished. Alice was interested to see how a linelander could be imprisoned between two points.

"He never thinks to go around one of the obstacles," she said.

"The line is his world," said Phool. "One never thinks of going out of the world to get around an obstacle."

"If I could communicate with him, could I teach him about a second dimension?"

"He has no apperceiving mass," said Phool laconically.

"Very good," said Alice, laughing; "surely he has no mass. Then he can get out of his narrow world only by accident?"

"Accident!" repeated Phool, affecting surprise, "I thought you were a philosopher."

"No," replied Alice, "I am only a college girl."

"But," said Phool, "you are a lover of wisdom. Isn't that what 'philosopher' means? You see I'm a stickler for etymologies."

"All right," said Alice, "I am a philosopher then. But tell me how that being can ever appreciate space outside of his world."

"He might evolve a few dimensions."

Alice stood puzzled for a minute, though she knew that Phool was jesting. Then a serious look came into his face, and he continued:

"One-dimensional beings can learn of another dimension only

by the act of some being from without their world. But let us see something of a broader world."

So saying, he waved the wand, and they were in a country where the inhabitants had length and breadth, but no appreciable thickness.

Alice was delighted. "This is Flatland," she cried out. Then after a minute she said, "I thought the Flatlanders were regular geometric figures."

Phool laughed at this with so much enjoyment that Alice laughed too, though she saw nothing very funny about it.

Phool explained: "You are thinking of the Flatland where all lawyers are square, and where acuteness is a characteristic of the lower classes while obtuseness is a mark of nobility. That would, indeed, be very flat; but we spell that with a capital *F*. This is flatland with a small *f*."

Alice fell to studying the life of the two-dimension people and thinking how the world must seem to them. She reasoned that polygons, circles and all other plane figures are always seen by them as line-segments; that they can not see an angle, but can infer it; that they may be imprisoned within a quadrilateral or any other plain figure if it has a closed perimeter which they may not cross; and that if a three-dimensional being were to cross their world (surface) they could appreciate only the section of him made by that surface, so that he would appear to them to be two-dimensional but possessing miraculous powers of motion.

Alice was pleased, but curious to see more. "Let's see other dimensional worlds," she said.

"Well, the three-dimensional world, you're in all the time," said Phool, at the same time moving the wand a little and changing the scene, "and now if you will show me how to wave this wand around through a fourth dimension, we'll be in that world straightway."

"Oh! I can't," said Alice.

"Neither can I," said he.

"Can anybody?"

"They say that in four-dimensional space one can see the inside of a closed box by looking into it from a fourth dimension just as you could see the inside of a rectangle in flatland by looking down into it from above; that a knot can not be tied in that space; and that a being coming to our world from such a world would seem to us three-dimensional, as all we could see of him would be a section made by our space, and that section would be what we

call a solid. He would appear to us—let us say—as human. And he would be not less human than we, nor less real, but more so; if 'real' has degrees of comparison. The flatlander who crosses the linelander's world (line) appears to the native to be like the one-dimensional beings, but possessed of miraculous powers. So also the solid in flatland: the cross-section of him is all that a flatlander is, and that is only a section, only a phase of his real self. The ability of a being of more than three dimensions to appear and disappear, as to enter or leave a room when all doors were shut, might make him seem to us like a ghost, but he would be more real and substantial than we are."

He paused, and Alice took occasion to remark:

"That is all obtained by reason; I want to see a four-dimensional world."

Then, fearing that it might not seem courteous to her guide to appear disappointed, she added:

"But I ought to have known that the wand couldn't show us anything we might wish to see; for then there would be no limit to our intelligence."

"Would unlimited intelligence mean the same thing as absolutely infinite intelligence?" Phool asked.

"That sounds to me like a conundrum," said Alice. "Is it a play on words?"

"There goes Calculus, " said Phool. "I'll ask him.—Hello! Cal."

Alice looked and saw a dignified old gentleman with flowing white beard. He turned when his name was called.

While Calculus was approaching them, Phool said in a low tone to Alice: "He'll enjoy having an eager pupil like you. This will be a carnival for Calculus."

When that worthy joined them and was made acquainted with the topic of conversation, he turned to Alice and began instruction so vigorously that Phool said, by way of caution:

"Lass! Handle with care."

Alice did not like the implication that a girl could not stand as much mathematics as any one. But then she thought, "That is only a joke," and she seemed vaguely to remember having heard it somewhere before.

"If you mean," said Calculus, "to ask whether a variable that increases without limit is the same thing as absolute infinity, the answer is clearly No. A variable increasing without limit is always nearer to *zero* than to absolute *infinity*. For simplicity of illustra-

tion, compare it with the variable of uniform change, time, and suppose the variable we are considering doubles every second. Then, no matter how long it may have been increasing at this rate, it is still nearer zero than infinity."

"Please explain," said Alice.

"Well," continued Calculus, "consider its value at any moment. It is only half what it will be one second hence, and only quarter what it will be two seconds hence, when it will still be increasing. Therefore it is *now* much nearer to zero than to infinity. But what is true of its value at the moment under consideration is true of any, and therefore of every, moment. An infinite is always nearer to zero than to infinity."

"Is that the reason," asked Alice, "why one must say 'increases without limit' instead of 'approaches infinity as a limit'?"

"Certainly," said Calculus; "a variable can not approach infinity as a limit. Students often have to be reminded of this."

Alice had an uncomfortable feeling that the conversation was growing too personal, and gladly turned it into more speculative channels by remarking:

"I see that one could increase in wisdom forever, though that seems miraculous."

"What do you mean by miraculous?" asked Phool.

"Why—" began Alice, and hesitated.

"People who begin an answer with 'Why' are rarely able to give an answer," said Phool.

"I fear I shall not be able," said Alice. "An etymologist" (this with a sly look at Phool) "might say it means 'wonderful'; and that is what I meant when speaking about infinites. But usually one would call that miraculous which is an exception to natural law."

"We must take the young lady over to see the curve tracing," said Calculus to Phool.

"Yes, indeed!" he replied. Then, turning to Alice, "Do you enjoy fireworks?"

"Yes, thank you," said Alice, "but I can't stay till dark."

"No?" said Phool, with an interrogation. "Well, we'll have them very soon."

"Fireworks in the daytime?" she asked.

But at that moment Phool made a flourish with the wand, and it was night—a clear night with no moon or star. It seemed so natural for the magic wand to accomplish things that Alice was not *very* much surprised at even this transformation. She asked:

"Did you say you were to show me curve tracing?"

"Yes," said Phool. "Perhaps you don't attend the races, but you may enjoy seeing the *traces*."

During this conversation the three had been walking, and they now came to a place where there was what appeared to be an enormous electric switchboard. A beautiful young woman was in charge.

As they approached, Calculus said to Alice, "That is Ana Lytic. You are acquainted with her, I presume."

"The name sounds familiar," said Alice, "but I don't remember to have ever seen her. I should like to meet her."

On being presented, Alice greeted her new acquaintance as 'Miss Lytic'; but that person said, in a very gracious manner:

"Nobody ever addresses me in that way. I am always called 'Ana Lytic,' except by college students. They usually call me 'Ana Lyt.' I presume they shorten my name thus because they know me so well."

In spite of the speaker's winning manner, the last clause made Alice somewhat self-conscious. Her cheeks felt very warm. She was relieved when, at that moment, Calculus said:

"This young lady would like to see some of your work."

"Some pyrotechnic curve tracing," interrupted the talkative Phool.

Calculus continued: "Please let us have an algebraic curve with a conjugate point."

Ana Lytic touched a button, and across the world of darkness (as it seemed to Alice) there flashed a sheet of light, dividing space by a luminous plane. It quickly faded, but left two rays of light perpendicular to each other, faint but apparently permanent.

"These are the axes of coordinates," explained Ana Lytic.

Then she pressed another button, and Alice saw what looked like a meteor. She watched it come from a great distance, cross the ray of light that had been called one of the axes, and go off on the other side as rapidly as it had come, always moving in the plane indicated by the vanished sheet of light. She thought of a comet; but instead of having merely a luminous tail, it left in its wake a permanent path of light. Ana Lytic had come close to Alice, and the two girls stood looking at the brilliant curve that stretched away across the darkness as far as the eye could reach.

"Isn't it beautiful!" exclaimed Alice.

Any attempt to represent on paper what she saw must be poor and inadequate. Figure 1 is such an attempt.

Suddenly she exclaimed: "What is that *point* of light?" indi-

cating by gesture a bright point situated as shown in the figure by P.

"That is a point of the curve," said Ana Lytic.

"But it is away from all the rest of it," objected Alice.

Going over to her apparatus and taking something—Alice could not see what—Ana Lytic began to write on what, in the darkness, might surely be called a blackboard. The characters were of the usual size of writing on school boards, but they were characters of light and could be plainly read in the night. This is what she wrote:

$$y^2=(x-2)^2(x-3).$$

Stepping back, she said: "That is the equation of the curve."

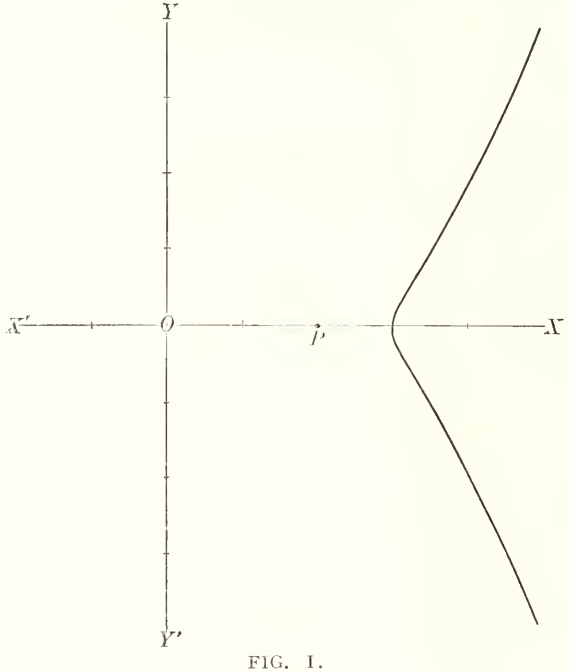

FIG. 1.

Alice expressed her admiration at seeing the equation before her and its graph stretching across the world in a line of light.

"I never imagined coordinate geometry could be so beautiful," she said.

"This is throwing light on the subject for you," said Phool.

"The point about which you asked," said Ana Lytic to Alice, "is the point (2,0). You see that it satisfies the equation. It is a point of the graph."

Alice now noticed that units of length were marked off on the

dimly seen axes by slightly more brilliant points of light. Thus she easily read the coordinates of the point.

"Yes," she said, "I see that; but it seems strange that it should be off away from the rest."

"Yes," said Calculus, who had been listening all the time. "One expects the curve to be continuous. Continuity is the message of modern scientific thought. This point seems to break that law—to be 'miraculous,' as you defined the term a few minutes ago. If all observed instances but one have some visible connection, we are inclined to call that one miraculous and the rest natural. As only that seems wonderful which is unusual, the miraculous in mathematics would be only an isolated case."

"I thank you," said Alice warmly. "That is the way I should like to have been able to say it. An isolated case is perplexing to me. I like to think that there is a universal reign of law."

"*Evidently*," said Phool, "here is an exception. It is *obvious* that there are several alternatives, such as, for example, that the point is not on the graph, that the graph has an isolated point, *and so forth*."

Calculus, Ana Lytic and Phool all laughed at this. To Alice's inquiry, Phool explained:

"We often say 'evidently' or 'obviously' when we can't give a reason, and we conclude a list with 'and so forth' when we can't think of another item."

Alice felt that the remark might have been aimed at her. Still she had not used either of these expressions in this conversation, and Phool had made the remark in a general way as if he were satirizing the foibles of the entire human race. Moreover, if she felt inclined to resent it as an impertinent criticism from a self-constituted teacher, she remembered that it was only the jest of a jester and treated it merely as an interruption.

"Tell me about the isolated point," she said to Calculus.

He proceeded in a teacher-like way, which seemed appropriate in him.

Calculus. For $x=2$ in this equation, $y=0$. For any other value of x less than 3, what would y be?

Alice. An imaginary.

Calculus. And what is the geometric representation of an imaginary number?

Alice. A line whose length is given by the absolute, or arithmetic, value of the imaginary and whose direction is perpendicular to that which represents positives and negatives.

Calculus. Good. Then—

Alice (bounding with delight at the discovery). Oh! I see! I see! There must be points of the graph outside of the plane.

Calculus. Yes, there are imaginary branches, and perhaps Ana Lytic will be good enough to show you now.

That young lady touched something on her magic switchboard, and another brilliant curve stretched across the heavens. The plane determined by it was perpendicular to the plane previously shown. (The dotted line in figure 2 represents in a prosaic way what Alice saw.)

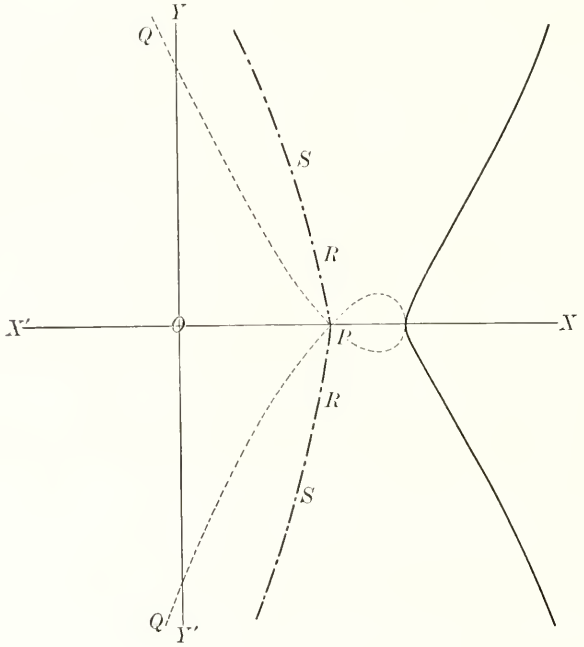

FIG. 2.

"O, I see!" exclaimed Alice. "That point is not isolated. It is the point in which this 'imaginary' branch, which is as *real* as any, pierces the plane of the two axes."

"Now," said Calculus, "if, instead of substituting real values for x and solving the equation for y, you were to substitute real numbers for y and solve for x, you would, in general, obtain for each value of y one real and two complex numbers as the values of x. The curve through all the points with complex abscissas is neither in the plane of the axes nor in a plane perpendicular to it. But you shall see."

(The dot-and-dash line in figure 2 represents these branches.)

When Ana Lytic made the proper connection at the switchboard, these branches of the curve also stood out in lines of light.

Alice was more deeply moved than ever. There was a note of deep satisfaction in her voice as she said:

"The point that troubled me because of its isolation is a point common to several branches of the curve."

"The supernatural is more natural than anything else," said Phool.

"The miraculous," thought Alice, "is only a special case of a higher law. We fail to understand things because they are connected with that which is out of our plane."

She added aloud: "This I should call the *miracle curve*."

"Yet there is nothing exceptional about this curve," said Calculus. "Any algebraic curve with a conjugate point has similar properties."

Then Calculus said something to Ana Lytic—Alice could not hear what—and Ana Lytic was just touching something on the switchboard when there was a crash of thunder. Alice gave a start and awoke to find herself in her own room at midday, and to realize that the slamming of a door in the corridor had been the thunder that terminated her dream.

She sat up in the big chair and, with the motion that had been characteristic of her as a little girl, gave "that queer little toss of her head, to keep back the wandering hair that *would* always get into her eyes," and said to herself:

"There aren't any curves of light across the sky at all! And worlds of one or two dimensions exist only in the mind. They are abstractions. But at least they are thinkable. I'm glad I had the dream. Imagination *is* a magic wand.—The future life will be a *real* wonderland, and—"

Then the ringing of a bell reminded her that it was time to start for an afternoon lecture, and she heard some of her classmates in the corridor calling to her, "Come, Alice."