8-12-2016

# KNOTS AND KNOT GROUPS

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by

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B.S., University of Peradeniya, 2011

A Research Paper
Submitted in Partial Fulfillment of the Requirements for the
Master of Science

Department of Mathematics
in the Graduate School
Southern Illinois University Carbondale
December, 2016

**RESEARCH PAPER APPROVAL**

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A Research Paper Submitted in Partial

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for the Degree of

Master of Science

in the field of Mathematics

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Graduate School
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August 10, 2016

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## INTRODUCTION

In this report, my aim is to present a fundamental concepts of knot theory.

Chapter 1 gives a brief discossion of history of knot theory.

Chapter 2 gives background containing definitions of knots and links in the space $\mathbb{R}^3$ and different definitions of knot equivalence.

In chapter 3, knot projections and regular diagrams, Reidemeister moves and proof of Reidemester's theorem for piecewise linear knots and are also given define knot equivalence for knot diagrams.

In chapter 4, knot groups and examples to show that knot group is not completely knot invariant are given.

# CHAPTER 1

# HISTORY

Knot theory was originally used by physicists to study atoms. In the 1860's, W Thomson introduced the idea that atoms might be knots. This idea remained of interest to physicists until the twentieth century, before mathematicians began investigating the concept. In contemporary science, mathematical theories about knot theory are applied to the fields of biology and chemistry too.

## 1.1 EARLY WORKS

In the $19^{th}$ century, the knot theory was studied by Carl Friedrich Gauss from a mathematical point of view. But even in the $18^{th}$ century, there was a good deal of mathematical work in knot theory. For an example, French mathematician Alexandre-Theophile Vandermonde wrote a paper titled *Remarques sur less problems de situation (1771)* that explored what we would now call the topological features of knots and braids. That is, he was concerned not with "questions of measurements, but with those of positions".

Gauss made the first step toward the study of what we now refer to as knot theory. He developed the "Gauss Linking Integral" for computing the linking number of two knots. He was fascinated by this discovery and went on to prove that the linking number is unchanged under ambient isotopy. This is the earliest discovered link invariant. This was the first method for studying about the two non-equivalent links from each other.

Johann Benedict Listing who was a student of Gauss in the 1830's was interested in knots during his study of topology. He was interested in the chirality of knots, or the equivalence of a knot to its mirror image. His paper included the significant result about trefoil knot with the statement that the right and left trefoil knots are not equivalent, or not amphichiral. Later, he showed that the figure eight knot and its mirror image are equivalent, or amphichiral.

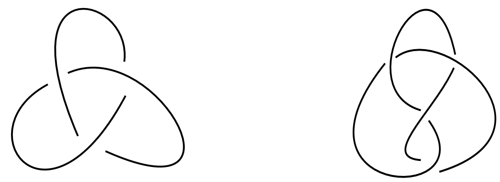



Figure 1.1. Examples of Knots

Knot theory was popular among physicists as well as mathematicians. In 1867, the English physicist, Sir William Thomson (Lord Kelvin) theorized that atoms were knots of swirling vortices in the aether. He suggested that atoms could be classified by the knots that they resembled and the representative knot would help to identify some physicochemical properties of the atom. In 1858 the work of physicist Hermann Von Helmboltz presented a foundation for Thomson's theory of vortex atoms. He had written a paper titled "On the integrals of hydrodynamic equations to which vertex motions conform" involving the concepts about "aether". Helmholtz analyzed the idea that vortices of this theoretical aether, an ideal fluid, were stable. It followed that these stable vortices could become knotted and still retain their original identities.

A friend of Thomson's the physicist James Clerk Maxwell, also developed a strong interest in knots. He was interested in the fact that knots could be used in the study of electricity and magnetism. In 1873, he wrote a paper entitled *Treatise on Electricity and Magnetism* using the idea of Gauss in relating knots to physics. During the period of his course work, he studied Listing's work on knots and remodeled Gauss' linking integral in terms of electromagnetic theory. Also, he created the knot diagrams with over and under crossings, and then explored how change the diagram without affecting the knot type. Maxwell analyzed a region bounded by three arcs and defined the three Reidemeister moves before they were named in the 1920's.

Peter Guthrie Tait, Thomas Kirkman, and Charles Newton Little made a great contribution by tabulating all possible knots with fewer crossings. Physicist Tait began making the first table of knots in 1867. With Thomson's theory of vortex atoms, Tait needed to classify knots according to the number of their crossings.

Mathematician Thomas Kirkman made the first major contribution to the task of classifying knots. He was only interested to classify knots for alternating knots and tabulated diagrams for alternating knots with up to eleven crossings. Although some diagrams which were found by Tait to be equivalent, this was a still a significant early development. After categorizing the table of alternating knot diagrams, he saw that there were some duplicate diagrams. He has used a similar method to the second Reidemeister move. As a result, he made an accurate table of knots.

Kirkman's work on the classification of the knots was continued by Tait and Charles Newton Little. They found some repeated knots in this table and remodeled it again after examining with the different methods of notation, including Listing's notation and Gauss' "scheme of knots". Consequently, they have published the table of alternating knots with up to ten crossings. However, Little was quite interested in classifying non-alternating knots. After six years of hard work, in 1899, Little published a table of forty-three ten-crossing, non-alternating knots, including 551 variations of the already classified diagrams.

Among the early knot theorists, Tait's contributions were important and varied. He defined the reduced knot diagram. Then Tait partnered with Little to work on the classification of knots, he was quite interested about the properties of reduced knot diagram, and how to obtain them. He defined a new concept called *a nugatory* crossing as a crossing that divides a diagram into two non-intersecting parts, as in Figure 1.2.

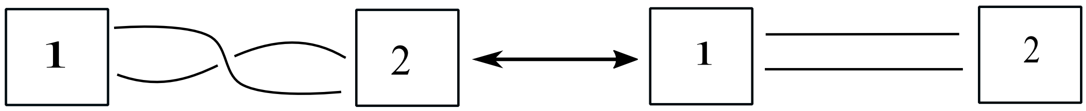



Figure 1.2. Removal of a nugatery crossing by twisting

He stated that the removable crossing could be added or removed from a diagram by twisting. When he tabulated knots using reduced knot diagrams, he came up with what we now know as the *Tait's three conjectures*. Now all of then have proved and most of them are true only for alternating knots.

Tait's knot conjectures:

1. Reduced alternating knot diagrams have minimal link crossing number.
2. Any two reduced alternating diagrams of a given knot have equal writhe.
3. The number of crossings is the same for any reduced knot diagram of an alternating knot.

## 1.2 LATER WORK

Mathematicians were very much interested in studying knot theory as a new subject area. In the 1920's, a mathematician was interested into apply knot theory for studying another subject areas. Thus braid theory was presented by Emil Artin in the early 1920's. A braid is defined as "a set of n strings, all of which are attached to a horizontal bar at the top and at the bottom such that each string intersects any horizontal plane between the two bars exactly once,"[2].

We can draw the projection of the braid on the plane. Then we close the open ends of the braid by adding arcs around one side of the braid. This is called the closure of the braid.

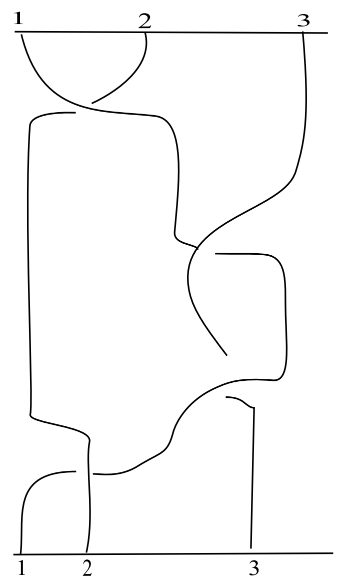



Figure 1.3. Example of a braid

In 1888, James Waddell Alexander (1888-1971) was also becoming interested in knot theory and he noticed a relation between knots and braids. He did experiments to find how to obtain knots and links by the closure of braid. Finally in 1923, he proved that every link can be expressed as a closed braid. As a result of this, every knot can be represented as a closed braid. Alexander discovered the first knot polynomial and he proved that it was a knot invariant in 1928, which allowed him to find a valuable means to distinguish many non-ambient isotopic knots from one another. It was essentially the only knot polynomial invariant until discovery of the Jones polynomial in 1984. The Alexander polynomial defined by

$\Delta_k(t) \doteq det(\Delta_k(t))$,

where $\doteq$ represents equality up to factors of the form $\pm t^n$ [2], and

$\Delta_k(t)$ is the reduced matrix obtained from the Alexander matrix of an oriented diagram by deleting the last two columns. It was a major finding in knot theory, even though it was not a complete invariant. That is there is some non-ambient isotopic knots with same Alexander polynomial. Other than this, it is difficult to identify the chirality of knots using the alexander polynormial. In the 1960's, John Conway reworked the Alexander polynomial, making it unique, and capable of identifying the chirality in some cases.

Kurt Reidemeister was an ingenious mathematician. He made significant contributions to development of knot theory in the 1920's. His work basically related to planar diagrams of knots. At the beginning, he struggled to develop new methods to categorize knots. But his effort was useless as neither an analytic nor a combinatorial approaches give sufficient information to manipulate the knot or draw a knot diagram. Reidemeister changed his point of view towards the methods of classification by diagram. Tait, Little, and Kirkman, published knot diagrams specified with over and under crossings. So, he has used those diagrams to explain equivalent between two knot diagrams. Finally, he proved that "Two knots $K, K'$ with diagrams $D, D'$ are equivalent if and only if their diagrams are related by a finite sequence $D = D_0, D_1, D_2, ...D_n = D'$ of intermediate diagrams such that each differs from its predecessor by one of the following three Reidemester moves" [2].

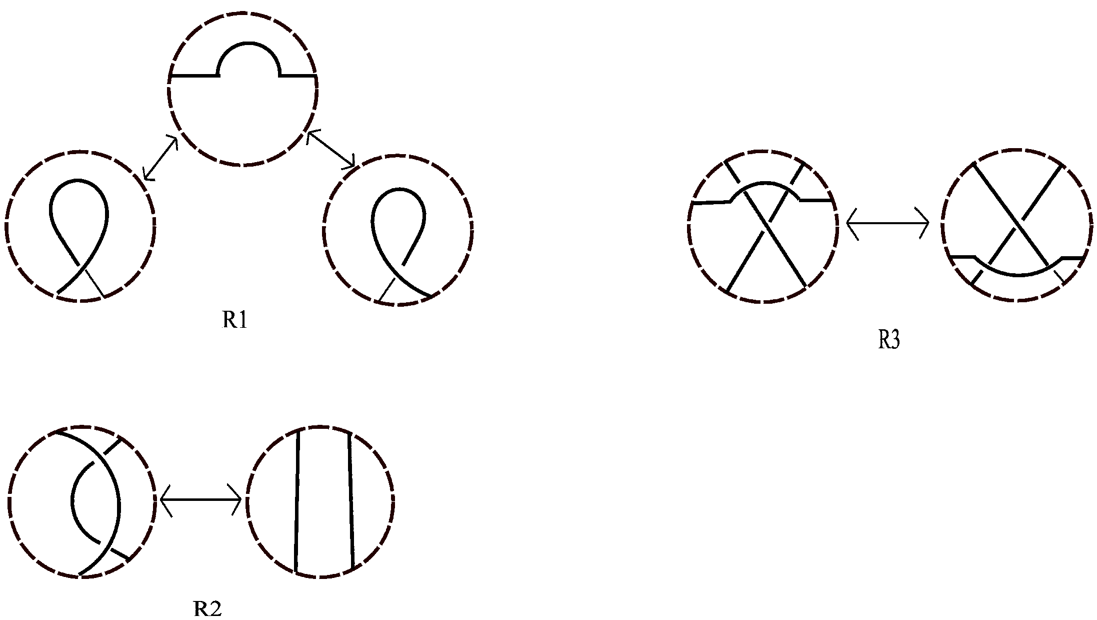



Figure 1.4. Three Reidemester moves

Regular isotopy is a relation defined by using only second and third Reidemeister moves. But if all three moves are used, it is referd to as "ambient isotopy".

The idea about Reidemeister moves had been given by Maxwell several years before. But it was not expressed clearly until Reidemeister. The most important remark in Reidemeister's study was the proof that these three moves were the only three needed to illustrate the equivalence of two knots [2].

## 1.3 CONTEMPORARY

English mathematician John Conway discoverd a new method for knot notation in the 1960's. Then, he had worked with the Alexander polynomial and normalized it. Conway's knot notation was based on the tangle which had been introduced by W.B.R.Lickorish. "A region in a knot or link projection plane surrounded by a circle such that the knot or link crosses the circle exactly four times" is known as the *tangle* [2].

Two tangles are said to be equivalent if it is possible to transform one into another by using a sequence of Reidemeister moves, while keeping the end points fixed [2].

Conway defined three axioms for the Conway polynomial, $\bigtriangledown(x)$:

1. Invariance: $K \sim K'$.

2. Normalization: $\bigtriangledown(O) = 1$, where O is any diagram of the unknot.

3. Skein Relation: $\bigtriangledown(K_+) - \bigtriangledown(K_-) = x \bigtriangledown (K_0)$.

Conway's skein relation uses the following notations:

$K_+, K_-, K_0$ are knot diagrams that differ only inside a disk in the manner shown in Figure 1.5. [5]

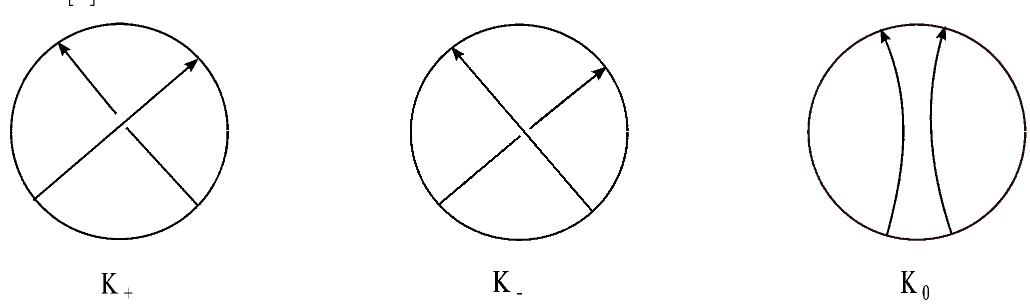



Figure 1.5. Skein relations

Conway's polynomial is related to Alexander's polynomial as follows, $\Delta_K(t) \doteq \nabla_K(\sqrt{t} - \frac{1}{\sqrt{t}})$ [5].

Conway tried to prove the knot invariance of his polynomial by using the Reidemeister moves. But there were some cases which have the same Conway polynomial for non-ambient isotopic knots. However, the chirality of knots in some cases can be analyzed by using the Conway polynomial, which could not be done using the Alexander polynomial. Since it is still not a complete invariant, mathematicians are significantly interested in searching for more sensitive polynomial.

There was critical finding in the mathematical field of knot theory, the Jones polynomial, which is a knot polynomial found by Vaughan Jones in 1984. Jones was awarded the Fields medal in 1990 for his work.

Let $L$ is an oriented knot (or link) and $K$ is a oriented regular diagram for $L$. Then the Jones polynomial of $L$, $V_L(t)$ is a polynomial in which satisfies the following three axioms:

1. The polynomial $V_L(t)$ is an invariant of $L$.

2. Normalization: $V_0(t) = 1$.

3. Skein Relation: $t^{-1}V_{K_+} - tV_{K_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{K_0}$, where $K_+, K_-, K_0$ are skein diagrams (cf. Figure 1.4) [4].

This is the only new polynomial knot invariant under the ambient isotopy since the Alexander polynomial, and it can be used to distinguish two non-equivalent knots from one another. After finding the Jones polynomial, it has been calculated for knots up to thirteen crossings and all of the knots had unique polynomials except for two knots, each with eleven crossings. After that, it was revealed that those two knots were equivalent and the knot table was corrected. Also, it was the first polynomial which can detect the knot's handedness, it distinguishes the right trefoil from the left trefoil. But Jones polynomial was not complete invariant and there exist non-ambient isotopic knots which have same Jones polynomial.

The Jones polynomial can be simplified to obtain an invariant called the Arf invariant, which always has value zero or one [2].

After Jones' finding, the HOMFLY/HOMFLY-PT polynomial was discovered by mathematicians- Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, J. Przytycki, and P. Traczyk. It is polynormial knot invariant. The HOMFLY polynomial, $P_K(a, z)$ has two variables, a and z which satisfies the following three axioms:

1. Invariance: $K \sim K' \Rightarrow P_K = P_{K'}$.

2. Normalization: If $K$ is the trivial knot $O$, then $P_O = 1$.

3. Skein Relation: $a^{-1}P_{K_+}(a, z) - aP_{K_-}(a, z) = aP_{K_0}(a, z)$, where $K_+, K_-, K_0$ are skein diagrams (cf. Figure 1.5) [7].

It is a generalization of the Jones polynomial, and in most cases detects chirality.

Still mathematicians were searching to find complete polynomial knot invariant since after finding the Alexander polynomial. Several findings are disclosed during that time and knot theorists were showing continued hard work to develop polynomial knot invariant since Jones' work. After that, the mathematician Louis H. Kauffman discovered another approach to the Jones polynomial in 1985. Before finding this, he found another polynomial called the BLIM/HO invariant, also called the bracket polynomial. He started by defining a polynomial in variables A, B and d which satisfied the following axioms:

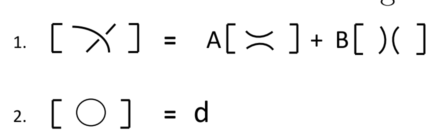



Figure 1.6. BLIM/HO invariant

In the second axiom, 0 represents the unknot [2].

Kauffman introduced the idea of a state, S, for a knot diagram, D, as a choice of a splitting marker for each vertex of D such that the two A regions which is the area on right one's if we travel along the bottom of the over-pass towards the crossing, and region B is the area on left.

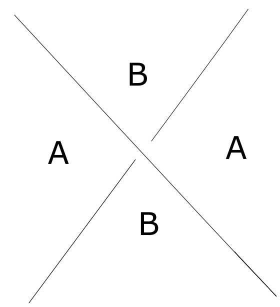



Figure 1.7. Regions

Then A regions are connected by A-splicing and the B regions are connected by the B-splicing. Two possible states are shown in Figure 1.8:

Figure 1.8. Splitting marker

Kauffman's bracket polynomial could be defined by the following "state sum" formula:

$$[K] = \sum_{S} A^{\alpha(S)-\beta(S)} d^{|S|-1}$$

Where the sum is taken over all states S of the diagram D of the knot $K$ and $\alpha(S)$ is the number of $A$-splicing, $\beta(S)$ is the number of $B$-splicing, and $|S|$ is the number of components of the state [2].

Also, it is easy to verify that if we set $B = A^{-1}$ and $d = (-A^2 - A^{-2})$, then

$$[K] = \sum_{S} A^{\alpha(S)-\beta(S)} (-A^2 - A^{-2})^{|S|-1}$$

$[K]$ is a regular isotopy invariant, that is, it is invariant under the second and third Reidemeister moves and but is not ambient isotopy invariant, i.e. first Reidemester move produces the following changes to bracket polynomial:

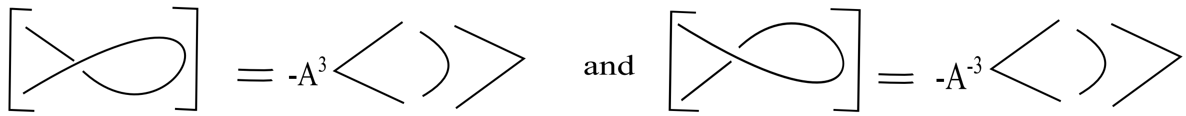



Figure 1.9. Kauffman's bracket polynomial is not ambient isotopy invariant

Since the Kauffman's bracket is not invariant under $R1$, he did some modifications on his polynomial using the writhe of an oriented knot diagram.

The writhe of an oriented diagram is the sum of the signs of all its crossings; by assigning the $+1$ for the crossing

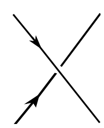


Figure 1.10. Crossing 1

and $-1$ for the crossing

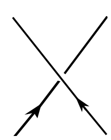


Figure 1.11. Crossing 2

The writhe is regular isotopy invariant.

So he defined new polynomial, actually a Laurent polynomial, associated with the writhe.

Let $L$ be an oriented link and let $K$ be an oriented diagram of $L$ with writhe $w(K)$. Then,

$f[L] = (-A)^{3w(K)}[K]$.

$f[L]$ is a ambient isotopy invariant knot polynomial. He initially thought that his discovery was an original invariant of links, but soon be realized that he had discovered a different method for obtaining the Jones polynomial [2].

# CHAPTER 2

# BACKGROUND

Mathematicians have the described the knot concept in various ways. In this chapter we give several definitions of knots and links, starting with the most general one.

**Definition.** A subset K of a space X is a knot if K is homeomorphic with a sphere $S^p$. More generally, $K$ is a link if $K$ is homeomorphic with a disjoint union $S_1^p \cup_{\dots} \cup S_n^p$ of one or more spheres [3].

In this paper we will consider the special case, a knot is a continuous, one-to-one map $K : S^1 \to S^3$ or $\mathbb{R}^3$.

Since $S^3$ is homeomorphic with the one point compactification of $\mathbb{R}^3$, these two knot theories are essentially the same.

Figure 2.1 contains some examples of knot and link diagrams.

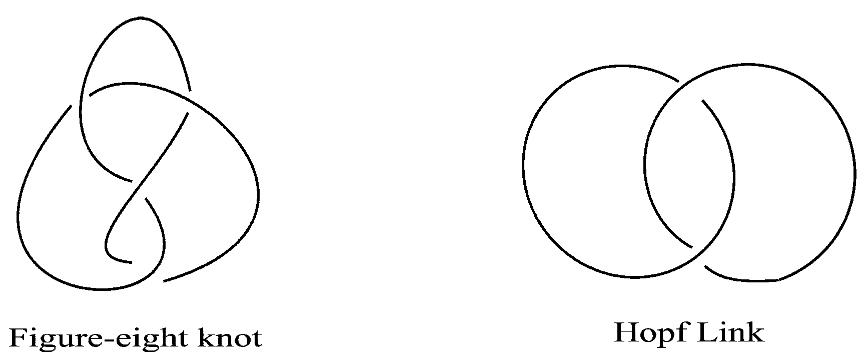



Figure 2.1. Figure-eight knot and Hopf link

## 2.1 KNOT EQUIVALENCE

Knot equivalence is an important notion when we are going to study two knots considered the same even if they are positioned quite differently in the space. The notion of equivalence satisfies the definition of an equivalence relation; it is reflexive, symmetric, and transitive. Knot theory consists of the study of equivalence classes of knots.

The notion of knot equivalence has been defined for various cases.

**Definition.** (map equivalence)

Two knots $K_1$ and $K_2$ are map equivalent if there exist a homeomorphism of $X \to X$, where $X = S^3$ or $\mathbb{R}^3$ such that $h(K_1) = K_2$ [3].

In general, it is difficult to study whether two knots are equivalent or not. Another formal mathematical definition is given below.

**Definition.** (oriented equivalence)

Two knots $K_1$ and $K_2$ are oriented equivalent or, $K_1$ is oriented equivalent to $K_2$, if there exists an orientation-preserving homeomorphism of $\mathbb{R}^3$ to itself that maps $K_1$ to $K_2$ [3].

Note: Let $f : X \to Y$ be a map from an oriented manifold $X$ to an oriented manifold $Y$. If the orientation of $f(X) \subset Y$ induced by $f$ is same as the orientation it inherits from $Y$, then $f$ is orientation preserving.

I.e. $h$ should be preserved orientation of the space.

For example figure eight knot and its mirror image is oriented equivalence.

**Definition.** (amphicheiral)

A knot $K$ is said to be amphicheiral if there exist an orientation reversing homeomorphism $h$ of $\mathbb{R}^3$ onto itself such that $h(K) = K$.

Figure eight knot is amphicheiral.

**Theorem 2.1.1.** *A knot $K$ is amphicheiral if and only if there exist an orientation preserving homeomorphism of $\mathbb{R}^3$ onto itself which maps $K$ onto its mirror image.*

Hence it is followed from the theorem that figure eight knot and its mirror image is oriented equivalence.

**Definition.** Let $K : S^1 \to S^3$ be a topological knot. Parameterize $S^1$ using the standard polar angle $\theta$, that is $(1,0)$ has $\theta = 0$ and then $\theta$ increases through $[0, 2\pi)$ going counterclockwise. Let $K^+ : [0, 2\pi) \to S^3$ be a one-to-one continuous function whose image is $K(S^1)$ that can be continuously extended to $[0, 2\pi]$ with $K^+(2\pi) = K^+(0)$. define $K^- : (0, 2\pi] \to S^3$ by $K^-(\theta) = K^+(2\pi - \theta)$. Then we can regard $K^+$ and $K^-$ as *oriented knots*.

**Definition.** A topological isotopy from $K_1$ to $K_2$ is a continuous map. $i : I \times S^1 \to S^3, I = [0, 1]$ such that $i(0, -) = K_1, i(1, -) = K_2$ and $i(t, -) = K_t$ is a knot for all $t \in I$.

For example, trefoil knot and its mirror image are map equivalent, but not ambient isotopic. See Figure 2.2.

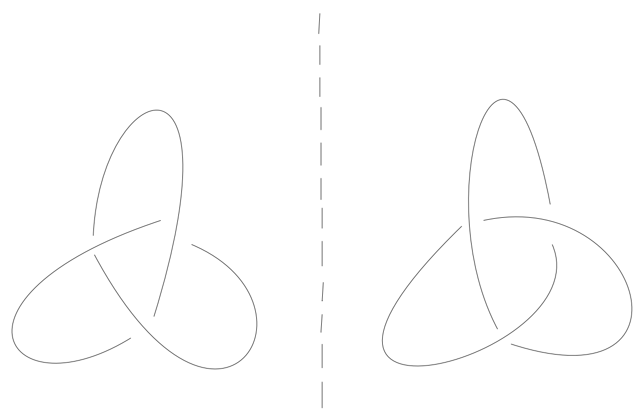


Figure 2.2. Trefoil knot and its mirror image

I.e. we can not find a homeomorphism $h$ which can get one from another. But the figure-eight knot and its mirror image are isotopic. We can get one from another by manipulating a string tie as on the left and transform it to being tied as on the right as shown in Figure 2.3.

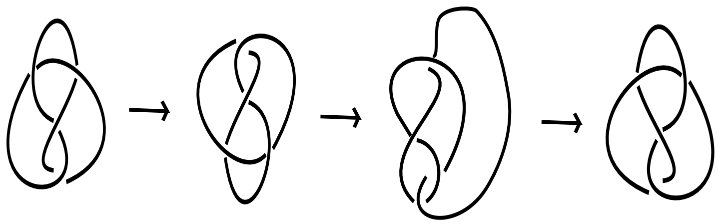


Figure 2.3. figure-eight knot and its mirror image is isotopic

Thus the Figure eight knot and its mirror image are map equivalent.

Note : The equivalence class of knot or link in a space X is called its knot type or link type.

But it is difficult to study about knot equivalence using knots in the 3-dimensional space. So, we are going to knot diagrams to check whether two knots are equivalent or not. The elementary knot moves are applied to PL knots in $\mathbb{R}^3$. We will pursue this topic in chapter 3.

**Definition.** (Topological ambient isotopy)

A *topological ambient isotopy* carrying $K_0$ to $K_1$ is a continuous map $h : I \times S^3 \to S^3$ such that $h(t, -)$ is a homeomorphism of $S^3$ for all $t \in I$, $h(0, -)$ is the identity map, and $h(1, -) \circ K_0 = K_1$ [1]

In addition $h(t, -) \circ K_0$ defines an isotopy from $K_0$ to $K_1$, and $h(1, -)$ is a homeomorphism from $K_0$ to $K_1$. This definition gives a non-trivial notion of equivalence.

**Definition.** (Smooth Knot)

A *smooth knot* is a smooth embedding $K : S^1 \to S^3$.

In particular, $K'$ is never vanishing. All smooth knots are tame topological knots. Wild knot is an example which is topological knot, but not smooth.

Note: Polygonal knot $K : S^1 \to \mathbb{R}^3$ is a knot whose image in $\mathbb{R}^3$ is the union of finite set of line segments. Tame knot is any knot equivalent to the polygonal knot. Knots which are not tame is called wild knot. Wild knots are not considered in this report.

**Definition.** (Smooth isotopy)

A smooth isotopy from $K_0$ to $K_1$ is a smooth map $h : I \times S^1 \to S^3$ such that $h(0,-) = K_0$, $h(1,-) = K_1$, and $h(t,-) = K_t$ is a knot for all $t \in I$.

**Definition.** (Oriented equivalence for smooth knots)

A diffeomorphism between $K_0$ and $K_1$ is an orientation-preserving diffeomorphism $h : S^3 \to S^3$ such that $h \circ K_0 = K_1$.

All spaces are endowed with orientations, all of which $h$ is required to preserve.

**Definition.** (Smooth ambient isotopy)

A smooth ambient isotopy carrying $K_0$ to $K_1$ is a smooth map $h : I \times S^3 \to S^3$ such that $h(t,-)$ is a diffeomorphism of $S^3$ for all $t \in I$, $h(0,-)$ is the identity map, and $h(1,-) \circ K_0 = K_1$.

**Definition.** (Piece-wise Linear Knot)

Let $K : S^1 \to S^3$ be a knot. A knot $K$ is piecewise linear or PL if its image in $S^3$ is a union of finite number of line segments.

A piecewise linear knot can be represented by a diagram whose arcs are straight lines. Eg:

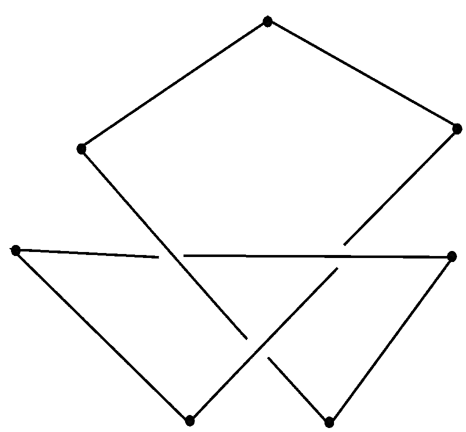


Figure 2.4. Example for PL knot

**Definition.** (Oriented Equivalence for PL knots)

Two PL knots $K_0$ and $K_1$ are equivalent if there exists an orientation preserving isomorphism $h : S^3 \to S^3$ such that $h \circ K_0 = K_1$.

We can define equivalent (or ambient isotopy) for PL knots and it is same definition applies to equivalence in topological category and the smooth category. But the map $h$ should be isomorphism.

Remark: There is a problem with topological isotopy definition. For example, consider trefoil knot in $\mathbb{R}^3$. If two joining ends of trefoil knot is pulled, then knot is pulled, then knotted part is shrinked to a point and thus it is unknot.

i.e. Trefoil knot is equivalent to an unknot according to the definition.

Also, definition for ambient isotopy defined for each category is slightly different as map $h$ should be homeomorphism for topological knots, diffeomorphism for smooth knots, and isomorphism for PL knots.

Note: Smooth knots and PL knots are obviously topological knots and it can be shown that they are always tame.

# CHAPTER 3

# PROJECTIONS AND DIAGRAMS

## 3.1 PROJECTIONS

A knot is an embedding of a circle in 3-dimensional Euclidean space ($\mathbb{R}^3$) or the 3-sphere. A knot in $\mathbb{R}^3$ can be projected onto the plane $\mathbb{R}^2$. This projection is almost always *regular*, i.e. it is injective everywhere, except at a finite number of double points, also called crossing points. Projections of a knot onto the plane allow the representation of a knot as a *knot diagram* on the $xy$-plane. It is difficult to study knots inspace, $\mathbb{R}^3$. So in order to study knots it is useful to consider projections of the knots on the $xy$-plane.

Let $p$ be the map that projects the point $P(x, y, z)$ in $\mathbb{R}^3$ onto the point $P(x, y, 0)$ in the $xy$-plane [5].

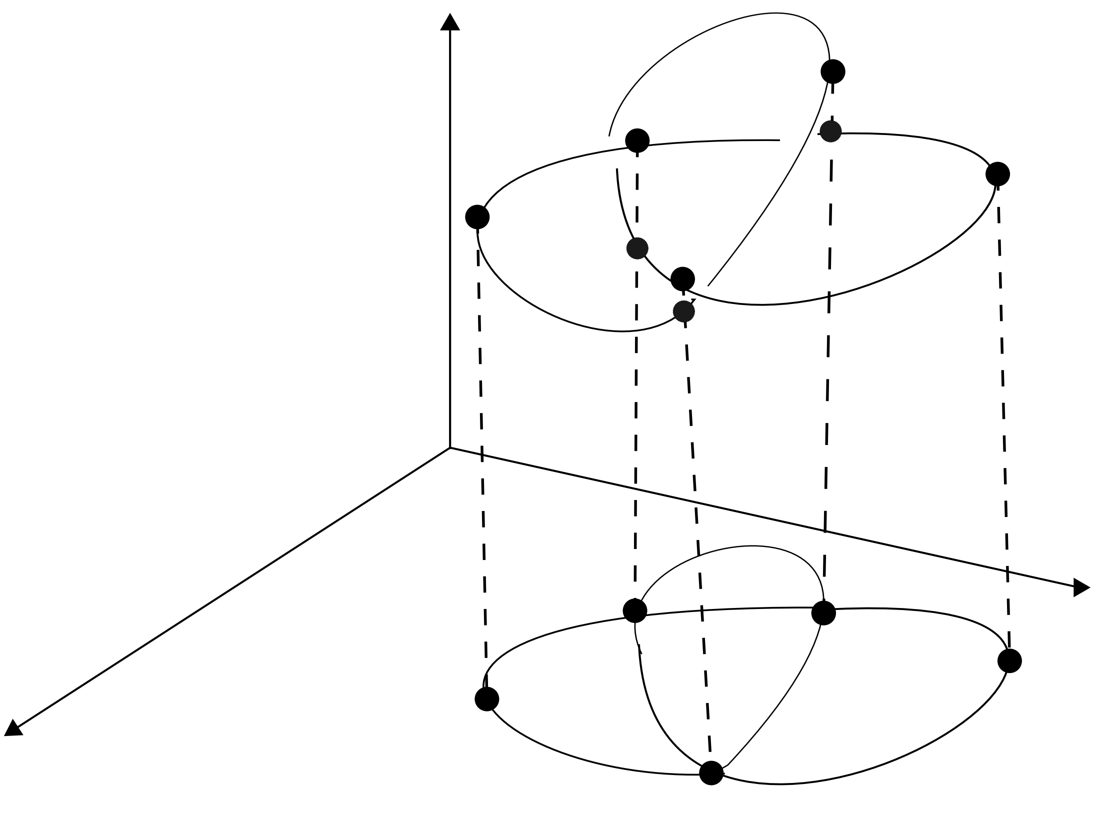


Figure 3.1. Projection of Trefoil knot in the $xy$-plane

We shall say that $p(K) = \hat{K}$ is the projection of $K$. If $K$ has orientation assigned , then $\hat{K}$ inherits its orientation from the orientation of $K$.

**Definition.** (Regular Projection)

Let $\hat{K}$ be the projection of $K$ in the $xy$-plane. A knot projection is called a *regular projection* if

1. all double points occurs at transverse crossings, and

2. for PL knots, no vertex of $K$ is mapped onto a double point.

Knot projections can cause some information to be lost, for example at a double point of a projection, it is not clear whether the knot passes over or under itself. So we slightly change the drawing of the projection close to each double point, drawing the projection so that it appears to have been cut. A diagram of a knot is the drawing of its regular projection in which are left gaps to remedy this fault. We say that the vertices have been decorated. For a particular knot type, the number of regular diagrams is uncountable.

A *knot diagram* is the regular projection of a knot onto a plane with over/under decorated vertices. A knort diagram give us a information about how the knot lies in 3-dimensions, and also we can use it to recover information lost in the projection. We call the arcs of this diagram edges and the points that correspond to two double points in the projection crossings.

Figure 3.1($b$) shows a projection of the figure-eight knot, while 3.1($a$) shows the decorated vertices.

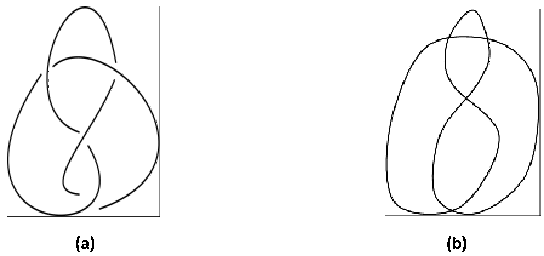



Figure 3.2. A projection of the figure-eight knot and the decorated vertices

Figure 3.3 shows the possible crossing points on a regular diagram.


Figure 3.3. crossing points on a regular diagram

The notion of equivalence between two knots can be studied using knot diagrams, i.e. when two different diagrams can represent the same knot. First of all, it is necessary to define the elementary knot moves.

**Definition.** On a given PL knot $K$ the following four operations are called elementary knot moves.

1. We may devide an edge, $AB$, in space of $K$ into two edges, $AC, CB$, buy placing a point $C$ on the edge $AB$, see Figure 3.4.1.

2. [The converse of (1)] If $AC$ and $CB$ are two adjecent edges of $K$ such that if $C$ is erased $AB$ becomes a straight line, then we may remove the point $C$, see Figure 3.4.1.

3. Suppose $C$ is a point in space that does not lie on $K$. If the triangle $ABC$, formed by $AB$ and $C$, does not intersect $K$, with the exception of the edge $AB$, then we may

remove $AB$ and add the two edges $AC$ and $CB$, see Figure 3.4.2.

4. [The converse of (2)] If there exists in space a triangle $ABC$ that contains two adgecent edges $AC$ and $CB$ of $K$, and this triangle does not intersect $K$, except at the edges $AC$ and $CB$, then we may delete the two edges $AC, CB$ and add the edge $AB$, see Figure 3.4.2.

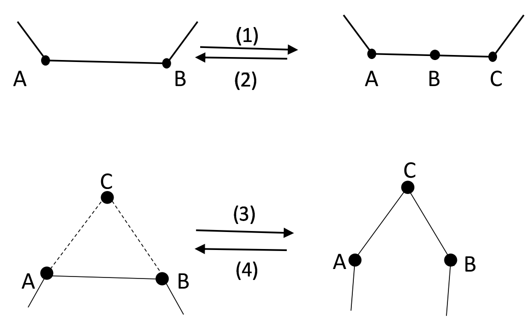



Figure 3.4. 3.4.1 and 3.4.2

[5]

Elementary knot moves are applied on smooth knots too, because every smooth knot is ambient isotopic to a PL knots. So if elementary knot moves are performed several times on a given knot, then it can seem to be a completely different knot. For example, the Figure 3.5 shows the resultant diagram after applying several the knot moves on the trefoil knot.

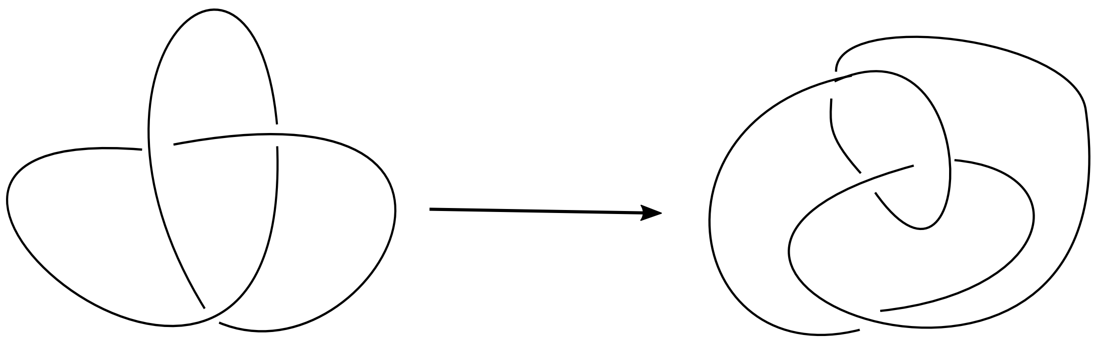


Figure 3.5. Elementary knot moves performed on trefoil knot

Knots that can be obtained one from another by applying the elementary knot moves are said to be equivalent or equal. Therefore, the two knots in Figure 3.5 are equivalent.

**Definition.** A knot $K$ is said to be equivalent (or equal) to a knot $K'$ if we can obtain $K'$ from $K$ by applying the elementary knot moves a finite number of times [5].
We shall denote this equivalence by $K \approx K'$.

One of the fundamental results of knot theory characterizes ambient isotopy in terms of an equivalence relation between knot diagrams. In 1920's the mathematician Kurt Reidemeister introduced the three Reidemeister moves. These are operations that can be performed on the knot diagram without changing the corresponding knot. In each of case, knot diagram has changed but, the knot type is unchanged.
The following figure shows the three Reidemeister moves.

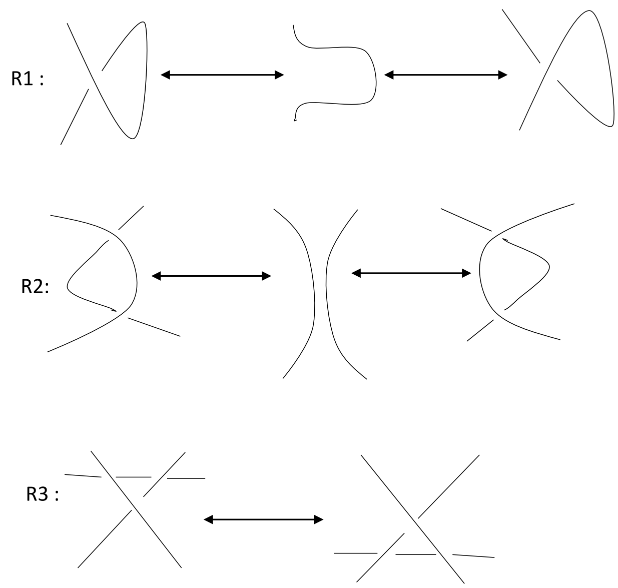



Figure 3.6. Three Reidemeister moves

Note: The diagrams $D$ and $D'$ are ambient isotopic if $D$ can be transformed into the $D'$ by using the moves $R_1, R_2$ and $R_3$, and regular isotopic if they differ by a sequence of Redemeister moves of types $R_2$ and $R_3$.

**Theorem 3.1.1.** *Two knots or links are equivalent if and only if their diagrams are related by a sequence of Reidemeister moves [5].*

*Proof.* We give the proof for the PL knots. Let $D$ be a knot diagram corresponding to the knot $K$, and $D'$ be a diagram corresponding to the knot $K'$.
Suppose $K$ is equivalent to $K'$, i.e. $K \approx K'$.
Then, there is a sequence of knots $K = K_0, K_1, K_2, ..., K_n = K'$ with each $K_{i+1}$ is an elementary deformation of $K_i$. Each $K_i$ can be projected to a plane.
Let $AB$ be an edge of $K$.
Suppose $K'$ is obtained from $AB$ of $K$ by adding two edges $AC \cup CB$.

The triangle along which the elementary deformation is performed is projected to a triangle in the plane as seen on the figure. Also, we can divide knot diagram into a small triangle containing only one crossing or segment in each triangle. Since $D'$ is obtained from

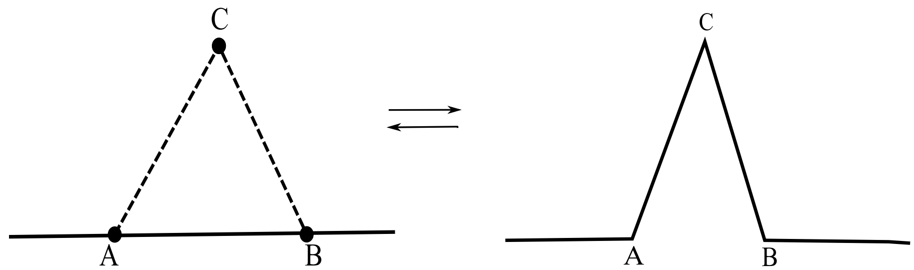



Figure 3.7. Replace $AB$ edge with two edges $AC$ and $CB$

$D$ by performing elementary deformation, we can view this elementary deformation as the composition of a series of other elementary deformation performed on smaller triangles. Next check whether we can change $AC \cup CB$ to the segment $AB$ by repeatedly using Reidemester moves.

**Case** $(i)$

When the intersection with the triangle is a segment and this segment is adjacent to the segment being deformed.

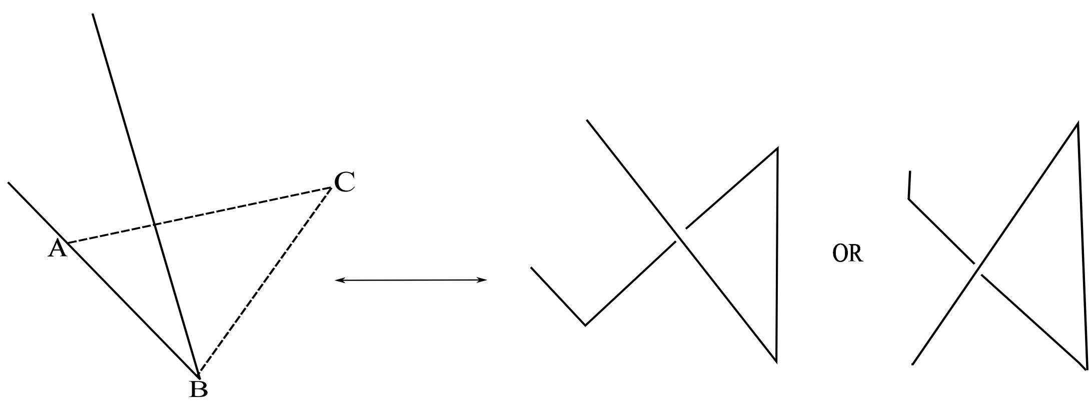



Figure 3.8. Intersection of $ABC$ triangle with a line adjacent to the $AB$ line

This process is the Redemester move $R_1$.

**Case** $(ii)$

When the intersection with the triangle is segment and that segment is not adjacent to the segment being deformed.

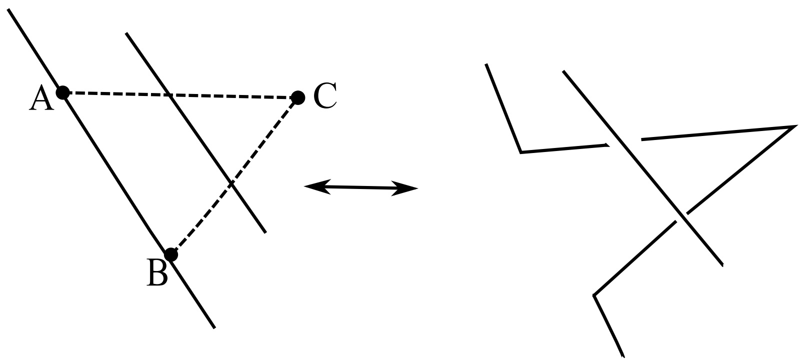



Figure 3.9. Intersection of $ABC$ triangle with a line segment

This process is the $R_2$ move.

<u>**Case** *(iii)*</u>

When the intersection with the triangle is a crossing.

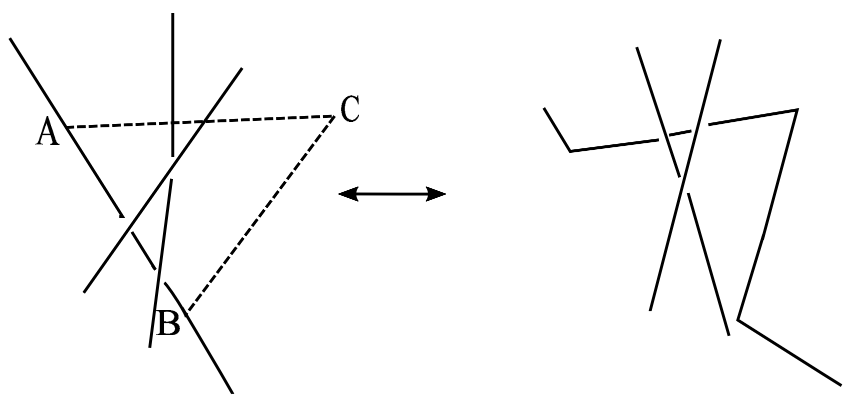



Figure 3.10. Intersection of $ABC$ triangle with a crossing

This process is the $R_3$ move.

□

A similar result hold for oriented knot. PL knots are ambient isotopic to a smooth knots and tame topological knots. So this theorem is true for smooth knots.

The proof is beyond he scope of this report.

### 3.2 KNOT INVARIANTS

The concept of a knot invariant plays important role when we are going to discuss whether two knots are equivalent or not. A knot invariant is a quantity defined for each knot and it is same for equivalent knots. Tricolorability is a particularly simple example. Most knot invariants are defined for knot diagrams in the plane and are unchanged under the Reidemeister moves.

The following shows some examples for the knot invariant.

Tricolorability

A knot diagram is 3-*colorable* if we can assign colors to its arcs such that

3C1 : Each arc is assigned one color,

3C2 : Exactly three colors are used in the assignment,

3C3 : At each crossing, either all the arcs have the same color, or arcs of all three colors meet [4].

Figure 3.11 gives an example of a 3- coloring of a knot diagram.

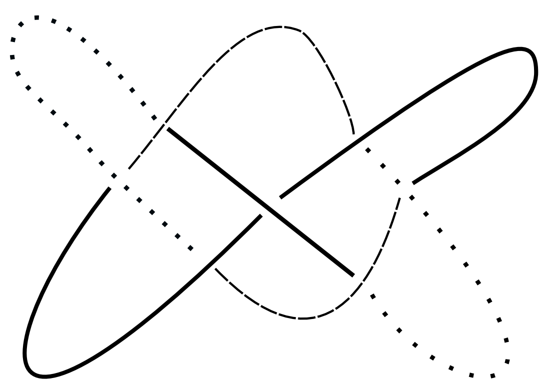


Figure 3.11. Example for 3-colorable knot diagram

If the knot diagram is 3-colorable, then we apply any Reidemester move on that diagram, resultant diagram is also 3-colorable.

Jones Polynormial

Jones polynormial defined for oriented knot and link diagrams.

JP1 : $V_U(t) = 1$, where U is the oriented unknot, JP2 : $t^{-1}V(K_+) - tV(K_-) = (t^{1/2} - t^{-1/2})V(K_0)$ where $K_+$, $K_-$, and $K_0$ are three oriented link diagrams that differ only inside a small disk in the manner shown in Figure 3.12.

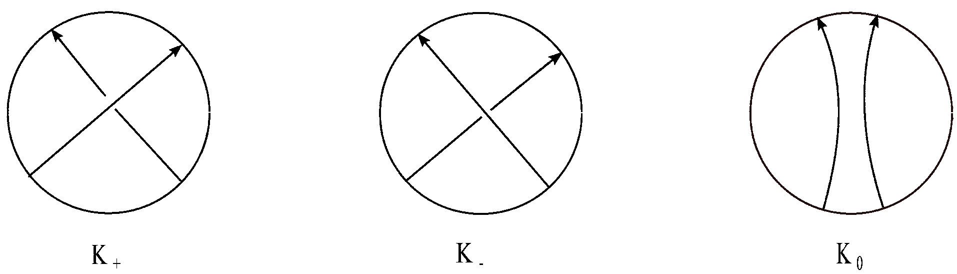



Figure 3.12. Skein Relations

Jones polynormial is an isotopy invariant of oriented knots or links. It may have same same polynomial for the knot diagrams in the same equivalence class.

# CHAPTER 4

# KNOT GROUP

The knot group is a important knot invariant defined for each knot. the knot group can be used to show that certain pairs of knots are not equivalent. But we can not always use knot groups to show that two knots are inequivalent. For example, the right-hand trefoil and its mirror image, the left-hand trefoil, have the same knot group, while they are non-equivalent.

**Definition.** (Knot Group)

The knot group of a knot $K$ is the fundamental group of the knot complement of $K$ in $S^3$, $\pi_1(S^3 \setminus K)$ [6].

**Theorem 4.0.1.** *The knot group is an invariant of ambient isotopy [6].*

We say two knots are equivalent if one can be transformed into another via an ambient isotopy of $S^3$ upon itself. By definition, two topological objects are equivalent if they are homeomorphic. So, if two knots are equivalent, the complements of knots under ambient isotopy are homeomorphic. Homeomorphic topological spaces have isomorphic fundamental groups. Hence knot group is invariant under ambient isotopy.
So this is one of the common method to distinguish inequivalent knots.

## 4.1 THE WIRTINGER PRESENTATION

In the beginning of the 20th century, Wilhelm Wirtinger found a general method for calculating the knot group for any tame knot in $\mathbb{R}^3$ or $S^3$. To calculate the knot groups, we are using knot diagram on the $xy$ - plane.

We need to start with an oriented knot diagram of a knot $K$ to construct the Wirtinger presentation. For any knot diagram of a tame knot consists of finitely many arcs with finite number of crossings at the ends where one arc bridges under another.

Let $K$ be a knot with $n$ number of arcs $a_0, a_1, a_2, ..., a_{n-1}$ and $m$ is the number of crossings.

At each crossings, the over pass arc is unbroken and thus each side is part of a same arc, while the under pass arc is broken, each side is associated with two different arcs (or in some cases, the two ends of the same arc). If $K$ is true knot (as opposed to a link), then $a_{i+1}$ is the arc that comes after $a_i$ with the given orientation.

Figure 4.1 shows the oriented figure-eight knot with labeled arcs.

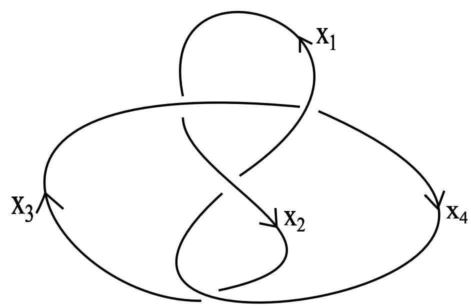



Figure 4.1. Oriented figure-eight knot with labeled arc

In order to construct the Writinger presentation, we need to consider the crossings of the knot. The Figure 4.2 shows the two possible crossings, based on orietations of the knot diagram:

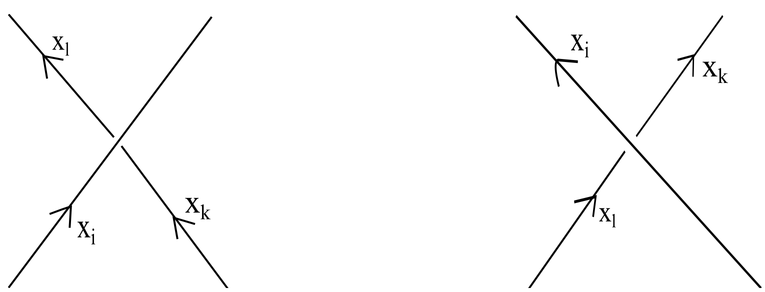



Figure 4.2. Crossings of the knot diagram

At each arc, write down the corresponding generator with exponent $+1$ if the arc is entering the crossing and $-1$ if the arc is leaving the crossing.

We define the relations between the group generator as follows:

1. $x_l = {x_i}^{-1} x_k x_i$ (called the $r_l$ relation)

2. $x_k = x_i x_l {x_i}^{-1}$ (called the $r_k$ relation)

The symbol $x_i$ represents the loop that starting from a base point in the complement of knot, goes straight to the $i^{th}$ arc, froming a circle around it in a positive direction and returns directly to the base point. The resulting presentation is called the Wirtinger presentation of the knot group.

**Theorem 4.1.1.** *The group $\pi_1(\mathbb{R}^3 - K)$ is generated by the (homotopy classes of the) $x_i$ and has presentation*

$$\pi_1(\mathbb{R}^3 - K) = (x_1, x_2, x_3, ..., x_n | r_1, r_2, ..., r_n)$$

Moreover any one of the $r_i$ may be omitted and the above remains true [3].

Example (1)(The granny knot)

The granny knot $K$ is obtained by taking the connected sum of two identical trefoil knots.

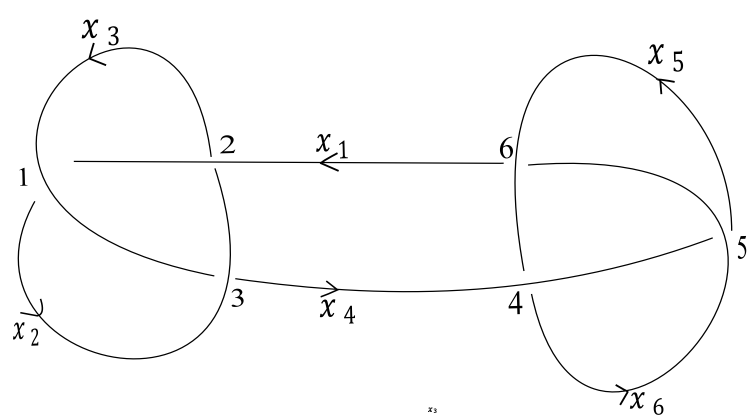
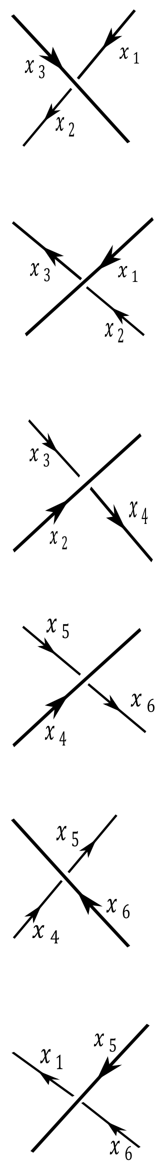



Figure 4.3. The granny knot

Let $\{1, 2, 3, 4, 5, 6\}$ denote the crossings and $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ denote the group generators corresponding to the over passing arcs.

Since we have 6 over passes, we know there are 6 at most generators. So we will use the Wirtinger presentation to find all the relations for this group.




$$x_2 = x_3^{-1} x_1 x_3$$

$$x_3 = x_1^{-1} x_2 x_1$$

$$x_4 = x_2^{-1} x_3 x_2$$

$$x_6 = x_4^{-1} x_5 x_4$$

$$x_5 = x_6^{-1} x_4 x_6$$

$$x_1 = x_5^{-1} x_6 x_5$$

Figure 4.4. Wirtinger presentation of the granny knot

$x_4 = {x_2}^{-1} x_3 x_2,$

$\Rightarrow x_2 x_4 = x_3 x_2$ .........(1).

Sbstitute $x_3 = {x_1}^{-1} x_2 x_1$. Then we have,

$x_2 x_4 = {x_1}^{-1} x_2 x_1 x_2,$

$\Rightarrow x_1 x_2 x_4 = x_2 x_1 x_2.$

Consider

$x_2 = {x_3}^{-1} x_1 x_3,$

$\Rightarrow x_3 x_2 = x_1 x_3$ .........(2).

Also, $x_3 = {x_1}^{-1} x_2 x_1,$

$\Rightarrow x_1 x_3 = x_2 x_1$ .........(3).

By equation (1), (2), and (3),

$x_2 x_4 = x_2 x_1.$

Thus we have,

$x_1 x_2 x_1 = x_2 x_1 x_2.$

Consider $x_1 = {x_5}^{-1} x_6 x_5,$

$\Rightarrow x_5 x_1 = x_6 x_5.$

Sbstitute $x_6 = {x_4}^{-1} x_5 x_4$. Then we have,

$x_5 x_1 = {x_4}^{-1} x_5 x_4 x_5,$

$\Rightarrow x_4 x_5 x_1 = x_5 x_4 x_5,$

$\Rightarrow {x_2}^{-1} x_3 x_2 x_5 x_1 = x_5 {x_2}^{-1} x_3 x_2 x_5$ (Since $x_4 = {x_2}^{-1} x_3 x_2$ ),

$\Rightarrow {x_2}^{-1} x_2 x_1 x_5 x_1 = x_5 {x_2}^{-1} x_2 x_1 x_5$ (By (2) and (3), we have $x_3 x_2 = x_2 x_1$),

$\Rightarrow x_1 x_5 x_1 = x_5 x_1 x_5.$

$\Pi_1(s^3 - K) = \{x_1, x_2, x_5 \mid x_1 x_2 x_1 = x_2 x_1 x_2, x_1 x_5 x_1 = x_5 x_1 x_5\}.$

Example (2)(The Square Knot)

The square knot $K'$ is a composite knot (knot is decomposable) obtained by taking the connected sum of a trefoil knot with its reflection.

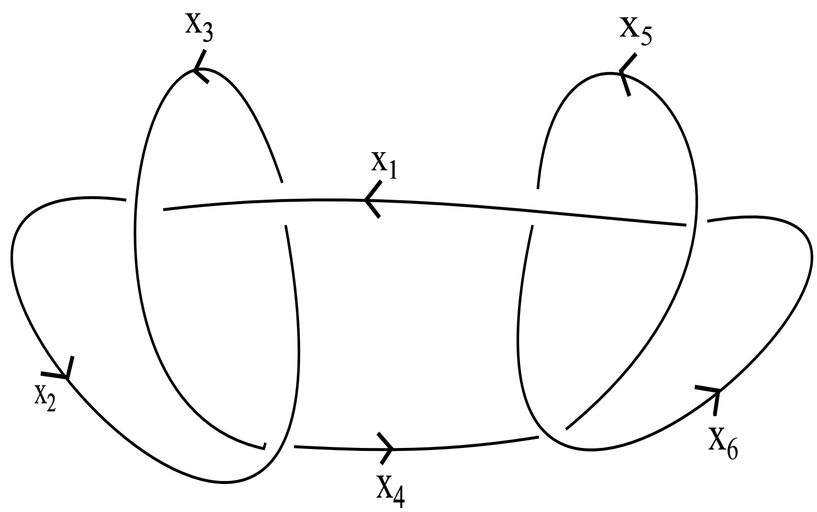
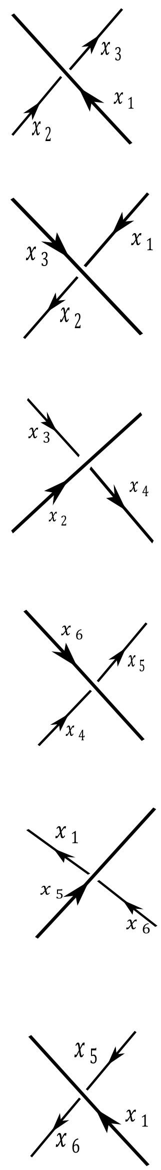



Figure 4.5. The Square Knot

We will use the Wirtinger presentation to find all the relations for the knot group of the square knot.



$$x_3 = x_1^{-1}x_2x_1 \quad \text{.......... (1)}$$

$$x_2 = \bar{x}_3^{1}x_1x_3 \quad \text{.......... (2)}$$

$$x_4 = \bar{x}_2^{1}x_3x_2 \quad \text{.......... (3)}$$

$$x_5 = \bar{x}_6^{1}x_4x_6 \quad \text{.......... (4)}$$

$$x_6 = x_1x_5x_1^{-1} \quad \text{.......... (5)}$$

$$x_1 = x_5x_6\bar{x}_5^{1} \quad \text{.......... (6)}$$

Figure 4.6. Wirtinger presentation of the square knot

Relations (1), (2), and (3) are same as the first three relations which is obtained in the example (1). Thus we have,

$$x_1x_2x_1 = x_2x_1x_2.$$

$(6) \Rightarrow x_6x_1 = x_1x_5,$

$$x_1x_5 = x_5x_1x_5{x_1}^{-1},$$

$$x_1x_5x_1 = x_5x_1x_5.$$

$$\Pi_1(s^3 - K') = \{x_1, x_2, x_5 \mid x_1x_2x_1 = x_2x_1x_2, x_1x_5x_1 = x_5x_1x_5\}.$$

Note : square knot and granny knot are not oriented equivalent or equivalent. But knot groups are isomorphic. This is the another example for showing that the knot group is not a complete invariant.

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