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Analyzing Predator-Prey Models Using Systems of Ordinary Linear Differential Equations

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Project Abstract:

The main operating concern of all species in any ecosystem or natural environment is rooted in the battle for survival. This constant battle for survival is most highlighted in the two main modes of species interaction; categorized as predation or competition. This research focused on applying biological mathematics to analyzing predation relationships, especially the relationship between the Canadian Lynx and the Snowshoe Hare. This predation relationship is quite special, because these species interact in a relatively isolated manner compared to others, meaning their populations fluctuated in a regular cycle due to lack of significant external variables on the relationship. These population fluctuations can be defined and analyzed mathematically using systems of linear ordinary differential equations, built of course upon several minimizing assumptions in order to exclude incalculable variables. This mathematical model, the Lotka-Volterra, can then be analyzed analytically or using computer simulation to determine period lengths, phase portraits, critical points, and other practical information to the reality of the relationship. Ability to analyze and predict such relationships can be quite useful in the biology field when studying extinction based on predation, or even excessive coevolution based on population interaction.

Research Project Background:

Predator-Prey Interaction

   The main operating concern of all species in any ecosystem or natural environment is rooted in the battle for survival. If a species of fish migrates over time to a different area, it is likely because they are less hunted or because of a greater food supply. The main concern of squirrels is to find, crack, and collect nuts, a winter survival method. Scientists have discovered and observed specific species actions that define that species way of life, and this main action is always rooted in survival. Think about the daily life of the human race. Majority of time, effort, and energy is put forth toward making money in some way, shape or form. Whether that be by pursuing an education, working, or networking, the majority of a human life is spent making money, which equivocates to the survival of the individual. This primal concern on survival is what drives theories of natural selection, evolution, and coevolution, each of which will be discussed in further detail in this research study.

   It is important at this early juncture to discuss the definition of a species. Interestingly enough, this definition has been and continues to be disputed across the board in the field of biology; known as the 'species problem'. Two different working definitions are more heavily accepted than others, but each has its own ambiguity. From a stance of viewing species as a statistical phenomena, a species is defined as a separate lineage that creates it's own gene pool, based on DNA discrepancies and morphology. However, the traditional definition of a species is defined as a grouping of organisms that can interbreed and produce fertile offspring. This definition is widely accepted because interbreeding is impossible between two different species. Even if two species mated

successfully, the offspring would be infertile, a classic case being that of the Liger. For the sake of precision and clarity, this will be the accepted definition for the remainder of this report.

Species interaction in natural and wildlife environments is unavoidable. Different species occupying similar ecosystems and living amongst each other will regularly interact. Which leads to the question, how does this primal concern on survival impact the way in which species interact? Species can interact in several different ways. Mutualism is an interaction of mutual benefit, and an example would be pollination, both the insects and the plants benefit from the interaction. Parasitic interaction involves the benefit of the parasite while harming the host species, an example being leaches, tics, or tape worms on a human. Other interactions include, commensalism (one species benefits, the other is unharmed), antibiosis, or neutralism. However, the two main modes of species interaction can be categorized as predation or competition.

Competition is always rooted in the issue of organisms fighting for the same limited resource in an ecosystem. Interspecific competition is defined as two or more different species competing for the same resource, and intraspecific competition is between members of the same species. Forest trees are in constant competition for sunlight, and animals are always competing for food or shelter resources.

Predation is the interaction heavily involved in this research. Predation is the species interaction when one species, the predator, eats another species, the prey, as a source of food. Predation relationship exist in ecological niches throughout the world. This is the species interaction which will be mathematically analyzed and embodied in this research. The importance of studying predation lies in the direct possible effects of

predation relationships, especially coevolution, natural selection, and even the possibility of extinction. Weaker species are constantly under threat of extinction under conditions of extreme predation and competition. Therefore, the possibility of predicting that future outcome with present mathematical abilities could potentially save that tragedy.

Natural Selection and Coevolution

Often defined as 'survival of the fittest', natural selection is one of the defining elements of causal development over time in a population. However, the phrase 'survival of the fittest' does not fully encapsulate the dominant aspects of natural selection. Natural selection is defined as the process by which traits become more or less common in a population due to consistent effects upon the survival or reproduction of their bearers. Natural selection can accurately be labeled 'character trait selection' based on reproductive preference in a species.

Predators and prey exert intense natural selection on one another. Natural selection is theoretically the cause of evolutionary development in the areas of camouflage, danger coloration, mimicry, and even chemical warfare amongst species. A species will select a mate based on the best survival characteristics embodied by that mate, therefore ensuring the survival of traits in the gene pool. Therefore it is intuitive to understand that in the midst of predation, a prey species will select a mate who is more elusive, better camouflaged, or has advanced adaptations to the predation. The same is true for the predator species selecting to mate with other members who are better hunters, faster, stronger, and so on.

This intense natural selection within the context of predation leads to extreme levels of coevolution. Coevolution is change in a species over time resulting from the long term close relationship and interaction with another species. Coevolution between predators and their prey turns into somewhat of a biological arms race, in which each side evolves new adaptations in response to escalations by the other. Darwin used the example of wolves and deer: wolf predation selects slow or careless deer, leaving faster, cunning, and elusive deer to reproduce their genes to further generations. In response, weak and slow wolves cannot adequately acquire food from the new generation of enhanced deer and die off, causing yet another step in evolutionary change.

One of the most thoroughly recorded examples of coevolution resulting from predation is that of bats and moths. Bats use echolocation (also known as sonar) as a navigation system, and is primarily utilized in nighttime hunting. Bats emit a pulse of high frequency sound, and as their ears receive the echo of the sound waves, the difference in frequency creates an image of the scene around them. Moths are the main source of food for these bats. Over time through adaptation and natural selection, moths have developed ears that are sensitive to the frequency of echolocation sound waves, and moths have trained themselves to take elusive flying action upon hearing these sounds. Bats eventually developed a mechanism in their larynx to switch the frequency of the echolocation sound wave, therefore confusing the moths. Moth's over time developed their own means of creating high frequency 'clicks' which interfered with the sound frequency of the bats, giving them distorted images and confusion. Recently it has been discovered that bats have learned to switch off their echolocation all together and zero in on the 'clicking' of the moths, hunting them down by using their own defenses against them.

These examples show the severity of coevolution and natural selection within the predation of two different species. This is why analyzing predator prey interactions and population fluctuations using mathematical processes is extremely beneficial. Biological mathematics gives biologists projected future estimates for population growth and interaction, as well as being able to analyze vast fluctuations in populations from normal phase growth, and being able to investigate sources of population scarcity.

The Test Case: Canadian Lynx and Snowshoe Hare

This research focused on the predation relationship between two specific species in the central regions of Canada: The Canadian Lynx and the Snowshoe Hare. The Canadian Lynx (*Lynx canadensis*) has been found in regions ranging from Alaska, all through the Canadian provinces even into the northern states. The species prefers subalpine coniferous forest ecosystems for living, but will often hunt in more open space (based on the location of the Snowshoe Hare). The species amounts to nothing more than a large domestic cat, average size around 20 lbs. and 20 in. tall to the shoulders. They are readily identified by their long black ear tufts and short black tipped tails, along with large, rounded feet with furry pads to allow travel on snow surfaces.

The Snowshoe Hare (*Lepus americanus*), also known as the Varying Hare, is a species of 'morphing' hare found primarily in the northern regions of North America. It's name 'Snowshoe' refers to the large size of it's hind feet and the trail pattern left behind by the elusive species. The Snowshoe Hare actually morphs between the seasons as an adaptive form of camouflage from predators (primarily the Lynx). It's fur turns white during the winter and a deeper brown during the spring and summer.

This predation relationship, between Canadian Lynx and Snowshoe Hare, has been selected for a reason. This is a very special predation relationship, very unique from most predation relationships throughout the world. This relationship was selected to study because the Canadian Lynx relies almost solely on Snowshoe Hare for food, and the Snowshoe Hare is hunted almost solely by Canadian Lynx. Since the lynx preys almost exclusively on the snowshoe hare, their populations follow a natural cyclic pattern based on the lack of other external variables. These two species have immensely coevolved together, the lynx becoming a hare hunting specialist, and the hare becoming more elusive of the lynx. The Canadian Lynx will only turn to grouse, rodents, or other animals if hares become scarce, therefore these two populations naturally fluctuate in almost perfect harmony.

This natural cyclical pattern changes approximately every ten years from abundance to scarcity and back again. In seasons of hare scarcity, most adult lynx can survive based on hunting skill, but young kittens often die off. As a result, lynx population charts follow a similar pattern to that of the hare, however the data peaks and lows are off set by an average of 1-2 years. This pattern is what will be mathematically analyzed in this research project. These graphical population patters will be represented mathematically, and conclusions will be drawn based on the accuracy of the mathematical representation to real life situations.

Laying a Foundation in Differential Equations:

What is a Differential Equation?

Differential equations are a mathematical means to describing the natural world. Several laws defining the behavior of the natural world are relations involving rates at which things occur. In order to express these laws mathematically, these relations become equations and the rates become derivates. Any equations containing derivatives are differential equations. Differential equations are used in the analysis and defining of fluid movement, electric current, heat transfer, structural stability, material breaking points, seismic waves, analyzing the character of light and sound, as well as the increase or decrease of populations. These types of differential equations, which describe or define some physical process, are often called mathematical models, since they model real-time observations. These mathematical models are often referred to as the 'language' in which the laws of nature are expressed, and are fundamental to much of contemporary science and engineering.

Different Types and Classifications

The entire realm of differential equations is broken primarily into two main classes; ordinary differential equations (ODE) and partial differential equations (PDE). An ODE is a differential equation in which the unknown function is a function of a single independent variable, along with one or more of their derivatives with respect to that variable. A classic ODE example would be Newton's second law of motion, which leads to the differential equation:

$$m\frac{d^2x(t)}{dt^2} = F(x(t))$$

for the motion of a particle of constant mass *m*. Notice the force *F* depends on the unknown

function *x* depending on a single independent variable *t*. Most ordinary differential

equations are utilized in the context of geometry, mechanics, astronomy, or population

modeling (the primary interest of this research pursuit).

Partial differential equations involve partial derivatives of functions of several

variables, and are much more difficult to work with that ordinary differential equations.

These types of equations are used primarily in the propagation of sound, heat,

electrostatics, electrodynamics, elasticity and fluids. One example of a partial differential

equation would be the wave equation:

$$u_{tt} = c^2 u_{xx}$$

This equation is a partial differential equation because the unknown function *u(x,t)* is

comprised of multiple independent variables and partial derivates of those variables.

The next levels of typology and categorization for differential equations would be

linear or nonlinear, and order of equation.   The order of a differential equation is defined

as the highest number of derivatives taken. For example, the wave equation above is a

second order partial differential equation, because the highest number of derivatives taken

is two, both in the partial derivates with respect to *x* and with respect to *t*.

In a linear differential equation, ordinary or partial, all instances of the unknown

function and its derivatives in the differential equation must be multiplied by a linear

operator. In other words, in order to be linear, the dependent variable of the differential

equation cannot by multiplied by itself or its own derivates. An example of a linear

differential equation would be, once again, the wave equation mentioned above, because

the dependent variables are only multiplied by linear operators. For the purpose of

contrast, an example of a non-linear equation is provided below:

$$u_t + uu_x = x^2$$

This non-linear differential equation is nonlinear in the uu_x term, since the dependent

variable is being multiplied by its own derivative with respect to x.

The primary method of research in this project will be systems of first order linear

differential equations. These systems of equations are mathematical models used to

express real population change in the Canadian Lynx-Snowshoe Hare predation

relationship.

Mathematical Predator-Prey Models

Developing a Model

One can easily develop a fundamental mathematical model depicting the

relationship between field mice and owls, however naïve this model may be to real-life

based on the stripping away of significant variables. Consider a population of field mice

occupying some farm land. Excluding predation, an assumption can be made that the

mouse population will increase at a rate proportional to the current population, which is a

common initial hypothesis in population growth study (although not necessarily a law). If

we denote the time by t and the mouse population by $p(t)$, then the assumption about

population growth can be expressed by the equation:

$$dp/dt = rp$$

where r represents the growth rate (a constant). This is the beginning of the population

model, so what are other natural occurrences deterring a population that should be

accounted for? The obvious next step would be to include the natural interaction of this

population of mice with the owl population, the farm mouse's greatest predator. A new

variable, k, must be introduced as the predation rate, or the kill rate, of owls eating the

mice. This new variable to the mouse population model will be introduced:

$$dp/dt = rp\text{-}k$$

This equation represents a fundamentally developed population model, although quite

elementary. In order for this model to be accurate, variables such as human interaction,

climate variances, limited food sources, natural disaster, and other predators are being

ignored.

Predator-Prey Models

Predator-prey models are developed by focusing on primary population variables

and basing the models on the assumptions that other less impacting variables do not exist.

Predation models focus on factors such as the 'natural' growth rate, or birth rate, and the

carrying capacity of the environment in which the population resides. Once these variables

are established, the major population decreasing variables are added, namely in this

project the predation, or kill rate.

In order to keep models simplified, assumptions must be made that would be

unrealistic in most natural predator-prey situations. Specifically, the following

assumptions will be made in this project:

-The predator is completely dependent on the prey as the only food source

-The prey species has an unlimited food supply

-There is no threat to the prey besides the predator species being studied

These assumptions isolate the predator-prey interactions into a 'bubble' with zero climate or natural disaster effects, and the bubble only includes these two species. This type of "pure" predation relationship does not exist in the natural world, however a few specific predation relationships come closer than others. For example, data collected on a specific species of Fiume fish and sharks found in the Adriatic Sea in the early 20th century operated in an almost pure predation relationship. Also, as mentioned previously, the Canadian Lynx and Snowshoe Hare in the northern regions of Canada interact in a relatively pure manner, since the lynx relies almost completely on the hare and the hare has a wide food supply and few other significant predators.

The Lotka-Volterra Model

In the 1920's, a biologist named Humberto D'Ancona was doing a statistaical study on the numbers of each species sold at three main Italian ports. During his study of these fish species from 1914-1923, he came across a startling conclusion. He believed that these predator prey relationships between the sharks, rays, and fish were in their natural states outside of human interaction, namely before and after the war. D'Ancona asked his father-in-law, Vito Volterra, a very successful mathematician, to anazlyze the data and draw some conclusions. Volterra spent weeks developing a series of models for interactions of two or more species, the first and simplest of these is the subject of this development. Alfred J. Lotka was an American mathematical biologist who discovered and development many of the same conclusions and models as Volterra and around the same time. His primary study was on a species of plant and an herbivores species which relied heavily on that plant as a feeding source.

The variables *x* and *y* will be used to denote the populations of the prey and predator, respectively, at time *t*. The first assumption in building this model is that in the absence of the predator, the prey grows at a rate proportional to the current population; thus:

$$dx/dt = ax, a>0 \text{ when } y=0$$

The second assumption is that in the absense of the prey, the predator dies out, thus:

$$dy/dt = -cy, c>0 \text{ when } x=0$$

The third and final assumption is that when predators and prey interact, their encounters are proportional to the product of their populations, and each encounter tends to promote the growth of the predator and inhibit the growth of the prey. Therefore, the predator population will increase by *zxy* and the prey population will decrease by *-zxy*. As a consequence of these assumption, the Lotka-Volterra model is created:

$$dx/dt = ax - bxy = x(a\text{-}by)$$

$$dy/dt = -cy + zxy = y(\text{-}c\text{+}zx)$$

The constants *a*, *b*, *c,* and *z* are all positive where *a* and *c* are the growth rate constants and *b* and *z* are measures of the effect of their interactions.

In order to find trajectories of the system, the variable *t* must be eliminated, and the following equation is discovered:

$$dy/dx = y(\text{-}c\text{+}zx)/x(a\text{-}by)$$

If one simply separates the variables and integrates, an equation can be found to define the trajectories, and the constants can be changed and varied for different trajectories. A sample plot of the phase portrait of the Lotka-Volterra Model representing several different trajectories is shown below in Figure 1.

**Figure 1: Sample Lotka-Volterra Phase Portrait**
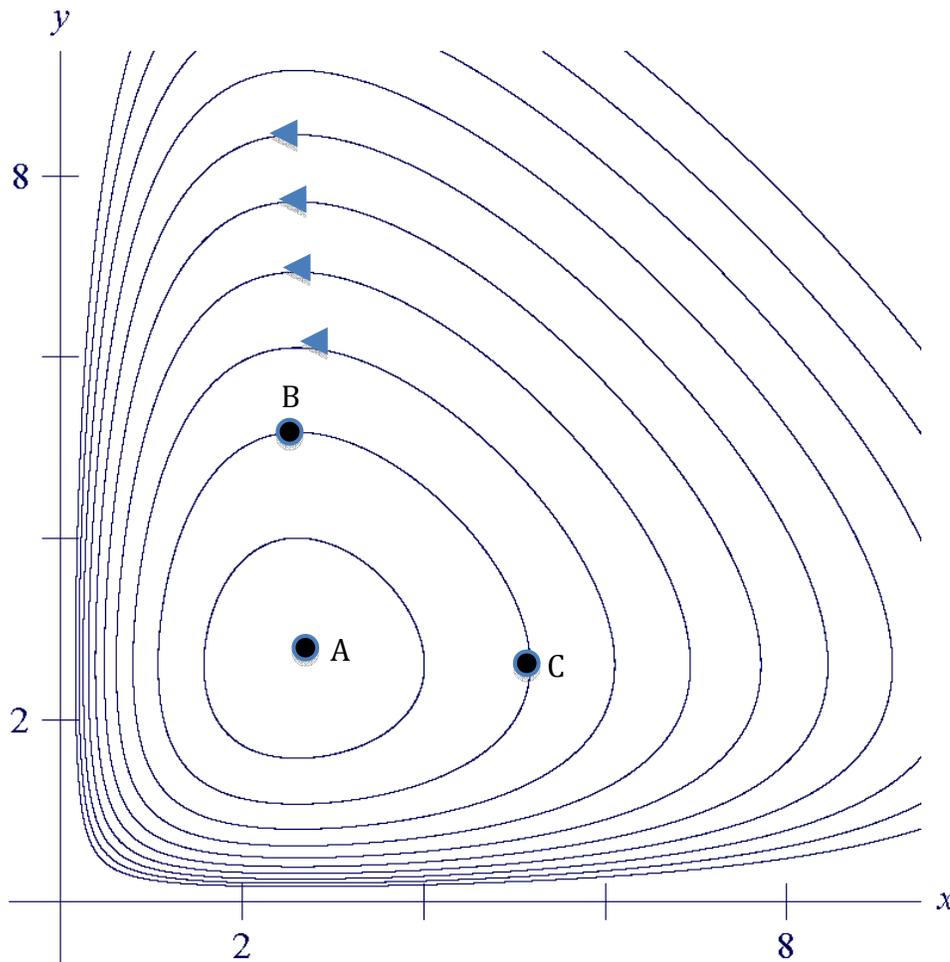
In this plot, the x-axis represents population volume of the prey species, and the y-axis represents the population volume of the predator species. Point A on the phase portrait represents a critical point of the system, sometimes referred to as an equilibrium point. In order to find critical points of the system, each equation of the system must be set to equal zero, and all points that satisfy that condition are critical points. Almost all predator-prey models will have two critical points: an equilibrium point somewhere in the first quadrant of the Cartesian plane, and the origin. Through analysis of the general solution of the linear system, it is discovered that the origin is a saddle point, with a shape

defined by two linear trajectories known as the coordinate axes. Point A is a much different critical point however. By analyzing this critical point using Jacobian Matrix analysis, the general solution shows this critical point of the linear system as a center, which is a stable (not necessarily asymptotically stable) critical point surrounded by infinitely many cyclic trajectories.

Two points B and C were included in order to discuss phase portrait function. The purpose of this phase portrait is to show the cyclic fluctuations of the predator and prey species with respect to each other without showing the change in time. Let C=(5,3) and let B=(3,5) (not perfectly to scale as shown). These two points are on the same trajectory of the system, and with the direction of the trajectory, point C advances to point B as time goes on. One must understand that each point does not just represent arbitrary numbers, but each point represents the state of predator and prey population. Therefore, point C represents the state of the predation relationship at an arbitrary time, and as the time progresses the populations of each species shifts eventually to the state represented by point B. Given more time, eventually the population would return to the state of point C, which would complete one periodic oscillation. Find a different species community with the same constants (birth rate, death rate, kill rate), but different population volumes, and that predation relationship will be located on a different trajectory of the same phase portrait.

Another extremely important plot stemming from the Lotka-Volterra model is the predator-prey cycle chart, representing periodic activity in the population fluctuation. This diagram is generated by plotting the *y-t* and *x-t* curves on the same plot, showing the fluctuation of predator and prey populations with respect to time on the same chart,

therefore important characteristics of time can be analyzed. Figure 2 below depicts a

sample predator-prey cycle chart.
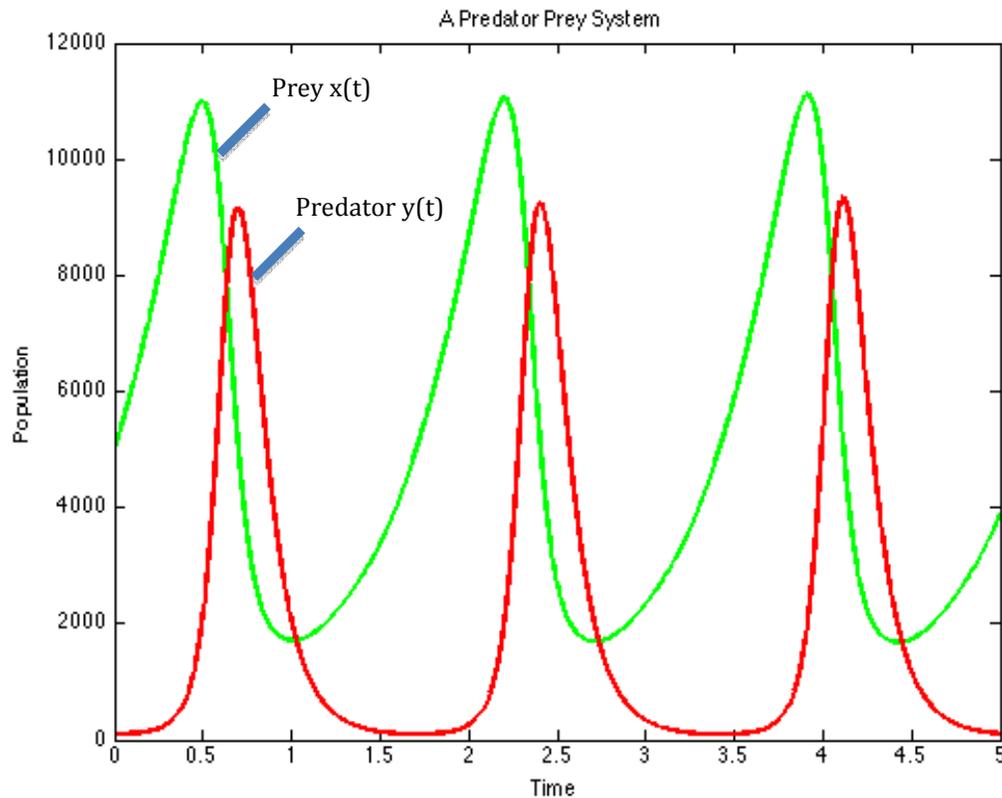
It is seen that as time progresses (in years), predator and prey populations clearly

fluctuate at cyclic interval. Notice that as the prey population peaks, predator population

begins to rise rapidly, yet as the predator population rises, the prey population falls rapidly.

Then follows a longer period as the prey population must slowly repopulate and the

predator population falls drastically. This cycle repeats itself over and over in reality,

which is why biological mathematics can attempt to recreate the pattern mathematically.

On this sample model, notice that the periodic oscilation is defined on roughly 1.75 year

intervals (which is quite unrealistic, but this is only an example).

Analyzing the Differential Equation:

Discovering Critical Values

Data analysis for this section will be based on the following system:

$$dx/dt = x(.5-0.02y)$$

$$dy/dt = y(-0.9+0.03x)$$

This system was created based on using the constants a=0.5, b=0.02, c=0.9, and d=0.03.

These constants come from determining the best possible parameters of the snowshoe

hare and Canadian Lynx population data shown in Figure 3 below.

Year	Hares (x1000)	Lynx(x1000)	Year	Hares (x1000)	Lynx (x1000)
1900	30	4	1911	40.3	8
1901	47.2	6.1	1912	57	12.3
1902	70.2	9.8	1913	76.6	19.5
1903	77.4	35.2	1914	52.3	45.7
1904	36.3	59.4	1915	19.5	51.1
1905	20.6	41.7	1916	11.2	29.7
1906	18.1	19	1917	7.6	15.8
1907	21.4	13	1918	14.6	9.7
1908	22	8.3	1919	16.2	10.1
1909	25.4	9.1	1920	24.7	8.6
1910	27.1	7.4			

Figure 3: Source http://www-rohan.sdsu.edu/~jmahaffy/courses/f00/math122/labs/labj/q3v1.htm

This data was inputted into Microsoft Excel and the Solver application was used in order to

discover the best possible fit parameters to the desired Lotka-Volterra model.

The first step in analyzing the system is discovering the critical points by setting

each of the two equations in the system to equal zero and determining analytically the

satisfying points. The two critical points of this equation are (0,0) and (30,25) (notice this

result in the origin and one point in the first quadrant as previously stated).

Computer Simulations

Shown below is a MATLAB rendered direction field of the system in Figure 4. Notice

that the point (0,0) is a saddle point and the critical point (30,25) is a center point, or
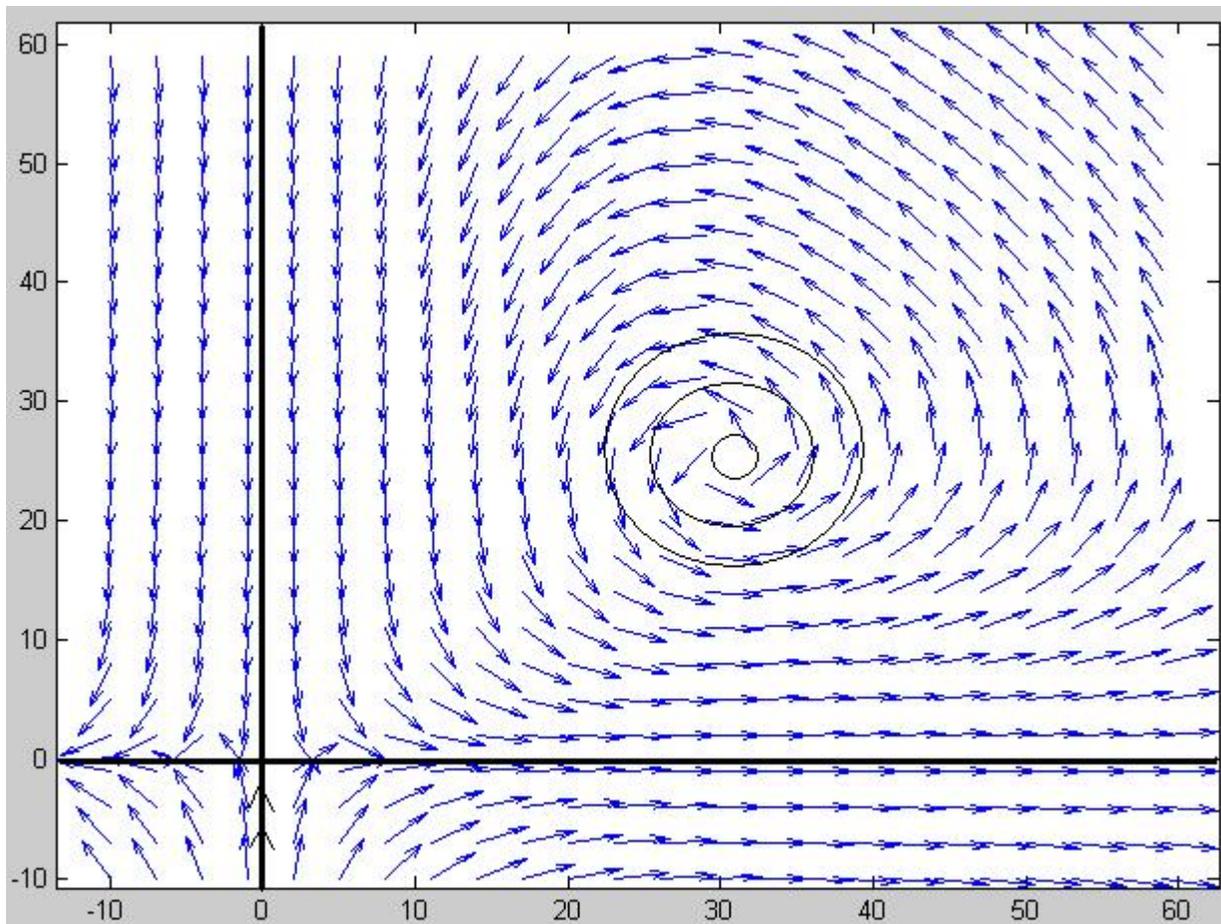
equilibrium point.

Figure 4: MATLAB rendered direction field of the system, notice critical point at (30,25)

By examining the corresponding linear systems to the solutions near each critical

point we can determine local behavior of each point. The general solution of the linear

system corresponding to the origin shows that, based on local behavior, this point is a

saddle point. However, the critical point at (30,25) is much more difficult to analytically

determine the behavior. Using Jacobian Matrix analysis, the corresponding linear system

can be obtained, and the eigenvalues of this system are imaginary. Imaginary eigenvalues

equivocates to the critical point being a center. The nonlinear systems of each of these

critical points share the behavior of their linear counterparts.

This direction field shows the general direction of the entire field at large. As

shown, the infinitely many trajectories that exist outside of the critical points take on a

similar directional pattern. Since the point (30,25) is a center, all trajectories surrounding

this point are known as cyclic variations, or periodic oscillations. While not accounting for

time, the main purpose of this visual is to notice the ever-changing pattern of the two

different populations with respect to one another.

The next critical graph to examine is the Predator-Prey Cycle Chart shown in Figure

5 below. This graph depicts population (in thousands) versus time (in years) of both

species on the same graph. This is quite useful in order to visualize the population

fluctuations of each species also with respect to time. The average time of the periodic

oscillation can be determined graphically in this way, and general population variation

characteristics can be determined. Notice the reality represented by the graph, as the prey

population peaks, the predator population begins to rapidly increase. Just as the predator

population peaks, the prey population begins to drop rapidly. The curve for the Lynx

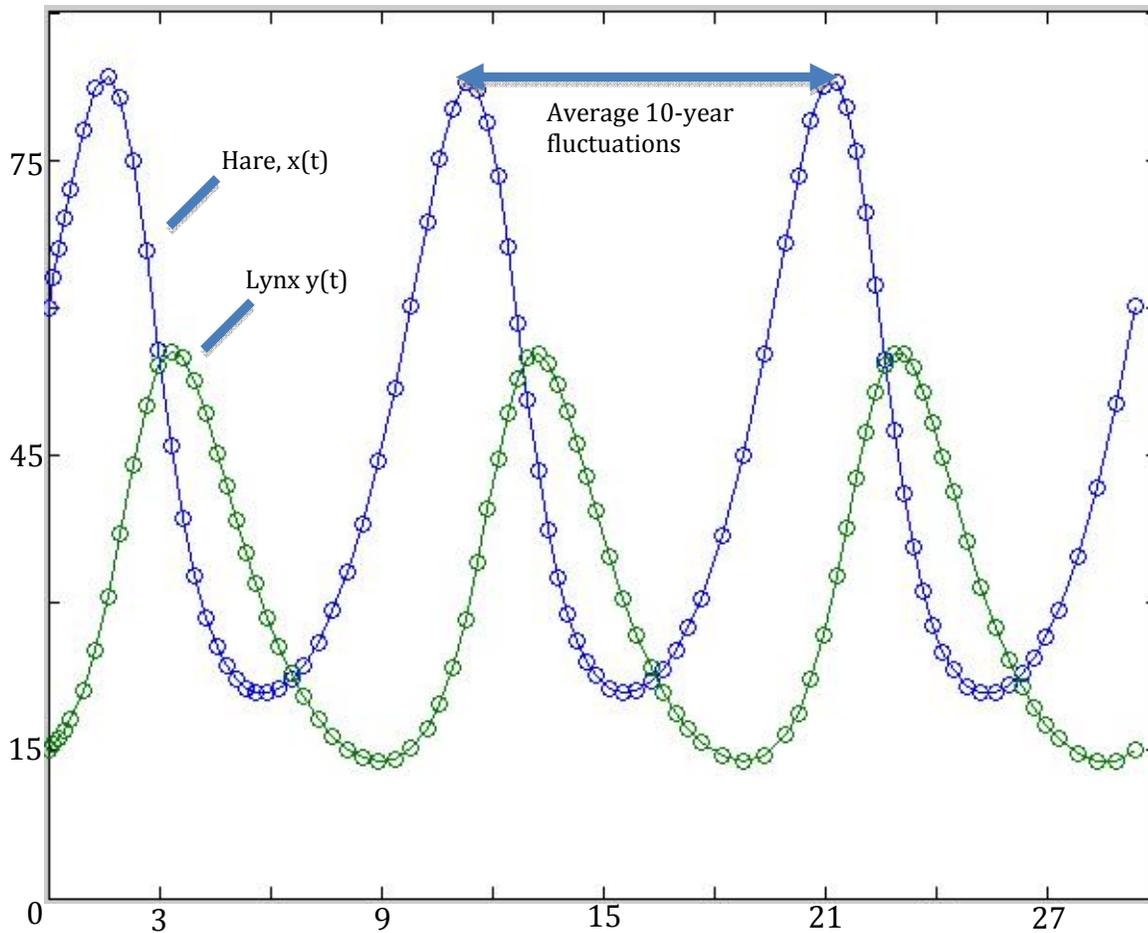follows behind the Hare curve in the same shape and pattern.

Figure 5: a MATLAB rendered Predator-Prey Cycle Chart of the system.

One of the most important values found in this graph is the average periodic oscillation. By

analyzing the same point in sequential phases and finding the time in between them, the

periodic oscillation can be determined.   Often peaks of the curves are used for this. As

shown in Figure 5 above, the average periodic oscillation for the Snowshoe Hare and

Canadian Lynx is a ten year fluctuation, which is confirmed by further research in this

realm of work.

It is quite interesting to match up this graph to the graph of the real data. Shown below in Figure 6 is the real data line graph for the Predator Prey Cycle chart of the data used in this project.
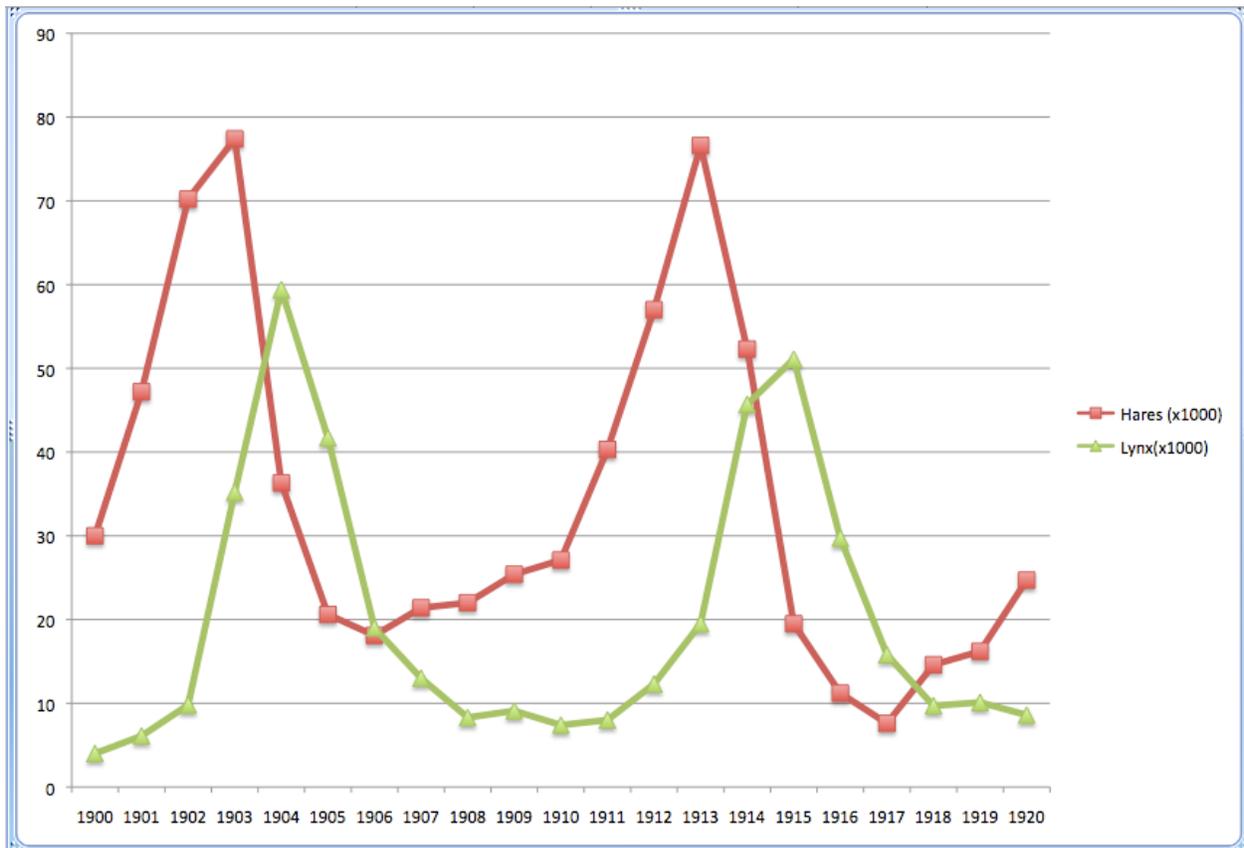

Figure 6: An Excel Rendering of the exact population data used to develop this system.

Comparing these two graphs it is seen that the overall form, shape, and character of the two curves are very much similar, however, not precise. Each graph has approximately 10 year periodic oscillations, with similar population maximum numbers. This graph comparison, at the same time as being proving and helpful, also depicts visually some of the limitations of the biological model. No mathematical model will ever be precise to reality, although generally accurate. When it comes to this specific kind of mathematics, there will always be

percent error involved when analyzing the difference between theoretical and analytical data.

Restrictions and Limitations

Although the Lotka-Volterra model is the best model currently available to accurately portray mathematically the population variation dynamics of a predation relationship, there are still several holes in the science. All of the assumptions that go into this kind of model development naturally inflict some 'holes' at the same time. Restrictions and limitations of this research exist in the assumptions it is built on.

In my view, the greatest limitation this model has against it is its lack of relevance to the majority of predation relationships. In order for a this model to accurately portray the reality of population variance, the two species involved must satisfy (or close to satisfy) those major pieces to the puzzle. The model works for Snowshoe Hares and Canadian Lynx, and for Fiume fish and Sharks in the Mediterranean, but what about birds and worms in Carbondale? The model is limited to predation relationships that rely explicitly on one another, free of the variables of climate change, natural disaster, hunting, or alternative food supplies.

Conclusions and Future Work:

First of all, one of the main conclusions of this research is that in the realm of biological mathematics, it is possible to mathematically represent the population variations of a predation relationship to a certain extent of accuracy. This can be done using the Lotka-Volterra Model. This system of linear first order differential equations can be used to

interpret analytically and graphically the cyclic fluctuations of the species populations. There is a level of error in the theoretical versus actual data, but the overall form, structure, and time of the fluctuations is consistent, and errors are most probably due to external variables unaccounted for in the model.

After analyzing some population data, it was found that Snowshoe Hare and Canadian Lynx populations fluctuate on an average of ten year periodic oscillations, which can be confirmed by real time data. This information can be incredibly useful to the biological fields focused on extinction prevention, or coevolution studies. While being able to determine population fluctation characteristics, the results can show extinction prevention specialists the years and seasons where extinction is naturally possible, preparing them ahead of time to do intense tagging and developing natural habitats as safety precautions. Coevolution studies could also use this data to be able to determine the number of cycles it required for each species to develop new means of survival, and adapt to the competition.

Future work could be done in several different ways. There are an infinite amount of new and diverse models that could develop from this very basic starting point. The Lotka Volterra Model can serve as a stepping stone in the biological mathematics field to several other new means of depicting the ecological world mathematically. By starting with this model, one could simply account for one more variable, for instance hunting, and have an entirely new model. This would also help to increase the relevance of the science, because the more predation relationships this level of study can apply to, the more relevant the science becomes. Accounting for more variables can increase the adaptability of the model to increasingly more species. Other options for future work could include studying

other predation relationships for which the Lotka-Volterra Model may apply, particularly species which rely on one another as resources while lacking substantial external variables.

This model has simply broken the ground of endless possibility in the biological mathematics world. It is the backbone of the science, proven by tests and time, and the future of this science has incredible potential.

Works Cited

Audesirk, Terry and Gerry and Bruce Byers. <u>Life on Earth</u>. Eds. Beth Wilbur and Star
       Mackenzie. 5th edition. San Francisco: Pearson Education, Inc. publishing as
       Pearson Benjamin Cummings, 2009.

Boyce, William and Richard DiPrima. <u>Elementary Differential Equations and Boundary</u>
       <u>Value Problems</u>. Ed. David Dietz. 9th edition. Hoboken, NJ: John Wiley & Sons, Inc.,
       2009.

Braun, Martin. Differential Equations and Their Applications. Vol. 15 of Applied
       Mathematical Sciences. New York: Springer-Verlag, 1975.