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ENUMERATING LABELLED GRAPHS WITH CERTAIN NEIGHBORHOOD PROPERTIES

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Abstract. Properties of (connected) graphs whose closed or open neighborhood families are Sperner, anti-Sperner, distinct or none of the proceeding have been extensively examined. In this paper we examine 24 properties of the neighborhood family of a graph. We give asymptotic formulas for the number of (connected) labelled graphs for 12 of these properties. For the other 12 properties, we give bounds for the number of such graphs. We also determine the status (a.a.s. or a.a.n.) in $\mathcal{G}_{n,1/2}$ of all 24 of these properties. Our methods are both constructive and probabilistic.

1 Introduction

A simple graph G has *vertex set* $V(G)$ and *edge set* $E(G)$. The *order* of G is $\#V(G)$ and the *size* of G is $\#E(G)$. The sets $N_G(v) = \{w \in V(G) : vw \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{v\}$ are the *open neighborhood* and *closed neighborhood* of v in G, respectively.

The collection $N_G[v]_{v\in V(G)}$ (respectively, $N_G(v)_{v\in V(G)}$) is the *family of closed (respectively, open) neighborhoods* of the graph G. Briefly, we say the *closed (open) neighborhood family* of G. In [5]-[8] graphs whose closed neighborhoods are distinct were studied. In [10] and [20, 21], graphs whose open neighborhoods are distinct were studied. Graphs whose closed neighborhood or open neighborhood family satisfy a particular property (CNAS and NAS; see definitions) have been extensively examined in [12]–[17]. We give results for all these properties in this paper. Results about graphs whose induced open neighborhood subgraphs consists of, for example, only paths or only cycles, can be found in [9] and [18]. Further results may be found in [2]-[4].

We focus on 24 properties (see definitions below) of the closed neighborhood or open neighborhood family of a graph. We examine the number of (connected) labelled graphs whose neighborhood families have these properties. Table 2 in Section 4 contains our main results. It gives asymptotic formulas for the number of labelled graphs with 12 of the properties (lines 1,2,5, and 6) and bounds for the other 12 properties. The properties are quite delicate: deleting a vertex, or adding or deleting an edge generally destroys the property. Hence finding generating functions for the number of labelled graphs with these properties by decomposing the graph into subgraphs with similar properties does not appear promising (cf. Chapter 5 of [19]). Consequently, elementary results about random graphs (with $p = 1/2$) aid in our enumeration. Along the way we determine the status (a.a.s. or a.a.n.) of all these properties for these random graphs (see Corollaries 2.5 and 3.5).

A set is a collection of distinct elements. A family $\mathcal{F}$ of subsets of a fixed set is *Sperner* if $A \not\subseteq B$ and $B \not\supseteq A$ for all $A, B \in \mathcal{F}$.

We say a graph G is:

(1) *closed neighborhood Sperner* (CNS) $\Leftrightarrow$ the closed neighborhood family of G is Sperner $\Leftrightarrow \forall u \neq v \in V(G)$, $N_G[u] \not\subseteq N_G[v]$ and $N_G[v] \not\subseteq N_G[u]$,

whose negation is

(2) $\neg$ *closed neighborhood Sperner* ($\neg$CNS) $\Leftrightarrow$ the closed neighborhood family of G is not Sperner $\Leftrightarrow \exists u \neq v \in V(G)$, $N_G[u] \subseteq N_G[v]$;

(3) *closed neighborhood anti-Sperner* (CNAS) $\Leftrightarrow$ the closed neighborhood family of G is anti-Sperner $\Leftrightarrow \forall u\, \exists v \neq u \in V(G)$, $N_G[u] \subseteq N_G[v] \Leftrightarrow$ every maximal chain in the closed neighborhood family of G ordered by set inclusion has at least two maximum elements,

whose negation is

(4) $\neg$ *closed neighborhood anti-Sperner* ($\neg$CNAS) $\Leftrightarrow$ the closed neighborhood family of G is not anti-Sperner $\Leftrightarrow \exists u \forall v \neq u \in V(G)$, $N_G[u] \not\subseteq N_G[v] \Leftrightarrow$ there exists a maximal chain in the closed neighborhood family of G ordered by set inclusion with a unique maximum element;

(5) *closed neighborhood distinct* (CND) $\Leftrightarrow$ the closed neighborhood family of G is a set $\Leftrightarrow \forall u \neq v \in V(G)$, $N_G[u] \neq N_G[v]$,

whose negation is

(6) $\neg$ *closed neighborhood distinct* ($\neg$CND) $\Leftrightarrow$ the closed neighborhood family of G is not a set $\Leftrightarrow \exists u \neq v \in V(G)$, $N_G[u] = N_G[v]$.

Analogous definitions for these 6 properties hold with "closed" replaced with "open" , we suppress the "O" in the acronym.

A property P of graphs is a collection of graphs which is closed under isomorphism. For a property P, we say G is a conn$P$ graph if G is connected and has property P. Hence, we examine 2 (connected or arbitrary)$\times$ 2 (closed or open) $\times\, 6 = 24$ properties in this paper.

Let G_n denote the set of all graphs with vertex set $[n] = \{1, \ldots, n\}$. The probability space $\mathcal{G}_{n,p}$ consists of all graphs $G \in G_n$ where each edge of G is chosen independently with probability $p = p(n)$. Hence, $G \in \mathcal{G}_{n,p}$ has probability $p^m q^{N-m}$ when G has m edges and $N - m$ non-edges where $N = \binom{n}{2}$ and $q = 1 - p$. We write $G_{n,p}$ in place of $G \in \mathcal{G}_{n,p}$. A graph property P is *asymptotically almost sure* (a.a.s.) provided $\Pr(G_{n,p} \in P) \to 1$ as $n \to \infty$, and *asymptotically almost null* (a.a.n.) provided $\Pr(G_{n,p} \in P) \to 0$ as $n \to \infty$. We refer the reader to [1].

Since we have constructive lower bounds for the number of connCNAS and connNAS graphs in G_n (see [13] and [14] respectively) and

$$Pr(G_{n,1/2} \in P) = \frac{\#(P \cap G_n)}{2^{\binom{n}{2}}},$$

we focus on $\mathcal{G}_{n,1/2}$ only. We do not consider other values of p.

Table 1a

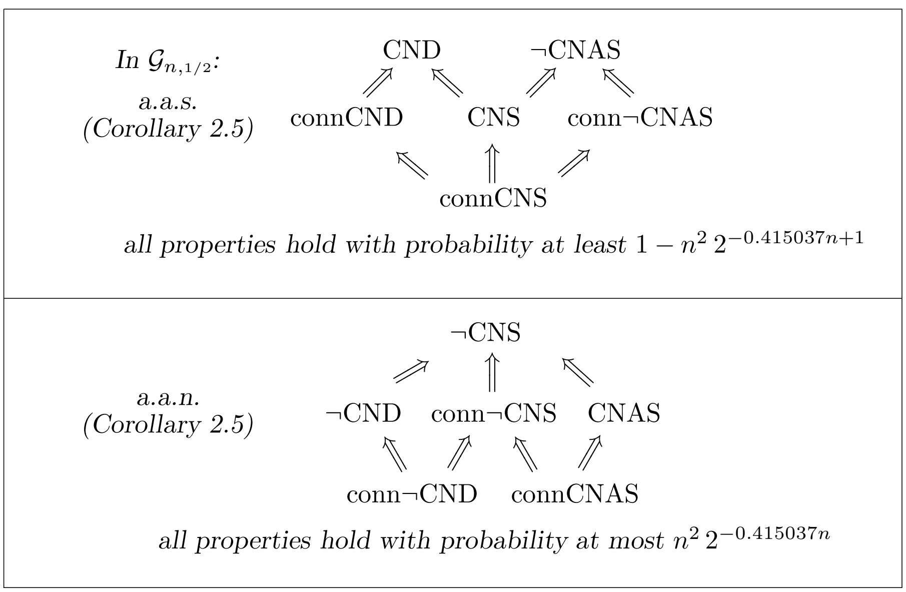

Table 1a gives the logical implications ($\Rightarrow$) and certain probabilistic bounds in $\mathcal{G}_{n,1/2}$ among the 12 closed properties. We do not show Table 1b which is identical to Table 1a but with "closed" replaced by "open". Tables 1a, 1b are the Hasse diagrams of the 4 sets of 6 properties we examine in this paper, partially ordered by logical implication, and where equality is logical equivalence. The probabilistic bounds in Tables 1a, 1b imply the probabilistic status (a.a.s. or a.a.n.) in $\mathcal{G}_{n,1/2}$ of these 24 properties (see Corollaries 2.5 and 3.5).

To conclude this Section we note that the threshold for $G_{n,p}$ to be connected is $p = \ln n/n$ (cf. [1]). We give a rough estimate for the probability that $G_{n,1/2}$ is connected that is adequate for our purposes.

Lemma 1.1. *We have*

$$\Pr(G_{n,1/2} \textit{ is connected}) \geq 1 - n^2 2^{-n}.$$

Proof. Fix $A \subseteq [n]$ with $|A| = a \in \{1, \ldots, n-1\}$. Let E_{A,A^c} denote the event "there are no edges between A and A^c". By independence, $\Pr(E_{A,A^c}) = 2^{-a(n-a)}$.

Let E denote the event $\bigcup_{\text{such } A,A^c} E_{A,A^c}$, *i.e.*, the event "$G_{n,1/2}$ is disconnected".

Then

$$\Pr(E) \leq \frac{1}{2} \sum_{a=1}^{n-1} \binom{n}{a} 2^{-a(n-a)}.$$

The terms in the sum decrease to $a = \lfloor (n+1)/2 \rfloor$ and increase thereafter; the maximum occurs at both $a = 1, n-1$. Hence, $\Pr(G_{n,1/2}$ is disconnected$) = \Pr(E) \leq n^2 2^{-n}$, which implies the result. ■

For the constants that appear in the paper we only write the first six digits of their infinite decimal expansions. We assume throughout that $n \geq 4$.

2 Closed neighborhood properties of a graph

We start with property $\neg$CNS since it lies at the top of the lower Hasse diagram in Table 1a.

Theorem 2.1. *We have*

$$\Pr(G_{n,1/2} \textit{ is } \neg CNS) \leq n^2\, 2^{-0.415037n},$$

where $2^{-0.415037} = 0.75$, $(-0.415037 = \dfrac{\ln 0.75}{\ln 2})$.

Proof. Fix $u \neq v \in [n]$, $A \subseteq B \subseteq [n] - \{u, v\}$. Let $E_{u,v,A,B}$ denote the event "$N_G[u] = A \cup \{u, v\}$ and $N_G[v] = B \cup \{u, v\}$", so $u \sim v$. Let $0 \leq a = |A| \leq |B| = b \leq n - 2$. Then, using independence,

$$\Pr(E_{u,v,A,B}) = 2^{-1}2^{-a}2^{-b}2^{-(n-a-2)}2^{-(n-b-2)} = 2^{-(2n-3)}.$$

Let $E_{u,v,a,b}$ denote the event $\bigcup_{\text{such } A,B} E_{u,v,A,B}$ for a fixed a and b. Then,

$$\Pr(E_{u,v,a,b}) \leq \binom{n-2}{a}\binom{n-2-a}{b-a}2^{-(2n-3)}.$$

Let $E_{u,v}$ be the event "$N_G[u] \subseteq N_G[v]$", *i.e.*, $\bigcup_{\substack{a=0,1,\ldots,n-2\\ a\leq b\leq n-2}} E_{u,v,a,b}$.

Then,

$$\begin{aligned}\Pr(E_{u,v}) &\leq \sum_{a=0}^{n-2}\binom{n-2}{a}\left\{\sum_{b=a}^{n-2}\binom{n-2-a}{b-a}\right\}2^{-(2n-3)}\\ &= \sum_{a=0}^{n-2}\binom{n-2}{a}2^{n-2-a}2^{-(2n-3)} = 2^{-n+1}\sum_{a=0}^{n-2}\binom{n-2}{a}2^{-a} = \frac{8}{9}(0.75)^n.\end{aligned}$$

Now $G_{n,1/2}$ is ¬CNS if there exists a pair $\{u, v\}$ of vertices with $N_G[u] \subseteq N_G[v]$. There are $\binom{n}{2}$ pairs $\{u, v\}$ so:

$$\Pr(G_{n,1/2} \text{ is } \neg\text{CNS}) \leq \frac{n^2}{2}\frac{8}{9}(0.75)^n \leq n^2(0.75)^n.$$

■

Theorem 2.2. *We have*

$$\Pr(G_{n,1/2} \textit{ is } \neg CND) \geq 2^{-n+1},$$

equivalently,

$$\Pr(G_{n,1/2} \textit{ is } CND) \leq 1 - 2^{-n+1}.$$

Proof. Fix $A \subseteq [n] - \{1, 2\}$ with $|A| = a$. Let E_A denote the event "$N_G[1] = N_G[2] = A \cup \{1, 2\}$". Similar to before, we have $\Pr(E_A) = 2^{-(2n-3)}$.

Let E denote the event $\bigcup_{\text{such } A} E_A$; note that the E_A are pair-wise disjoint events. Then

$$\Pr(E) = \sum_{a=0}^{n-2}\binom{n-2}{a}2^{-(2n-3)} = 2^{-n+1}.$$

Since $E \subseteq$ "$G_{n,1/2}$ is $\neg$CND", then $\Pr(G_{n,1/2}$ is $\neg\text{CND}) \geq \Pr(E) = 2^{-n+1}$, hence the result. ■

Recall that graphs G and H are *equal* $(G = H) \Leftrightarrow V(G) = V(H)$ and $E(G) = E(H)$. When $V(G) = V(H)$, G and H are *distinct* $(G \neq H) \Leftrightarrow E(G) \neq E(H)$, i.e., distinct edge-sets give distinct graphs.

We require a construction from [14]: Let $V(G) = [n]$, and $E(G) = \{12\} \cup \{1k, 2k : k \in [n] - \{1,2\}\}$. Let $\mathcal{B}$ denote the set of edges with both end-vertices in $[n] - \{1,2\}$, so $|\mathcal{B}| = \binom{n}{2}$. Now define a graph $G_{\mathcal{A}}$ with $V(G_{\mathcal{A}}) = [n]$ and $E(G_{\mathcal{A}}) = E(G) \cup \mathcal{A}$ where $\mathcal{A} \subseteq \mathcal{B}$.

Theorem ([13]). *The graph $G_{\mathcal{A}}$ is a CNAS graph, a connCNAS graph, and a conn¬CND graph for all $\mathcal{A} \subseteq \mathcal{B}$.*

Proof. Clearly $G_{\mathcal{A}}$ is connected. Observe that $N_{G_{\mathcal{A}}}[1] = N_{G_{\mathcal{A}}}[2] = [n] \supseteq N_{G_{\mathcal{A}}}[k]$ for all $k \in [n] - \{1,2\}$. ■

Theorem 2.3. *For P any of the properties CNAS, connCNAS, or conn¬CND*

$$\Pr(G_{n,1/2} \textit{ is } P) \geq 2^{-2n+3},$$

equivalently,

$$\Pr(G_{n,1/2} \textit{ is } \neg P) \leq 1 - 2^{-2n+3}.$$

Proof. Distinct edge-sets $\mathcal{A} \subseteq \mathcal{B}$ give distinct graphs $G_{\mathcal{A}}$, so $\#P_n \geq 2^{\binom{n-2}{2}}$. Each graph in $\mathcal{G}_{n,1/2}$ has probability $2^{-\binom{n}{2}}$, so $\Pr(G_{n,1/2}$ is $P) \geq 2^{-2n+3}$, hence the result. ■

Theorem 2.4. *We have*

$$\Pr(G_{n,1/2} \textit{ is connCNS}) \geq 1 - n^2 2^{-0.415037n+1}.$$

Proof. From Theorem 2.1 and Lemma 1.1,

$$\begin{aligned}\Pr(G_{n,1/2} \text{ is } \neg\text{CNS or disconnected}) &\leq n^2\, 2^{-0.415037n} + n^2 2^{-n} \\ &\leq n^2 2^{-0.415037n+1},\end{aligned}$$

which implies the result. ■

Corollary 2.5. *For P any of the properties on a fixed line of the table below $\Pr(G_{n,1/2}$ is $P)$ is bounded by the values in the left-hand and right-hand columns.*

			P		
1	$1 - n^2\, 2^{-0.415037n+1}$	$\leq$	*CNS, connCNS, CND, connCND,*	$\leq$	$1 - 2^{-n+1}$
2	$1 - n^2\, 2^{-0.415037n+1}$	$\leq$	$\neg CNAS$, *conn*$\neg CNAS$	$\leq$	$1 - 2^{-2n+3}$
3	2^{-2n+3}	$\leq$	*conn*$\neg CNS$, *CNAS, connCNAS,* *conn*$\neg CND$	$\leq$	$n^2\, 2^{-0.415037n}$
4	2^{-n+1}	$\leq$	$\neg CNS$, $\neg CND$	$\leq$	$n^2\, 2^{-0.415037n}$

Proof. For the upper half of this table see the Hasse diagram in the upper half of Table 1a. Line 1 comes from Theorems 2.4 (lower bound) and 2.2 (upper bound) and line 2 from Theorems 2.4 and 2.3, and monotonicity. Hence, each of these six properties is a.a.s. in $\mathcal{G}_{n,1/2}$.

For the lower half see the lower half of Table 1a. Line 3 comes from Theorems 2.3 and 2.1 and line 4 from Theorems 2.2 and 2.1, and monotonicity. Hence, each of these six properties is a.a.n. in $\mathcal{G}_{n,1/2}$. ∎

Lines 1–4 of Table 2 follow from Corollary 2.5 upon multiplying by $2^{\binom{n}{2}}$.

3 Open neighborhood properties of a graph

We now consider open neighborhood properties of a graph, where our results are similar, but different, to before.

Theorem 3.1. *We have*

$$\Pr(G_{n,1/2} \text{ is } \neg NS) \leq n^2\, 2^{-0.415037n}.$$

Proof. Similar to the notation in the proof of Theorem 2.1, for $u \neq v$, let $E'_{u,v,A,B}$ denote the event "$N_G(u) = A$ and $N_G(v) = B$". Then $u \not\sim v$ and $\Pr(E'_{u,v,A,B}) = 2^{-(2n-3)}$. Then we define the "open" analogue of events $E_{u,v,a,b}$, namely $E'_{u,v,a,b}$, and $E_{u,v}$, namely $E'_{u,v}$: "$N_G(u) \subseteq N_G(v)$"; and the remainder of the proof is precisely as in Theorem 2.1. ∎

Theorem 3.2. *We have*

$$\Pr(G_{n,1/2} \textit{ is } \neg ND) \geq 2^{-n+1}$$

equivalently,

$$\Pr(G_{n,1/2} \textit{ is } ND) \leq 1 - 2^{-n+1}.$$

Proof. Again, similar to the notation in the proof of Theorem 2.2, let E'_A denote the event "$N_G(1) = N_G(2) = A$". We then define E' as the event $\bigcup_{\text{such } A} E'_A$ and the proof proceeds as in Theorem 2.2. ∎

Assume $n = 2m \geq 4$. Let $V(G) = [2m] = [m] \cup ([2m] - [m])$ and $E(G) = \{1k, 2k : k \in [2m] - [m]\} \cup \{(m+1)k, (m+2)k : k \in [m]\}$. Set $\mathcal{B} = \{ij : i \in [m], j \in [2m] - [m]\} - E(G)$, so $|\mathcal{B}| = (\frac{n-4}{2})^2 = 0.25n^2 - 2n + 4$. Let $V(G_{\mathcal{A}}) = V(G)$ and $E(G_{\mathcal{A}}) = E(G) \cup \mathcal{A}$ where $\mathcal{A} \subseteq \mathcal{B}$. A similar definition holds with $[2m]$ replaced by $[2m+1]$.

Theorem ([14]). *The graph $G_{\mathcal{A}}$ is a NAS graph, a connNAS graph, and a conn¬ND graph for all $\mathcal{A} \subseteq \mathcal{B}$.*

Proof. Clearly $G_{\mathcal{A}}$ is connected. Observe that $N_{G_{\mathcal{A}}}(1) = N_{G_{\mathcal{A}}}(2) = [2m] - [m] \supseteq N_{G_{\mathcal{A}}}(k)$ for all $k \in [m]$, and $N_{G_{\mathcal{A}}}(m+1) = N_{G_{\mathcal{A}}}(m+2) = [m] \supseteq N_{G_{\mathcal{A}}}(k)$ for all $k \in [2m] - [m]$. ∎

Theorem 3.3. *For P any of the properties NAS, connNAS, or conn¬ND*

$$\Pr(G_{n,1/2} \textit{ is } P) \geq 2^{-0.25n^2 - 1.5n + 4},$$

equivalently,

$$\Pr(G_{n,1/2} \textit{ is } \neg P) \leq 1 - 2^{-0.25n^2 - 1.5n + 4}.$$

Proof. For even $n \geq 4$ distinct edge-sets $\mathcal{A} \subseteq \mathcal{B}$ give distinct graphs $G_{\mathcal{A}}$, so $\#P_n \geq 2^{0.25n^2 - 2n + 4}$. And each graph in $\mathcal{G}_{n,1/2}$ has probability $2^{-\binom{n}{2}}$, so $\Pr(G_{n,1/2} \textit{ is } P) \geq 2^{-0.25n^2 - 1.5n + 4}$, which gives the result. A slight modification of this argument is needed for odd $n \geq 5$. ∎

Theorem 3.4. *We have*

$$\Pr(G_{n,1/2} \textit{ is connNS}) \geq 1 - n^2 2^{-0.415037n + 1}.$$

Proof. This follows from Lemma 1.1 and Theorem 3.1, as in the proof of Theorem 2.4. ∎

Corollary 3.5. *For P any of the properties on a fixed line of the table below* $\Pr(G_{n,1/2}$ *is* $P)$ *is bounded by the values in the left-hand and right-hand columns.*

			P		
1	$1 - n^2\, 2^{-0.415037n+1}$	$\leq$	*NS, connNS, ND, connND,*	$\leq$	$1 - 2^{-n+1}$
2	$1 - n^2\, 2^{-0.415037n+1}$	$\leq$	$\neg NAS$, $conn\neg NAS$	$\leq$	$1 - 2^{-0.25n^2-1.5n+4}$
3	$2^{-0.25n^2-1.5n+4}$	$\leq$	$conn\neg NS$, *NAS, connNAS*, $conn\neg ND$	$\leq$	$n^2\, 2^{-0.415037n}$
4	2^{-n+1}	$\leq$	$\neg NS$, $\neg ND$	$\leq$	$n^2\, 2^{-0.415037n}$

Proof. For the upper half of this table consider the Hasse diagram in the upper half of Table 1b (not shown). Line 1 comes from Theorems 3.4 (lower bound) and 3.2 (upper bound), and line 2 from Theorems 3.4 and 3.3, and monotonicity. Hence, each of these six properties is a.a.s. in $\mathcal{G}_{n,1/2}$.

For the lower half consider the lower half of Table 1b. Line 3 comes from Theorems 3.3 and 3.1 and line 4 from Theorems 3.2 and 3.1, and monotonicity. Hence, each of these six properties is a.a.n. in $\mathcal{G}_{n,1/2}$. ■

Lines 5–8 of Table 2 follow from Corollary 3.5 upon multiplying by $2^{\binom{n}{2}}$.

4 Main Results and Asymptotics

Recall,

$$f(n) \sim g(n) \text{ if and only if } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1,$$

and for a property P of graphs, and $P_n = P \cap G_n$ is the set of distinct graphs on $[n]$ having property P.

For P any of the properties on a fixed line of Table 2, $\#P_n$ is bounded by the values in the left-hand and right-hand columns (see the comments after Corollaries 2.5 and 3.5).

For the 12 properties P in lines $1, 2, 5,$ and 6 of Table 2, we have

$$\#P_n \sim 2^{\binom{n}{2}},$$

and for the 8 properties in lines $3, 4$, and 8 of Table 2, we have

$$\log_2 \#P_n \sim \frac{n^2}{2}.$$

Table 2

	$\#P_n$				
1	$2^{\binom{n}{2}}(1 - n^2\, 2^{-0.415037n+1})$	$\leq$	$\#\mathrm{CNS}_n$, $\#\mathrm{connCNS}_n$, $\#\mathrm{CND}_n$, $\#\mathrm{connCND}_n$,	$\leq$	$2^{\binom{n}{2}}(1 - 2^{-n+1})$
2	$2^{\binom{n}{2}}(1 - n^2\, 2^{-0.415037n+1})$	$\leq$	$\#\neg\mathrm{CNAS}_n$, $\#\mathrm{conn}\neg\mathrm{CNAS}_n$	$\leq$	$2^{\binom{n}{2}}(1 - 2^{-2n+3})$
3	$2^{0.5n^2-2.5n+3}$	$\leq$	$\#\mathrm{conn}\neg\mathrm{CNS}_n$, $\#\mathrm{CNAS}_n$, $\#\mathrm{connCNAS}_n$, $\#\mathrm{conn}\neg\mathrm{CND}_n$	$\leq$	$n^2\, 2^{0.5n^2-0.915037n}$
4	$2^{0.5n^2-1.5n+1}$	$\leq$	$\#\neg\mathrm{CNS}_n$, $\#\neg\mathrm{CND}_n$	$\leq$	$n^2\, 2^{0.5n^2-0.915037n}$
5	$2^{\binom{n}{2}}(1 - n^2\, 2^{-0.415037n+1})$	$\leq$	$\#\mathrm{NS}_n$, $\#\mathrm{connNS}_n$, $\#\mathrm{ND}_n$, $\#\mathrm{connND}_n$,	$\leq$	$2^{\binom{n}{2}}(1 - 2^{-n+1})$
6	$2^{\binom{n}{2}}(1 - n^2\, 2^{-0.415037n+1})$	$\leq$	$\#\neg\mathrm{NAS}_n$, $\#\mathrm{conn}\neg\mathrm{NAS}_n$	$\leq$	$2^{\binom{n}{2}}(1 - 2^{-0.25n^2-1.5n+4})$
7	$2^{0.25n^2-2n+4}$	$\leq$	$\#\mathrm{conn}\neg\mathrm{NS}_n$, $\#\mathrm{NAS}_n$, $\#\mathrm{connNAS}_n$, $\#\mathrm{conn}\neg\mathrm{ND}_n$	$\leq$	$n^2\, 2^{0.5n^2-0.915037n}$
8	$2^{0.5n^2-1.5n+1}$	$\leq$	$\#\neg\mathrm{NS}_n$, $\#\neg\mathrm{ND}_n$	$\leq$	$n^2\, 2^{0.5n^2-0.915037n}$

From lines 1 and 2 in the tables of Corollaries 2.5 and 3.5, for P any one of the 12 properties CNS, connCNS, CND, connCND, ¬CNAS, conn¬CNAS, NS, connNS, ND, connND, ¬NAS, or conn¬NAS, we have $\lim_{n\to\infty} \Pr(G_{n,1/2} \text{ is } P) = 1$. Hence

$$\lim_{n\to\infty} \Pr(G_{n,1/2} \text{ is } P)^{1/n} = 1.$$

From line 3 in the table of Corollary 2.5, for P any one of the 4 properties conn¬CNS, CNAS, connCNAS, or conn¬CND, we have

$$8\,(0.25)^n \leq \Pr(G_{n,1/2} \text{ is } P)\ \leq n^2\,(0.75)^n.$$

Hence

$$0.25 \le \liminf_{n\to\infty} \Pr(G_{n,1/2} \text{ is } P)^{1/n} \le \limsup_{n\to\infty} \Pr(G_{n,1/2} \text{ is } P)^{1/n} \le 0.75. \tag{1}$$

Similarly, from line 3 in the table of Corollary 3.5, for P any one of the 4 properties conn¬NS, NAS, connNAS, or conn¬ND, we have

$$0 \le \liminf_{n\to\infty} \Pr(G_{n,1/2} \text{ is } P)^{1/n} \le \limsup_{n\to\infty} \Pr(G_{n,1/2} \text{ is } P)^{1/n} \le 0.75. \tag{2}$$

Finally, from line 4 in the tables of Corollaries 2.5 and 3.5, for P any of the 4 properties ¬CNS, ¬CND, ¬NS, or ¬ND, then

$$0.5 \le \liminf_{n\to\infty} \Pr(G_{n,1/2} \text{ is } P)^{1/n} \le \limsup_{n\to\infty} \Pr(G_{n,1/2} \text{ is } P)^{1/n} \le 0.75. \tag{3}$$

Conjecture. We conjecture that $\lim_{n\to\infty} \Pr(G_{n,1/2} \text{ is } P)^{1/n}$ exists for each of the 12 properties considered in Equations (1), (2), and (3).

Observations 4.1 and 4.2 below and Fekete's lemma [11] may bear on the proof.

Suppose graphs G and H have disjoint vertex sets $V(G)$ and $V(H)$, respectively. The *join* $G \vee H$ of G and H is the graph with $V(G\vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G),\ v \in V(H)\}$.

For the remainder of this Section let P be any of our 24 properties except ¬CNAS, conn¬CNAS, CND, or connCND. Note that the 12 properties P considered in the above Conjecture are amongst these 20 properties.

Observation 4.1. *Suppose G_i and H_i are P graphs with $V(G_i) = V$ and $V(H_i) = W$ for $i = 1, 2$ where V and W are disjoint sets. Then $G_i \vee H_i$ is a connected P graph. If $G_1 \neq G_2$ or $H_1 \neq H_2$, then $G_1 \vee H_1 \neq G_2 \vee H_2$.*

Proof. We prove the Theorem for one property only, namely the CNAS property. We also show why the Theorem is false for the ¬CNAS property. The proofs for the remaining properties are similar.

For any $v \in V$, we have $N_{G_1\vee H_1}[v] = N_{G_1}[v] \cup W$. Now for an arbitrary $v \in V$, if $N_{G_1}[v] \subseteq N_{G_1}[u]$ for some $u \in V$ with $u \neq v$, then $N_{G_1\vee H_1}[v] \subseteq N_{G_1\vee H_1}[u]$. Similarly for an arbitrary $w \in W$. Thus $N_{G_1\vee H_1}$ is CNAS. Now observe that $G_1 \vee H_1$ is connected for all G_1 and H_1. Hence $G_1 \vee H_1$ is a connected P graph if G_1 and H_1 are P graphs. Clearly, $G_1 \vee H_1 \neq G_2 \vee H_2$ if $G_1 \neq G_2$ or $H_1 \neq H_2$.

Suppose $v \in V$ has $N_{G_1}[v] = V$ and for all $u \in V$ with $u \neq v$ then $N_{G_1}[u] \subset V$, and suppose $w \in W$ has $N_{H_1}[w] = W$ and for all $x \in W$ with $x \neq w$ then $N_{H_1}[x] \subset W$. Then both G_1 and H_1 are ¬CNAS but $N_{G_1 \vee H_1}$ is CNAS since $N_{G_1 \vee H_1}[v] = N_{G_1 \vee H_1}[w] = V \cup W$. ■

Let $f_P(n)$ denote the number of distinct P graphs on $[n]$.

Observation 4.2. *For $n, m \geq 1$, we have $f_P(n+m) \geq f_P(n)f_P(m)$.*

Proof. Let $\mathfrak{N}$ and $\mathfrak{M}$ be the set of P graphs on $[n]$ and $[n+m]-[n]$, respectively. Hence, $|\mathfrak{N}| = f_P(n)$ and $|\mathfrak{M}| = f_P(m)$. By Observation 4.1, $\{G \vee H : G \in \mathfrak{N},\ H \in \mathfrak{M}\}$ is a set of $f_P(n)f_P(m)$ distinct (connected) P graphs on $[n+m]$. Hence, $f_P(n+m) \geq f_P(n)f_P(m)$. ■

Slight improvements of the lower bound (a further quadratic factor in n) in Observation 4.2 can be made which we do not include.

5 Questions and Comments

We list several questions, with comments, that we find interesting.

(i) We have found an asymptotic formula, $\#P_n \sim 2^{\binom{n}{2}}$, for the number of labelled graphs of order n for the 12 properties P in lines 1,2,5, and 6 of Table 2. Find an asymptotic formula for the number of labelled graphs of order n for the 12 properties P in lines 3,4,7, and 8 of Table 2.

(ii) Find the correct quadratic term in the exponent of 2 for the properties P in line 7 of Table 2. Improve the linear term in the exponent of 2 for the properties in lines 3,4, and 8 of Table 2.

(iii) Prove or disprove the above Conjecture concerning the limits in (1), (2), and (3). If these limits exist, find their values.

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