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Neighborhood Champions in Regular Graphs

John McSorley
Southern Illinois University Carbondale, jmcsorley@math.siu.edu

Walter D. Wallis
Southern Illinois University Carbondale

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Neighborhood Champions in Regular Graphs

J. P. McSorley and W. D. Wallis,
Department of Mathematics,
Southern Illinois University,
Carbondale, IL 62901-4408. USA.

Abstract

For a vertex x in a graph G we define $\Psi_1(x)$ to be the number of edges in the closed neighborhood of x. Vertex x^* is a neighborhood *champion* if $\Psi_1(x^*) > \Psi_1(x)$ for all $x \neq x^*$. We also refer to such an x^* as a *unique champion*. For $d \geq 4$ let $n_0(1, d)$ be the smallest number such that for *every* $n \geq n_0(1, d)$ there exists a n vertex d-regular graph with a unique champion. Our main result is that $n_0(1, d)$ satisfies $d+3 \leq n_0(1, d) \leq 3d+1$. We also observe that there can be no unique champion vertex when $d = 3$.

1 Introduction

We assume the standard ideas of graph theory. All graphs considered in this paper will be connected. We sometimes specify the vertex-set V and edge-set E of a graph G by denoting the graph $G(V, E)$. $|V|$ and $|E|$ are respectively the *order* and *size* of G. The *distance* $d(v, u)$ between two vertices v and u in a graph is the number of edges in a shortest path from v to u. The *closed k-neighborhood* of a vertex v is defined as

$$N_k[v] = \{u \in V : d(v, u) \leq k\}.$$

When $k = 1$ we simply use the term "neighborhood".

The scale-k locality statistic $\Psi_k(x)$ of vertex x in a connected graph $G(V, E)$ was defined in [4] to be the size of the subgraph induced by the closed k-neighborhood of x. (This statistic is also called the number of edges *seen* by x.) The scale-k scan statistic $M_k(G)$ of G was then defined to be the maximum over $x \in V$ of the scale-k locality statistics:

$$M_k(G) = \max_{x \in V} \Psi_k(x).$$

In a mild abuse of notation, we define $\Psi_0(x)$ to be the degree of vertex x in G, and $M_0(G)$ to be the maximum degree in G.

A *champion* for scale k is a vertex x^* such that $\Psi_k(x^*) > \Psi_k(x)$ for all $x \neq x^*$. The reasons for studying champions are discussed in [4]; in that paper the authors discussed *dramatic champions*, for which $\Psi_k(x^*) >> \Psi_k(x)$ for all $x \neq x^*$. In [3] we construct some families of dramatic *neighborhood champions*, or champions for scale 1.

In this paper we focus on the pure graph theory of the situation, and consider the existence of neighborhood champions for scale 1 in connected regular graphs. We shall sometimes discuss graphs in which more than one vertex attains the maximum value $M_1(G)$. We shall use the word "co-champion" to denote these vertices.

1.1 Groupies

A related, but different, concept to a champion in G is a *groupie* in G. No confusion should arise between the two: A vertex in G is a groupie if the average degree of its neighbors is greater than or equal to the average degree of G. The concept was introduced in [1] (see also [2]).

2 Cubic graphs: $d = 3$

Theorem 1. *For $d = 1, 2$, and 3 there are no d-regular graphs with a unique neighborhood champion.*

Proof Clearly regular graphs of degrees $d = 1$ or 2 have no champions.

Now suppose G is a cubic graph: If $M_1(G) = 3$ then *every* vertex attains $M_1(G)$. If $M_1(G) = 4$, then any vertex x with $\Psi_1(x) = 4$ lies in exactly one triangle, and the other vertices of the triangle are co-champions; so G contains at least three co-champions. If $M_1(G) = 5$ and vertex x sees five edges, the configuration must be as shown in Figure 1, where y is a co-champion. And if $M_1(G) = 6$ (the maximum) we have $G = K_4$, and every vertex is a co-champion. Thus no cubic graph has a champion.

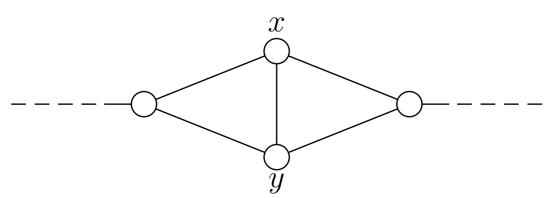

Figure 1: A cubic configuration

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However, one can construct cubic graphs with precisely two co-champions, or *twin champions*, for every even number $n \geq 10$ of vertices. From above we must have $M_1(G) = 5$.

A short exhaustive search shows that this is impossible for fewer than 10 vertices (the graphs may be found on page 127 of [5]).

For every even $n \geq 10$ we construct a cubic graph G on n vertices for which $\Psi_1(x) = M_1(G) = 5$ for precisely two vertices. Our technique is to implant the graph shown in Figure 2 as a subgraph of a host graph. The implant graph H has six vertices a, b, p, q, y, z and adjacencies ap, bq, yp, yq, zp, zq, yz.

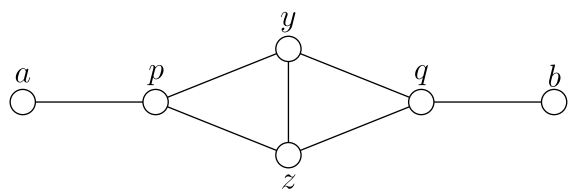


Figure 2: The graph H to be implanted

Construction Select any triangle-free cubic graph on $n-4$ vertices and choose any edge ab in that graph. Delete this edge. Then identify vertices a, b with the vertices a, b of H. See Figure 3 for an example of this; the value of $\Psi_1(x)$ is shown on each vertex and the champion is emphasized.

$\Psi_1(y) = \Psi_1(z) = 5$, $\Psi_1(p) = \Psi_1(q) = 4$ and $\Psi_1(x) = 3$ otherwise.

To show that the construction is always possible for $n \geq 10$, we observe

Lemma 1. *If $n - 4 = 2s \geq 6$, there is a triangle-free cubic graph on $n-4$ vertices.*

Proof Take the integers modulo $2s$ as vertices. For each i, let vertex i be adjacent to vertices $i-1$, $i+1$, and $i+s$ (modulo $2s$). ■

(This graph is called a *Möbius ladder* [5, p263].)

So we have

Theorem 2. *For every even $n \geq 10$ there exists a cubic graph on n vertices with precisely two co-champions.* ■

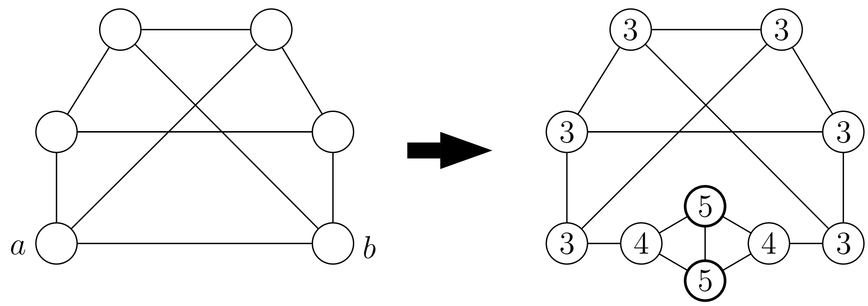


Figure 3: Example for 10 vertices

3 General Constructions: $d \geq 4$

For even $d \geq 4$ let $n_0(c,d)$ be the smallest number such that for *every* $n \geq n_0(c,d)$ there exists a n vertex d-regular graph with precisely c neighborhood co-champions; for odd d we require existence for even $n \geq n_0(c,d)$ only.

In this section we discuss $n_0(1,d)$.

The complete graph on n vertices is denoted K_n, while $K_{m,n}$ denotes the complete bipartite graph with vertex-sets of sizes m and n. A *one-factor* is a graph consisting of disjoint edges; in particular, given two ordered sets of vertices $Y = \{y_0, y_1, \ldots, y_{n-1}\}$ and $Z = \{z_0, z_1, \ldots, z_{n-1}\}$, we define the one-factor $F_j^n(Y,Z)$ to consist of the edges

$$y_0 z_j, y_1 z_{1+j}, , \ldots, y_{n-1} z_{n-1+j},$$

where subscripts are reduced modulo n. Then $K_{n,n}$ can be represented as

$$F_0^n(Y,Z) \cup F_1^n(Y,Z) \cup \ldots \cup F_{n-1}^n(Y,Z).$$

Lemma 2. *Suppose $d \geq 4$. For every $t \geq 0$ there exists a d-regular graph, with a neighborhood champion, on $n = 3d + 2t + 1$ vertices.*

Proof Let H represent the complete graph on the $d+1$ vertices $x_0, x_1, \ldots, x_d$ with the d edges of the cycle $x_0 x_1 x_2 \ldots x_{d-1}$ deleted. Take a copy of $F_0^n(Y,Z) \cup$

$F_1^n(Y,Z) \cup \ldots \cup F_{d-1}^n(Y,Z)$, where $n = d+t$, and delete the edges $y_0z_0, y_1z_1, \ldots, y_{d-1}z_{d-1}$. Adjoin this to H by adding edges $x_0y_0, x_0z_0, x_1y_1, x_1z_1, \ldots, x_{d-1}y_{d-1}, x_{d-1}z_{d-1}$.

In this graph,

$$\begin{aligned}\Psi_1(x_d) &= d(d-1)/2,\\ \Psi_1(x_i) &= (d^2-5d+14)/2, \text{ for } 0 \le i \le d-1,\\ \Psi_1(y_j) &= \Psi_1(z_j) \;=\; d, \text{ for } 0 \le j \le n-1.\end{aligned}$$

Then x_d is a champion provided $d(d-1)/2 > (d^2-5d+14)/2$, that is $d \geq 4$.

■

The above construction gives graphs whose order is of opposite parity to d. When d is odd, this provides all possible orders from some point on, because regular graphs of odd degree must have even order. However, for even degree, another construction is needed for even orders.

Suppose G is the graph of Lemma 2 in the case where $d \geq 4$ is even. We modify G to form $\hat{G}$ as follows: Add a vertex $\hat{x}$. Delete the $d/2$ edges x_0y_0, $x_2y_2, x_4y_4, \ldots, x_{d-2}y_{d-2}$, and add the d edges $\hat{x}x_0, \hat{x}y_0, \hat{x}x_2, \hat{x}y_2, \hat{x}x_4, \hat{x}y_4, \ldots, \hat{x}x_{d-2}, \hat{x}y_{d-2}$.

The Ψ_1 values of all vertices of G are unchanged. We have $\Psi_1(\hat{x}) = d + \binom{d/2}{2} = (d^2+6d)/8$. Thus vertex x_d is still the champion, and we have

Lemma 3. *Suppose $d \geq 4$ is even. For every $t \geq 0$ there exists a d-regular graph, with a neighborhood champion, on $n = 3d + 2t + 2$ vertices.* ■

Theorem 3. *Suppose $d \geq 4$. Then*

$$d + 3 \leq n_0(1,d) \leq 3d + 1.$$

Proof For any $d \geq 4$ the only d-regular graph with $d+1$ vertices is K_{d+1}, which clearly doesn't have a unique champion. And for odd $d \geq 4$ there is no d-regular graph with $d+2$ vertices, so $n_0(1,d) \geq d+3$. And for even $d \geq 4$ the only d-regular graph with $d+2$ vertices is K_{d+2} minus a one-factor, which again doesn't have a unique champion; so $n_0(1,d) \geq d+3$ here also. Hence, for any $d \geq 4$, we have $n_0(1,d) \geq d+3$. The upper bound $n_0(1,d) \leq 3d+1$ comes from Lemmas 2 and 3. ■

4 Small degrees: $d = 4, 5$

By inspection, there are no 4-regular (quartic) graphs on $n = 7$ or 8 vertices with a unique champion (see [5, p145]). From Theorem 3 and the examples for orders $n = 9, 10, 11$ and 12 shown in Figure 4 we see that $n_0(1, 4) = 9$, *i.e.,* there is a 4-regular graph, with a neighborhood champion, on n vertices whenever $n \geq 9$.

Similarly, at degree 5, inspection shows (see [5, p154]) there are no quintic graphs on $n = 6$ or 8 vertices with a unique champion. We present examples on 10, 12 and 14 vertices in Figure 5, showing that $n_0(1, 5) = 10$. Thus there is a 5-regular graph, with a neighborhood champion, on n vertices for every even $n \geq 10$. So the cases of $d = 4$ or 5 are completely solved.

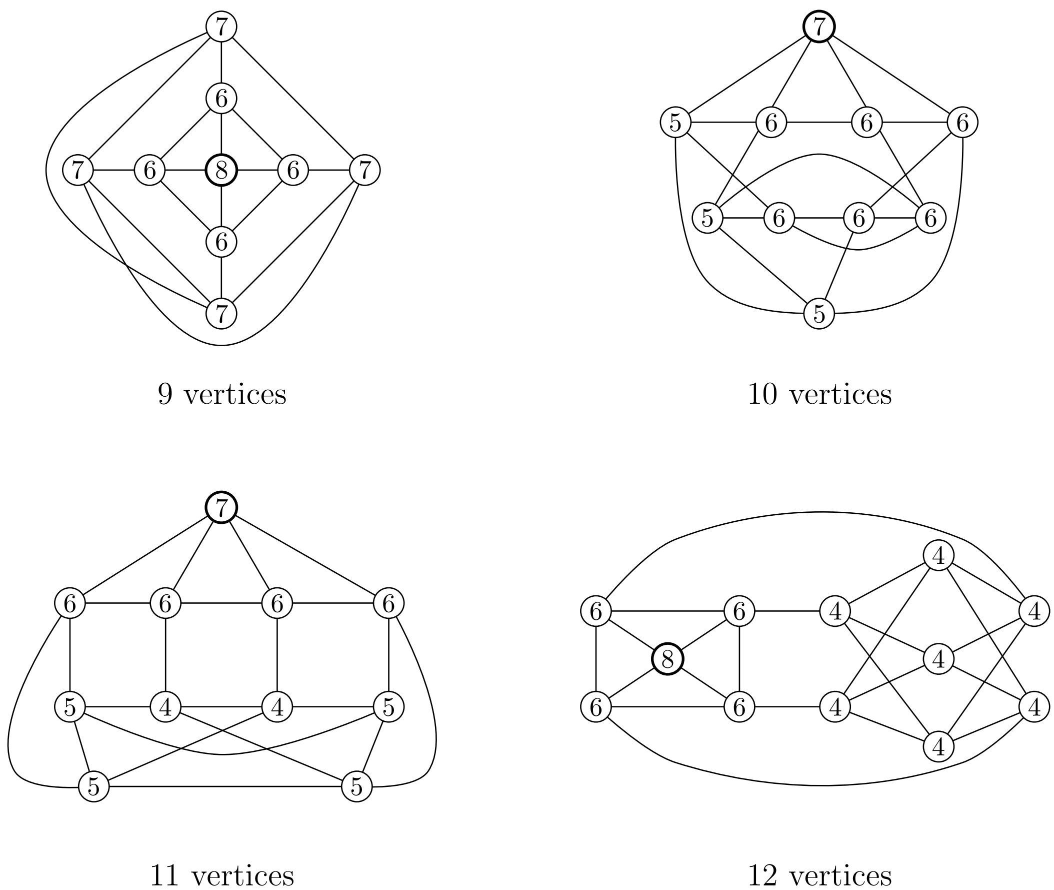
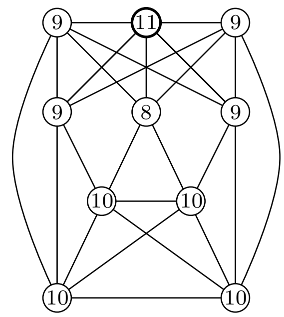
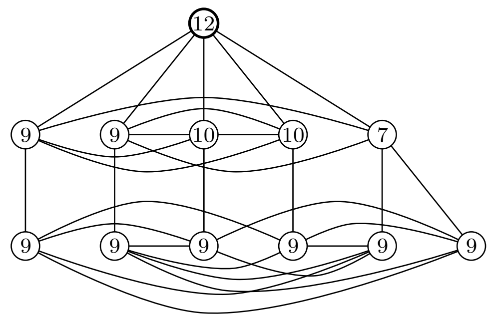


Figure 4: Small quartic graphs, each with a champion

10 vertices



12 vertices



14 vertices

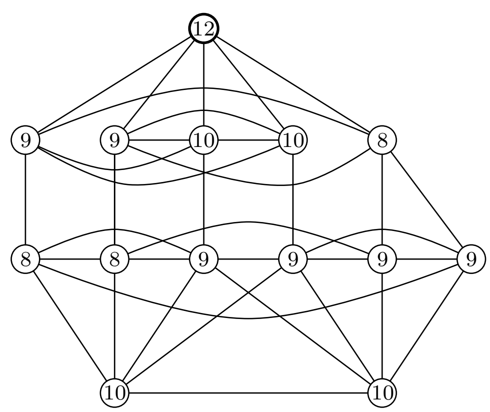


Figure 5: Small quintic graphs, each with a champion

5 Directions for future work

In our constructions above of a d-regular graph with a unique champion we allowed the function Ψ_1 to take on as many values as the construction demanded. In some applications it is desirable to restrict the range of Ψ_1 to just two values, namely $\Psi_1(x^*)$, the value of Ψ_1 at the champion vertex x^*, and $\Psi_1(x)$ for all other vertices $x \neq x^*$. Some graphs with this property were constructed in [4], and the existence of further graphs is an area for future research.

Another problem is the existence of regular graphs with precisely two co-champions. We discussed the $d = 3$ case earlier; some further results have been obtained and will be the subject of a future paper.

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