2000

Visually Building Smale Flows in S^3

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Published in *Topology and Its Applications*, 106(1), 1-19.

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Abstract

A Smale flow is a structurally stable flow with one dimensional invariant sets. We use information from homology and *template* theory to construct, visualize and in some cases, classify, nonsingular Smale flows in the 3-sphere.

Key words: Flows, knots, Smale flows, templates.
AMS Subject Classification (1991): 58F25, 57M25.

0 Introduction

The periodic orbits of a flow in S^3 form knots. For *Morse-Smale flows* there are only finitely many periodic orbits. Wada [15] has classified all links that can be realized as a collection of closed orbits of nonsingular Morse-Smale flows on S^3. Further, Wada's scheme includes an indexing of the components of the link according to whether the orbit is an attractor, a repeller, or a saddle.

In a *Smale flow*, by contrast, the saddle sets may contain infinitely many closed orbits, while the attractors and repellers must still be collections of finitely many orbits. Franks [8] devised an abstract classification scheme for nonsingular Smale flows on S^3 using a device he called the *Lyapunov graph* of a flow. Each vertex of a Lyapunov graph corresponds to an attractor, repeller or saddle set (the *basic sets* of the flow). The saddle vertices are labeled with an *incidence matrix* (determined non-uniquely by the first return map on a cross section). A simple algorithm is used to decide if a given Lyapunov graph can be realized by a nonsingular Smale flow on S^3. However, the Lyapunov graph contains no explicit information about the embedding of the basic sets. In contrast to Wada's study of Morse-Smale flows, Franks' work does does not allow one to *see* Smale flows. It was our curiosity to visualize Smale flows that motivated this paper. It is however worth noting that Wada's results have provided tools for understanding bifurcations between Morse-Smale flows [3] and it is likely that some of our results may sheld light on bifurcations between Smale flows and form Morse-Smale flows to Smale flows. Also see [11] for an example.

The project of this paper is to visually construct examples of Smale flows and in some special contexts classify all the possible embedding types. Our primary tools will be the theory of *templates*, branched 2-manifolds which model the saddle sets [1] and certain earlier results of Franks that do give some information about the embedding of closed orbits. Specifically, computations of linking numbers and Alexander polynomials are employed.

Sections 1 and 2 contain background information. Our main classification theorem (Theorem 9) is in section 3. Various generalizations and applications follow in sections 4 and 5. The author wishes to thank John Franks and Masahico Saito for helpful conversations.

1 Knots and links

A knot k is an embedding of S^1 into S^3. It is traditional to use k to denote both the embedding function and the image in S^3. A knot may be given an orientation. We will always use a flow to induce an orientation on our knots. The *knot group* of k is the fundamental group of $S^3 \backslash k$. A link of n components is an embedding of n disjoint copies of S^1 into S^3.

Two knots k_1 and k_2 (or two links) are equivalent if there is an isotopy of S^3 that takes k_1 to k_2. When we talk about a knot we almost always mean its equivalence class, or *knot type*. A *knot diagram* is a projection of a knot or link into a plane such that any crossings are transverse. If the knot has been given an orientation the crossings are then labeled as positive or negative according to whether they are left-handed or right-handed respectively.

The knot group can be calculated from a diagram and, unlike the diagram, is invariant. If a knot has a diagram with no crossings then it is called an *unknot* or a *trivial knot*. The following proposition will be of use to us.

Proposition 1 (The Unknotting Theorem, [14, page 103]) *The knot group of k is infinite cyclic if and only if k is the unknot.*

Given a diagram of a two component link $k_1 \cup k_2$ the *linking number* of k_1 with k_2 is half the sum of the signs of each crossing of k_1 under k_2 and is denoted $lk(k_1, k_2)$. The linking number is a link invariant, and thus is independent of the choice of the diagram. The Hopf link shown in Figure 1. Its linking number is ± 1, depending on the choice of orientations.

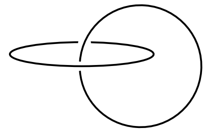

Fig. 1. The Hopf link

A knot $k \subset S^3$ is *composite* if there exists a smooth 2-sphere S^2 such that $S^2 \cap k$ is just two points p and q, and if γ is any arc on S^2 joining p to q then the knots

$$k_1 = \gamma \cup (k \cap \text{ outside of } S^2) \text{ and}$$

$$k_2 = \gamma \cup (k \cap \text{ inside of } S^2),$$

are each nontrivial, (i.e. not the unknot). We call k_1 and k_2 factors of k and write $k = k_1 \# k_2$. Of course the designation of the two components of S^3/S^2 as "inside" and "outside" is arbitrary. This implies $k_1 \# k_2 = k_2 \# k_1$. We call k the connected sum of k_1 and k_2. If a nontrivial knot is not composite then it is *prime*. Figure 2 gives an example. It shows how to factor the *square knot* into two trefoils. Trefoils are prime. It was shown by Schubert [2, Chapter 5] that any knot can be factored uniquely into primes, up to order. The unknot serves as a unit.

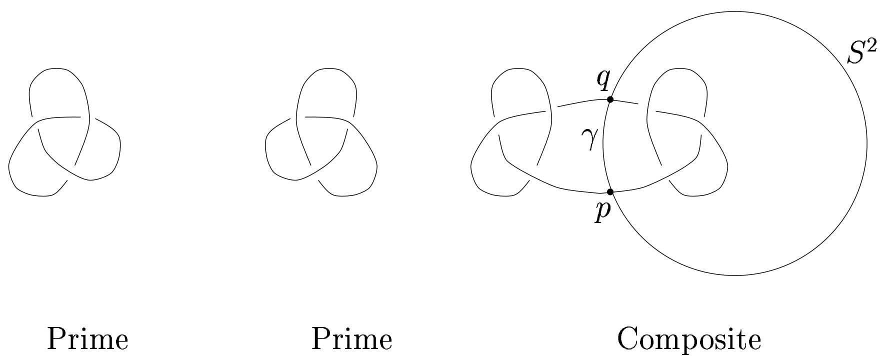

Fig. 2. The square knot is the sum of two trefoils.

A knot which fits on a torus and wraps about it p times longitudinally and q times meridianally (p and q must be relatively prime), is called a (p,q) *torus knot*. If k is any knot, then a (p,q) *cable* about k, where p and q are relatively prime, is defined as follows. Let $N(k)$ be a solid torus neighborhood of k, whose core is k. Let l be a *standard longitude* of $\partial N(k)$ for k, i.e. $lk(l,k) = 0$. Now consider a torus T with a (p,q) knot on it. Let $h : T \to \partial N$ be a homeomorphism that takes a standard longitude of T to l. The image of (p,q) under this map is said to be a (p,q) cable of k or, $(p,q)k$.

We shall extend the usual cabling notation a bit. Let $(0,1)k$ be a meridian of k, $(1,0)k$ be a standard longitude of k, and $(0,0)k$ be a loop bounding a disk on T. (A curve on a surface is *inessential* if it bounds a disk in the surface, and is *essential* if it does not.)

Thus, we have tools with which to build up new knots from old ones. In Wada's paper and in Sections 4 and 5 here, one builds new flows from old ones using processes based in part on taking connected sums and forming cables.

The cabling construction has been generalized in two ways. A knot k' is a *satellite* of a given knot k if k' lives inside a tubular neighborhood k and meets every meridianal disk. If the orientation of k' is always roughly the same as that of k (i.e. there is a fibration of $N(k)$ by meridianal disks which are always transverse to k'), we say that k' is a *generalized cable* of k. It is known that the satellite of a nontrivial knot is nontrivial [14], a fact which we shall make use of in Case 3 in the proof of Theorem 9.

Knot polynomials form an important class of knot and link invariants. We shall make use of the first known knot polynomial, the Alexander polynomial. It can be readily calculated from either a knot diagram or the fundamental group. The latter approach will be of special importance to us. The reader who wishes to check our polynomial calculations should be able to find all he or she needs in [4].

2 Dynamics of flows

A C^1 flow ϕ_t on a compact manifold M is called *structurally stable* if any sufficiently close approximation ψ_t in the C^1 topology is *topologically equivalent*, that is if there exists a homeomorphism $h : M \to M$ taking orbits of ϕ_t to orbits of ψ_t, preserving the flow direction. Structurally stable C^1 flows have a hyperbolic structure on their chain-recurrent sets [12]. We define these concepts next.

A point $x \in M$ is *chain-recurrent* for ϕ_t if for every $\epsilon > 0$ and $T > 0$ there exists a chain of points $x = x_0, \ldots, x_n = x$ in M, and real numbers $t_0, \ldots, t_{n-1}$ all bigger than T such that $d(\phi_{t_i}(x_i), x_{i+1}) < \epsilon$ when ever $0 \leq i \leq n-1$. The set of all such points is called the chain-recurrent set $\mathcal{R}$. It is a compact set invariant under the flow.

A compact invariant set K for a flow ϕ_t has a *hyperbolic structure* if the tangent bundle of K is the Whitney sum of three bundles E^s, E^u, and E^c each of which invariant under $D\phi_t$ for all t. Furthermore, the vector field tangent to ϕ_t spans E^c and there exist real numbers $C > 0$ and $\alpha > 0$ such that

$$\|D\phi_t(v)\| \leq Ce^{-\alpha t}\|v\| \text{ for } t \geq 0 \text{ and } v \in E^s,$$

$$\|D\phi_t(v)\| \leq Ce^{\alpha t}\|v\| \text{ for } t \leq 0 \text{ and } v \in E^u.$$

We also define the local stable and unstable manifolds associated to an orbit $\mathbf{O}$. They are respectively,

$$W^s_{\text{loc}}(\mathbf{O}) = \bigcup_{x \in \mathbf{O}} \{y \in M | d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to \infty \text{ and } d(x,y) \leq \epsilon\}$$

and

$$W^u_{\text{loc}}(\mathbf{O}) = \bigcup_{x \in \mathbf{O}} \{y \in M | d(\phi_t(x), \phi_t(y)) \to 0 \text{ as } t \to -\infty \text{ and } d(x,y) \leq \epsilon\}.$$

The global stable and unstable manifolds are defined similarly by removing the condition that $d(x,y) \leq \epsilon$.

It was shown by Smale that if the chain-recurrent set $\mathcal{R}$ of flow has a hyperbolic structure then $\mathcal{R}$ is the union of a finite collection of disjoint invariant compact sets called the *basic sets*. Each basic set $\mathcal{B}$ contains an orbit whose closure contains $\mathcal{B}$. The periodic orbits of a basic set $\mathcal{B}$ are known to be dense in $\mathcal{B}$. A basic may either consist of a single closed orbit or it may contain infinitely many closed orbits and infinitely other nonperiodic chain-recurrent orbits. In the later case any cross section is a Cantor set and the first return map is a *subshift of finite type*. In the former case any cross section is a finite number of points but the first return map is still a (trivial) subshift of finite type. Thus, each basic set is a suspension of a subshift of finite type. A nontrivial basic will be called *chaotic*.

Definition 2 *A flow ϕ_t on a manifold M is called a* Smale flow *provided*

(a) the chain-recurrent set $\mathcal{R}$ of ϕ_t has a hyperbolic structure,
(b) the basic sets of $\mathcal{R}$ are one-dimensional, and
(c) the stable manifold of any orbit in $\mathcal{R}$ has transversal intersection with the unstable manifold of any other orbit of $\mathcal{R}$.

Most references allow for fixed points but we will be working primarily with nonsingular flows. Smale flows on compact manifolds are structurely stable under C^1 perturbations but are not dense in the space of C^1 flows. It is easy to see that for dim $M = 3$, each attracting and repelling basic set is a closed orbit. The admissible saddle sets, however, may be chaotic. A Smale flow with no chaotic saddle sets is called a *Morse-Smale* flow.

For a chaotic saddle set of a Smale flow in a 3-manifold one can construct a neighborhood that is foliated by local stable manifolds of orbits in the flow. Collapsing in the stable direction produces a branched 2-manifold. With a semi-flow induced from the original flow, this branched 2-manifold becomes what is known as a *template*. The template models the basic saddle set in that the saddle set itself can be recovered from the template via an inverse limit process and that any knot or link of closed orbits in the flow is smoothly isotopic to an equivalent knot or link of closed orbits in the template's semi-flow. The proof of this is due to Birman and Williams [1] and can also be found in [9, Theorem 2.2.4].

A key tool in the analysis of hyperbolic flows is the concept of a Markov partition. We refer the reader to [7] for details. In our context a Markov partition is a finite, disjoint collection of disks transverse to a basic set of a flow. Each orbit of the basic set must pass through some element of the Markov partition in forward time.

Definition 3 *Given a Markov partition $\{m_1, \ldots, m_n\}$ for a suspended sub-shift of finite type with first return map ρ we define the corresponding $n \times n$* incidence matrix *A, by*

$$A_{ij} = \begin{cases} 1 \text{ if } \rho(m_i) \cap m_j \neq \phi \\ 0 \text{ otherwise.} \end{cases}$$

The incidence matrix, like a knot diagram, is not invariant but does contain invariant information. We can encode additional information about the embedding of a basic set by modifying the incidence matrix.

Definition 4 *Given a Markov partition for a basic set with first return map ρ, assign an orientation to each partition element. If the partition is fine enough the function*

$$O(x) = \begin{cases} +1 \text{ if } \rho \text{ is orientation preserving at } x, \\ -1 \text{ if } \rho \text{ is orientation reversing at } x, \end{cases}$$

is constant on each partition element. The structure matrix *S is then defined by $S_{ij} = O(x)A_{ij}$, where x is any point in the i-th partition element.*

The next proposition was proved by Franks in [5].

Proposition 5 (Franks, 1977) *Let ϕ_t be a Smale flow with a single attracting closed orbit a, and a single repelling closed orbit r, with saddle sets $\Lambda_1, \ldots, \Lambda_n$. Then the absolute value of the linking number of a and r is given* *by*

$$|lk(a,r)| = \prod_{i=1}^{n} |\det(I - S_i)|,$$

where $S_1, \ldots, S_n$ denote the respective structure matrices of the saddle sets.

The manner in which a saddle set "links" a closed orbit k, is described by modifying the structure matrix S to form a *linking matrix* L_k. Consider a Markov partition, $\{m_1, \ldots, m_n\}$, of the saddle with incidence matrix A. Pick a base point b in $S^3 - k$ and paths p_i from b to m_i, also in $S^3 - k$. For each $a_{ij} \neq 0$ let γ_{ij} be a segment of the flow going from m_i to m_j without meeting any of the other partition elements. Now form a loop consisting of γ_{ij}, p_i, p_j and, if needed, short segments in m_i and in m_j. If the disks have been chosen small enough, then the linking number of any such loop with k depends only on i and j. That this can always be done is shown in [6].

Definition 6 *The* linking matrix L *associated with a suitable Markov partition for a given closed orbit k is defined to by*

$$L_{ij} = S_{ij} t^q,$$

where q is the linking number of the loops formed from segments connecting m_i to m_j and k as described above.

The following proposition is a special case of Theorem 4.1 in [6].

Proposition 7 (Franks, 1981) *Let ϕ_t be a Smale flow in S^3 with one attracting closed orbit a, one repelling closed orbit r, and a single saddle set s. Let L_a and L_r be the linking matrices for s with respect to a and r respectively. Then the Alexander polynomials of a and r are given by $\Delta_a(t) = \det(I - L_a)$ and $\Delta_r(t) = \det(I - L_r)$ respectively, up to multiples of $\pm t$.*

Finally, we record a proposition about Morse-Smale flows. It is an obvious corollary to Wada's theorem [15], though it is easy to prove directly.

Proposition 8 *In a (nonsingular) Morse-Smale flow on S^3 with exactly two closed orbits, the link of closed orbits forms a Hopf link with one component a repeller and the other an attractor.*

3 Lorenz-Smale flows

By a *simple Smale flow* we shall mean a Smale flow with three basic sets: a repelling orbit r, an attracting orbit a, and a nontrivial saddle set. In this section we will show how to construct simple Smale flows in which the saddle set l can be modeled by an embedding of the Lorenz template shown in Figure 3. That is, there is an isolating neighborhood of the saddle set l foliated by local stable manifolds of the flow, such that when we collapse out in the stable direction we get an embedding of the Lorenz template. Call such flows *Lorenz-Smale flows*. In this section we classify all possible Lorenz-Smale flows.

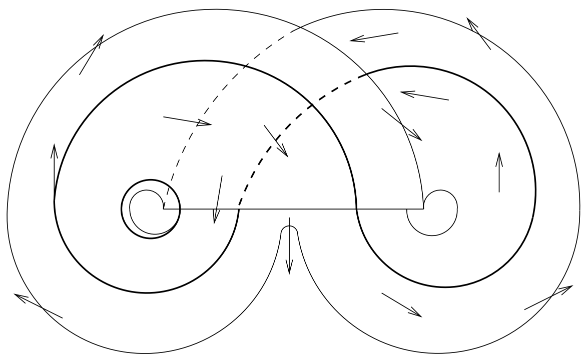

Fig. 3. The Lorenz template

We shall call the isolating neighborhood of the saddle set L. The set of points of ∂L where the flow is transverse outward is called the *exit set*. The backward orbits of these points approach orbits in the saddle set l. The exit set consists of two annuli, X and Y, connected by a long strip S. The core of the exit set is homeomorphic to the boundary of the Lorenz template. We shall refer to the cores of X and Y, as x and y, respectively. See Figure 4. The set of points of ∂L where the flow is transverse inward is called the *entrance set*. The entrance set also consists of two annuli connected by a long strip. Denote these two annuli by X' and Y' and their respective cores x' and y' and the connecting strip by S'. See Figure 4. The intersection of the closures of the entrance and exit sets consists of three closed curves where the flow is tangential to ∂L. Although the entrance set is harder to visualize, its topological type can be determined by an Euler characteristic argument. Notice that x and x' are isotopic to $\overline{X} \cap \overline{X'}$ and hence have the same knot type. Similarly, y and y' must have the same knot type.

Take tubular neighborhoods of a and r and denote them by A and R respectively.

To build a Smale flow from these building blocks, we first attach the closure of the exit set of L to ∂A. This gives a vector field on a new 3-manifold pointing inward along its entire boundary. That this can be done smoothly was shown

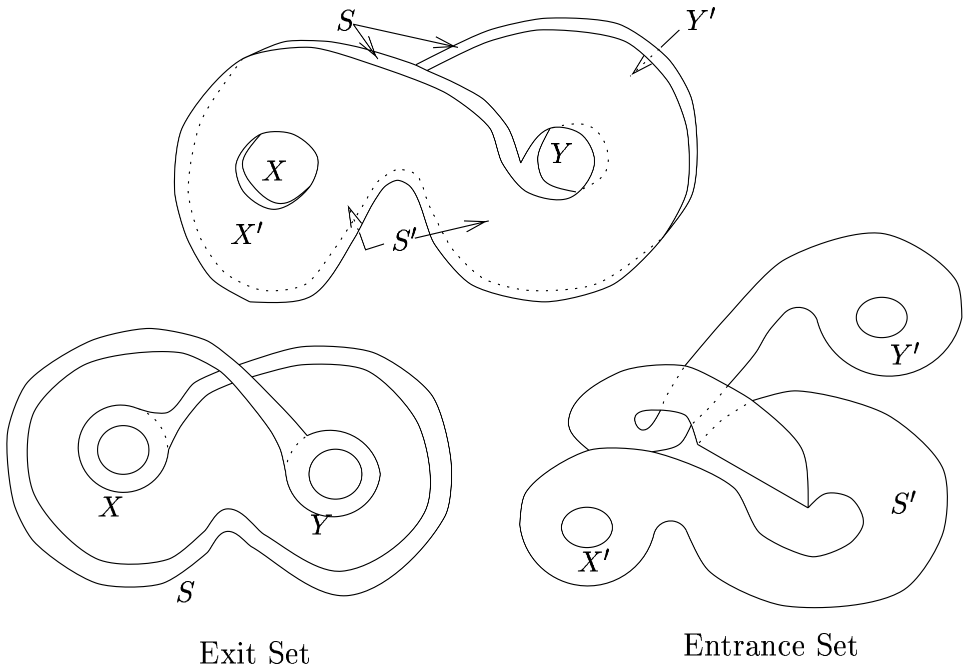

Fig. 4. Neighborhood of a Lorenz saddle set (top) with the exit set (bottom left) and entrance set (bottom right)

by Morgan in [13]. Next attach ∂R to the boundary of $A \cup L$ so that the union $A \cup L \cup R$ is S^3.

We know from Proposition 5 that the linking number between r and a must be ± 1, but what types of knots can a and r be? What are all the different ways the saddle set can be embedded so as to still have the Lorenz template as a model? This later question is made precise by asking, what types of knots can x and y be? Can they be linked? Can the annuli X and Y have any number of twists or are there restrictions? (This last question is equivalent to finding the *framing* of a knot.)

To answer these questions we use the following framework. First, we study what may happen when the attaching of the exit set of L to ∂A is such that x and y are both inessential in ∂A. Then we investigate the case where one of them, say y, is essential but the other is still inessential. Finally we consider the case where both x and y are essential in ∂A. The results of this analysis are then consolidated into the statement of Theorem 9. Our classification scheme is only up to isotopy of S^3, plus mirror images and flow reversal. Also, we shall not be concerned with the orientation, i.e. the flow direction, of a or r, since their orientations can be easily reversed by modifying the flow in a tubular neighborhoods of a and r, [15, first paragraph].

Theorem 9 *For a Lorenz-Smale flow in S^3 the following and only the following configurations are realizable. The link $a \cup r$ is either a Hopf link or a trefoil and meridian. In the latter case the saddle set is modeled by a stan-*

dardly embedded Lorenz template, i.e. both bands are unknotted, untwisted, and unlinked, with the core of each band a meridian of the trefoil component of $a \cup r$. In the former case there are three possibilities: (1) The saddle set is standardly embedded. (2) One band is twisted with n full-twists for any n, but remains unknotted and unlinked to the other band, which must be unknotted and untwisted. (3) One band is a (p, q)torus knot, for any pair of coprime integers, with twist $p+q-1$. The other band is unknotted, untwisted and unlinked to the knotted one.

PROOF. The proof is divided into three cases.

CASE 1: Suppose both x and y are inessential in ∂A. It follows that X and Y are untwisted, that is the linking number between each of the two components of ∂X and of ∂Y is zero. It is also obvious that x and y are unknotted and unlinked.

There are two subcases to consider. It could be that x and y are concentric in ∂A, or it could be that they are not. That is x and y may or may not form the boundary of an annulus in ∂A. In Figure 5 we show that both cases can be realized. The neighborhood L is attached to a 3-ball B along the closure of the exit set of L. Figure 5 also shows two ways one might attach handles to the 3-ball so as to turn it into a solid torus. Suppose we attach the handle to the small disks marked C and C' in the manner shown. Call the resulting solid torus A_1. If we take $L \cup A_1$ the result is still a solid torus, and the complement in S^3 is just another solid torus, R_1. We can now build a Smale flow with an attractor in A_1, a repeller in R_1 and a Lorenz saddle set in L. Upon further inspection the reader should be able to see that x and y are concentric.

Now, instead on attaching a handle at C and C', we attach one to B and B' as shown again in Figure 5. This time call the solid torus obtained A_2. As before $L \cup A_2$ is a solid torus with solid torus complement in S^3. Thus, we have constructed another Lorenz-Smale flow with x and y inessential on a tubular neighborhood of the attractor. Is it diffeomorphic to the Lorenz-Smale flow we constructed before? To see that the answer is no, study the loops x and y again. They are still both inessential, that is they both bound disks in ∂A_2. But they are no longer concentric. This can be seen from careful study of Figure 5.

In both these examples the attractor and repeller form a Hopf link. We claim that if both x and y are inessential in ∂A then a and r must form a Hopf link. Since R is a tubular neighborhood of a knot, the union of A and L is a knot complement. But we will show that $A \cup L$ is a solid torus whose core has the same knot type as that of a. Thus, we could remove $A \cup L$ from our flow and replace it with a solid torus containing just an attractor and no saddle set. This gives us a nonsingular Morse-Smale flow on S^3 with just two closed orbits whose link type is the same as $a \cup r$. But by Proposition 8 these must form a Hopf link.

We now show that $A \cup L$ is a solid torus of whose core is the same knot type as a. First, assume that x and y are not concentric. Then they bound disjoint disks D_x and D_y in ∂A. Expand D_x and D_y, if needed, so that they contain all of X and Y respectively but remain disjoint. Thicken D_x and D_y by pushing into A a little, forming two 3-balls B_x and B_y, which are disjoint and do not meet the orbit a. Let $L' = L \cup B_x \cup B_y$ and $A' = A - (B_x \cup B_y)$. It is clear that A' is a solid torus whose core is still a. The set L' is a 3-ball and $L \cup A = L' \cup A'$. But the union $L' \cup A'$ is taken along the disk $S \cup \overline{(\partial B_x - D_x)} \cup \overline{(\partial B_y - D_y)}$. Thus, there is a deformation retract from $L' \cup A'$ to A'. This proves our claim for x and y not concentric.

Now suppose x and y are concentric and assume x is inner most. Then construct the 3-ball B_x as before. Let $L' = L \cup B_x$ and $A' = A - B_x$. This time L' is a solid torus. It is attached to A' along the annulus $Y \cup S \cup (B_x - D_x)$ whose core y is inessential in $\partial A'$ and is a (1,0) longitude in L'. Thus we can retract L' to Y and push it into A' without changing the knot type of a. This finishes the proof of our claim.

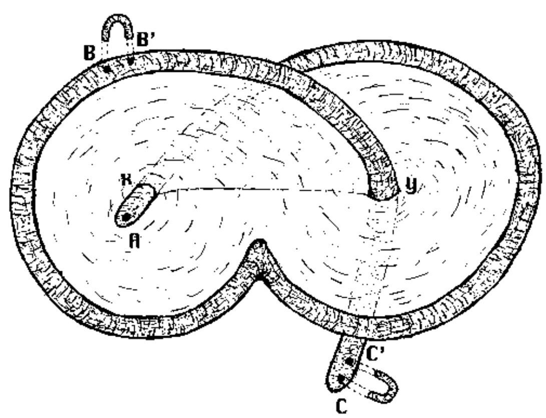

Fig. 5. Lorenz saddle set neighborhood attached to a ball.

CASE 2: Suppose that x is inessential but that y is essential. The opposite case is similar. Again, it is clear that X is untwisted and that x is unknotted and unlinked to y. We shall again show that a and r must form a Hopf link. It then follows that since y lives in a standardly embedded torus, ∂A, y is a torus knot or unknot. If y is a meridian (0,1), or a longitude (1,0) then Y is untwisted. If y is an unknot $(1, q)$ or $(q, 1)$ then Y has q full twists. For nontrivial torus knots the twisting in Y is uniquely determined by the knot type of y. If y is a (p, q) torus knot then the twist in Y is $p + q - 1$.

That any (p,q) torus knot can be realized by y is shown by construction in Figure 6. One places a (p,q) curve on a torus. Attach an annulus to this curve along one boundary component. Add a "Lorenz ear" to form a Lorenz template. Next a "finger" pushes out of the torus and snakes along the boundary of the template and finally pokes through the x loop. Thicken this complex up to get $A \cup L$. The repeller is then placed as a meridian in the complement. An example with y a $(2,1)$ curve is shown in Figure 7.

The argument that a and r must form a Hopf link is the same as in the concentric subcase of Case 1 above. The core y of the annulus Y is a $(1,q)$ cabling of the core of the solid torus L'. We can foliate L' with meridianal disks each of which meets Y in an arc. Thus, Y is a deformation retraction of L'. We then push Y into A'.

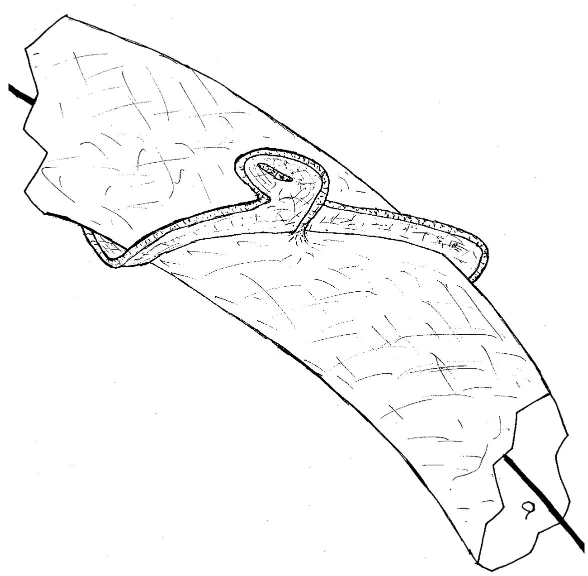

Fig. 6. y is a (p,q) cable of a

CASE 3: Suppose that both x and y are essential in ∂A. In Figure 8 we give an example. The loops x and y are meridians in ∂A. They are also standard longitudes in ∂R. The attractor a is a trefoil knot while r is unknotted and is a meridian of a. We claim that up to mirror images and flow reversal, this is the only possible configuration.

Now consider the general setting. The attaching map from ∂L to ∂A takes x and y to two copies of some (p,q) cable knot of the attractor, a. Here we allow p or q to be zero, but not both. Likewise, the attaching map from ∂L to ∂R takes x' and y' to some pair of $(p',q')r$ knots. Of course x and y are respectively ambient isotopic to x' and y' within ∂L, so all four have the same knot type.

For future reference, let $T = \langle x, y | xyx = yxy \rangle$. A knot k with $\pi_1(S^3 - k) = T$

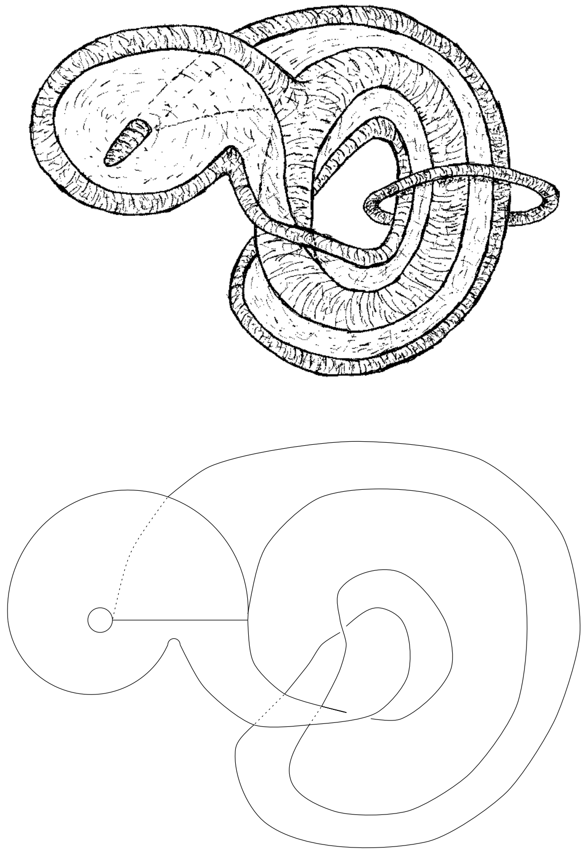

Fig. 7. The top figure has an attractor in the fat tube and a repeller in the thin tube. A template for the saddle set is shown below. The loop y is a (2,1) torus curve. Any (p, q) torus curve, knotted or unknotted, can be realized.

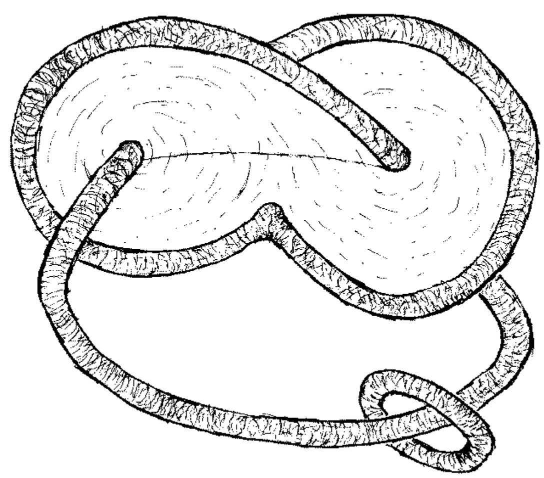

Fig. 8. Lorenz-Smale flow with a trefoil attractor

is a left or right handed trefoil. See [2, Lemma 15.37, Corollary 15.23].

Lemma 10 *The Alexander polynomials of the attractor and repeller are* $\Delta_a =$

$t^q - 1 + t^{-q}$ *and* $\Delta_r = t^{q'} - 1 + t^{-q'}$, *respectively*

PROOF. If the linking matrix for a is

$$\begin{bmatrix} t^q & t^q \\ t^{-q} & t^{-q} \end{bmatrix},$$

then the result follows by Proposition 7. The only difficulty in determining the linking matrix is the assignment of the signs to the powers of the t's. One can check our assignment explicitly for the $q = 1$ case by studying Figure 8. In general, if the powers are all of the same sign, the polynomial that results is not symmetric in t, nor is any $\pm t$ multiple. But it is well known that the Alexander polynomial of a knot is symmetric in t, up to multiples of $\pm t$. That is $\Delta(t) = \pm t^n \Delta(1/t)$, for some n. See [4]. □

Not ready:

Lemma 11 *The fundamental groups of* $L \cup A$ *and* $L \cup R$ *are* $\langle xy | x^p y x^p = y x^p y \rangle$ *and* $\left\langle xy | x^{p'} y x^{p'} = y x^{p'} y \right\rangle$, *respectively. The Alexander polynomials of* a *and* r *are* $\Delta_a = t^{p'} - 1 + t^{-p'}$ *and* $\Delta_r = t^p - 1 + t^{-p}$, *respectively.*

PROOF. We shall find $\pi_1(L \cup R)$ using the Seifert/Van Kampen Theorem. The calculations of $\pi_1(L \cup A)$ are similar. The Alexander polynomials can then be determined from Fox's Free Differential Calculus [4].

We must choose generators for L, R and $L \cap R$. The generators for $L \cap R$ and L are shown in Figure 9. The base point b, is in the "middle" of the strip S'. For L we shall abuse notation slightly and call the generators x and y, as they are isotopic to the x and y loops, however, we do not use the orientation of the flow. (By the proof of the previous lemma the images of x and y must wrap around ∂R in the same direction.) Denote the generators of $L \cap R$ by w and z. For R we shall use a loop isotopic to r but with base point $b \in \partial R \cap S'$. Again we abuse notation and call this new loop r.

The fundamental groups of interest are then $\pi_1(R) = \langle r \rangle$, $\pi_1(L) = \langle x, y \rangle$, and $\pi_1(L \cap R) = \langle w, z \rangle$. The homomorphisms induced by inclusion maps are $\alpha : \pi_1(L \cap R) \to \pi_1(R)$ and $\beta : \pi_1(L \cap R) \to \pi_1(L)$. These give $\alpha(w) = r^p$, $\alpha(z) = \overline{r}^p$, $\beta(w) = yx\overline{y}$, and $\beta(z) = xy\overline{x}$,. By Van Kampen's theorem $\pi_1(L \cup R) = \langle r, x, y | r^p = yx\overline{y},\ \overline{r}^p = x\overline{yx} \rangle \cong \langle r, y | \overline{y} r^p \overline{y} = r^p \overline{y} r^p \rangle$. □

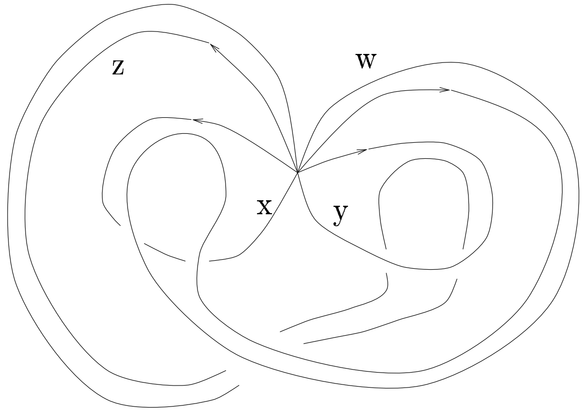

Fig. 9. The generators of $\pi_1(R \cap L)$ and $\pi_1(L)$.

It follows from Lemmas 10 and 11 that if x is a (p, q) curve on ∂A then x' is a $(\pm q, p)$ curve on ∂R.

If p or q is zero then the other is ± 1 since the curve is in a torus. Now suppose x is a $(0, \pm 1)$ curve on ∂A. Then $\pi_1(L \cup R) \cong T$, and so a is a trefoil knot. Since x is a meridian of a and x is isotopic to x' which in turn is isotopic to r, we see that r must be a meridian of a.

If x is a $(\pm 1, 0)$ curve on ∂A then the rolls of a and r are switched.

It is left only to show that p and q cannot both be nonzero. It shall be useful to study the attaching of the exit set of L to ∂A in terms of the boundary curves of the exit set. They consist of three loops denoted as α, β and γ. We take α to be isotopic to x and β to be isotopic to y. Then γ is the remaining curve. See Figure 10. Our strategy is to show that if p and q are both nonzero then γ bounds a disk in ∂A and that γ is a nontrivial knot. This contradiction will then prove our claim.

Let $\partial_+ L$ be the closure of the exit set of L. Clearly $(\partial A \backslash \partial_+ L) \cup \partial_+ L$ is torus. Now α and β bound an annulus $\alpha\beta$ in $(\partial A \backslash \partial_+ L)$. Thus, $(\partial A \backslash \partial_+ L)$ has two components, the annulus $\alpha\beta$ and another component we shall call D which has a single boundary component γ. Now, $\partial_+ L \cup \alpha\beta$ is a torus with a disk removed. Since $\partial_+ L \cup \alpha\beta \cup D$ must be a torus, D is a disk. Since this torus is embedded in S^3 it follows that γ, the boundary of D, is unknotted.

The Alexander polynomial calculations in Lemma 11 show that a is knotted, and thus even for the $(1, 1)$ case α and β are nontrivial knots. Now since α and β are parallel knots in ∂A we can deform L to appear as in Figure 10. By studying Figure 11 we see that γ is a satellite of α. This implies γ is nontrivial and completes the proof of Theorem 9. □

Corollary 12 *In any Lorenz-Smale flow there is a pair of unlinked saddle*

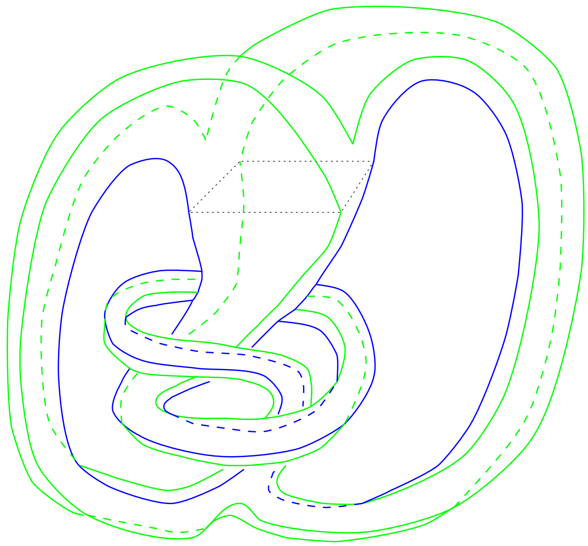

Fig. 10. The gray curve is γ.

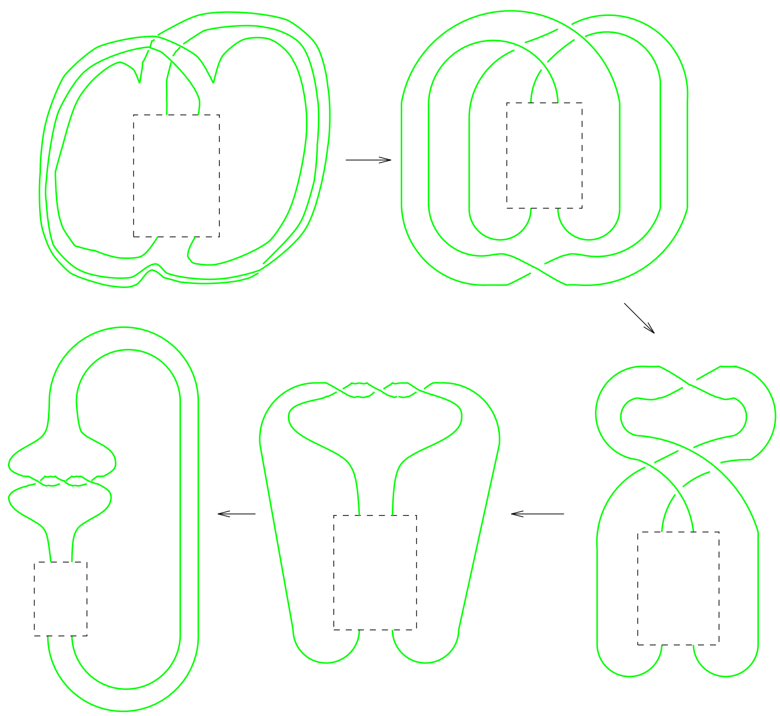

Fig. 11. γ is a satellite of α

orbits.

PROOF. In the Lorenz template there are two fixed points in the first return map of the branch line. These correspond to a pair of unlinked unknots in the semi-flow if the Lorenz template is standardly embedded. In all of the embeddings allowed for in Theorem 9 these two orbits remain unlinked. Thus, by the Birman-Williams template theorem [1], the saddle set also contains a pair of unlinked orbits. □

Corollary 13 *Consider a Smale flow on S^3 with an attracting fixed point, a repelling fixed point, a Lorenz saddle set and no other basic sets. There is only one possible configuration, and in it the template of the saddle set is a standardly embedded Lorenz template.*

PROOF. The proof is similar to Case 1 above. □

4 Connected sums

Wada's classification theorem for Morse-Smale flows is based on applying a series of *moves* to one or two existing Morse-Smale flows and building up new ones. Conclusion (a) of the next theorem establishes an operation that produces a new Smale flow from two existing ones that is similar to Wada's move IV [15].

Theorem 14 *Let ϕ_1 and ϕ_2 be nonsingular Smale flows on S^3 such that (1) they each have only one attracting closed orbit with knot types k_1 and k_2 respectively, (2) there is only one repelling closed orbit which is unknotted and is a meridian of the attractor and, (3) the repellers bound disks whose interiors meet the chain-recurrent sets in a single point. Then (a) and (b) below hold true.*

(a) There exists a nonsingular Smale flow on S^3 such that there is only one attracting closed orbit which has knot type $k_1 \# k_2$, and there is only one repelling closed orbit which is unknotted and is a meridian of the attractor.

(b) There exists a nonsingular Smale flow on S^3 such that there is only one attracting closed orbit which has knot type k_1, and there is only one repelling closed orbit which has knot type k_2. Furthermore, the attractor and the repeller have linking number one, and can be placed into solid tori whose cores are meridians of the each other.

PROOF. The proofs are simple cut and paste arguments. Some details are left to the reader. For (a) let V_i, $i = 1, 2$, be tubular neighborhoods of the repellers in ϕ_i, $i = 1, 2$, respectively. Let D_i, $i = 1, 2$, be the disks described in hypotheses (3). Thicken up these disks a bit by taking cross product with a small interval $I = [-1, 1]$. We require that each $D_i \times I$ meet the corresponding k_i in an unknotted arc. Let $D_i^{\pm} = (D_i \times \{\pm 1\}) \backslash V_i$. Choose the signs so that flows enter the thickened disks on the positive sides. See Figure 12. Delete from the 3-sphere of each flow the interior of the union of V_i and $D_i \times I$, for the corresponding $i = 1, 2$. We now have flows on two cylinders C_i, $i = 1, 2$. See Figure 13. The boundary of C_i is the union of D_i^+, D_i^-, and the annulus $A_i = \partial V_i \backslash D_i \times I$. The flow exits C_i on the interior of D_i^+, for $i = 1, 2$. Glue C_1 to C_2 by identifying D_1^+ with D_2^- and D_2^+ with D_1^-. This creates a solid torus, V. The flow induced on V is inward on $\partial V = A_1 \cup A_2$. Further, the identifications can be chosen so that the flow on V has an attracting orbit with knot type $k_1 \# k_2$, assuming V is standardly embedded. It is now clear how to construct the desired flow on S^3.

The proof of conclusion (b) is similar and in fact simpler and so is left as an exercise. Note that hypotheses (3) is not required. □

Fig. 12. Cut out these balls.

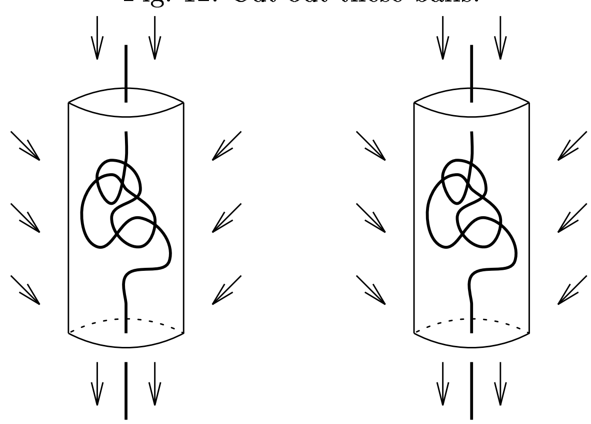

Fig. 13. Paste cylinders together

5 Attracting Links

The class of links which can arise in Morse-Smale flows (nonsingular on S^3) is, according to Wada's Theorem, quite limited. For Smale flows it is easy to construct examples in which every knot and link can be realized simultaneously as a saddle orbits. This is a consequence of the existence of *universal templates*, templates in which all links all realized as closed orbits, [10]; also see [9] and [16].

Franks has shown that any link can be realized as an attractor of a Smale flow [6, Propsition 6.1]. In the proof the link is realized as a braid in an unknotted solid torus whose entrance set is its entire boundary. Thus given a Smale flow with attractor k we can replace k with any generalized cable of k, though a new saddle set will typically be introduced.

In [9] it is shown that given a Smale flow ϕ with a saddle set modeled by a template T containing the closed orbit k, there exists another Smale flow ϕ' with the same basic sets as ϕ except that k is an attractor and and the template T has been replaced (as a model) with T', a template formed by "surgering" T along k (a standard template operation).

Turning our attention to simple Smale flows we shall use a similar idea to show that given any knot k there exists a simple Smale flow with attractor k and repeller a meridian of k. As a corollary of the construction we can give a "dynamics" proof that Alexander polynomials multiply under connected sums.

Theorem 15 *For any knot k there exists a simple Smale flow such that with attractor k and repeller a meridian of k that does not link any closed orbits in the saddle set.*

PROOF. The template U shown in Figure 14(a) is known to contain all knots as periodic orbits [10]. Thus we can suppose k has been realized as an orbit in U. We shall work with a variation of U shown in Figure 14(b) and denoted V. It has five "Lorenz ears". Notice however that the middle ear does not stretch all the way across; it is to extend only as far as an outer most arc of k. (Technically V is not a template, but it is still a branched manifold with a semi-flow).

Now consider the rather odd looking object in Figure 15. The dark gray circle represents the tubular neighborhood of a repeller. The the light gray tube has the same knot type as k (though only a portion of it is shown); we have only added an extra loop in an outermost strand of k. The dark region is a topological ball which meets the light gray tube at a single disk near the cusp of the fourth Lorenz ear. Their union is a solid torus A. The branched manifold V has been cut open along k and is now a true template T (compare with the proof of Theorem A.3.3 in [9]). The boundary of T is in the boundary of A. We thicken up T to get TT. Now we can regard TT as a neighborhood of a saddle set. Its exit set is attached to A as required. From the figure we can see that $A \cup TT$ is a solid unknotted torus. Thus, we can use a meridian of k as a repeller and build up the desired flow. □

Corollary 16 *The Alexander polynomial multiplies under connected sums of knots.*

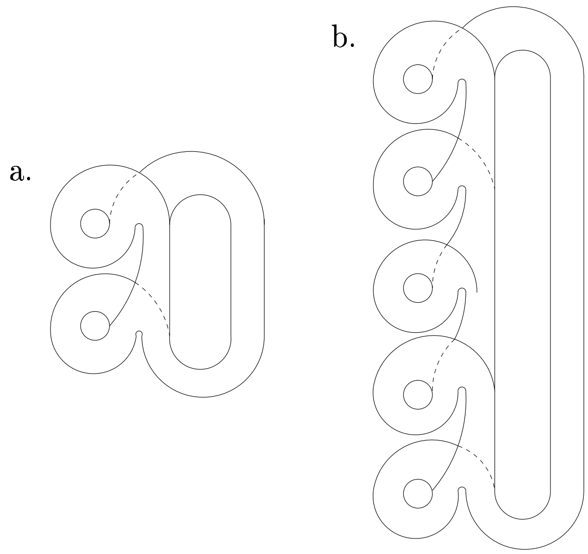

Fig. 14. Two templates containing all knots

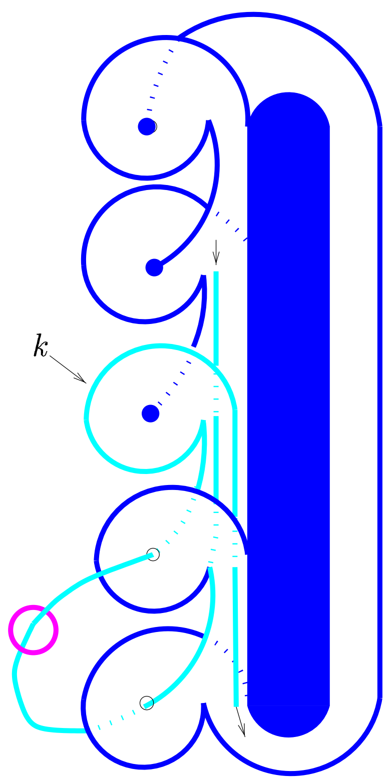

Fig. 15. The attractor k is inside the light gray tube, the repeller is in the dark gray tube while the dark region is a ball in the basin of attraction of k

PROOF. The claim is that given knots k_1 and k_2 then $\Delta_{k_1} \cdot \Delta_{k_2} = \Delta_{k_1 \# k_2}$. By Theorem 15 there exist Smale flows for k_1 and k_2 that satisfy the hypotheses of Theorem 14. We use conclusion (a) of Theorem 14 to construct a Smale flow with attractor $k_1 \# k_2$ and apply [6, Theorem 4.1] noting that there are only two saddle sets and hence only two linking matrices needed in the formula given in [6, Theorem 4.1]. □

Remark 17 (Concluding remarks) *We have in these last two sections given a variety of tools for building new Smale flows from old ones. Many other such results could be stated. But, we are nowhere close to developing a calculus of Smale flows along the lines of what Wada has done for Morse-Smale flows. Indeed we don't even know if there are any restrictions on the link type of* $a \cup r$ *for simple Smale flows.*

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