Fall 2010

Representations of Finite Groups

Joseph Hundley
Southern Illinois University Carbondale, jhundley@math.siu.edu

REPRESENTATIONS OF FINITE GROUPS

JOSEPH HUNDLEY

Contents

1. Basic Definitions 1
1.1. $\boldsymbol{G}$-Modules 3
1.2. A digression 3
2. Direct sum and tensor product 3
3. Irreducibles and complete reducibility 4
4. Irreducible representations of finite abelian groups 5
5. Irreducible Representations of S_3 5
6. Characters 6
References 9

1. Basic Definitions

I'm following [F-H]. First let me fill in a little bit of background material.

Definition 1.0.1. *(field) A* **field** *is a set F equipped with two binary operations $+$ and $\cdot$ such that $(F,+)$ and $(F\setminus\{0\},\cdot)$ are abelian groups, and*

$$x\cdot(y+z)=x\cdot y+x\cdot z \qquad (\forall x,y,z,\in F).$$

Examples 1.0.2. *$\mathbb{Q},\mathbb{R}$ and $\mathbb{C}$ are fields, while $\mathbb{Z}$ is not.*

Definition 1.0.3. (Vector space) *Let F be a field. A* **vector space** *over F is an abelian group V equipped with a function $F\times V\to V$ called* **scalar multiplication** *and written*

$$(\alpha,v)\in F\times V\mapsto \alpha v\in V,$$

such that

$$\begin{aligned}
\alpha(v+w)&=\alpha v+\alpha w &&(\forall \alpha\in F,\ v,w,\in V),\\
(\alpha+\beta)v&=\alpha v+\beta v &&(\forall \alpha,\beta\in F, v\in V),\\
(\alpha\beta)v&=\alpha(\beta v) &&(\forall \alpha,\beta\in F, v\in V),\\
1v&=v &&(\forall v\in V).
\end{aligned}$$

Examples 1.0.4. *Row vectors, column vectors, matrices, polynomials, functions.*

Definition 1.0.5. (linear function)*Let F be a field and let V,W be vector spaces over F. A function $L:V\to W$ is* **linear** *if*

$$L(\alpha v+w)=\alpha L(v)+L(w) \qquad (\forall \alpha\in F,\ v,w\in V).$$

Definition 1.0.6. ($\boldsymbol{GL(V)}$) *Let F be a field and let V be a vector space over F. Then $\boldsymbol{GL(V)}$ is the set of bijective linear functions $V\to V$, equipped with the binary operation $\circ$ (composition of functions). It is a group.*

Date: December 2, 2010.

Definition 1.0.7. **(GL_nF)** *Let F be a field and n be an integer. Then* $\boldsymbol{GL_nF}$ *is the set of $n \times n$ invertible matrices with entries in F, equipped with matrix multiplication. It is a group. Further, the function $A \mapsto$(multiplication by A) is an isomorphism $GL_nF \to GL(F^n)$, where F^n is realized as column vectors.*

Definition 1.0.8. *(representation) Let G be a group. A* **representation** *of G is a homomorphism ρ from G into $GL(V)$ for some vector space V, over some field F.*

We shall work mostly with representations on vector spaces over $\mathbb{C}$.

Examples 1.0.9.
- *The isomorphism $GL_nF \to GL(F^n)$ described above,*
- *The map $\sigma \mapsto M(\sigma)$, $S_n \to GL_nF$ defined by $M(\sigma)e_i = e_{\sigma(i)}$*
- *One can use the homomorphism which maps everything to the identity.*
- *One can realize F^n as* **row** *vectors. Then right multiplication by A is an element of $GL(F^n)$ for each A. However the function $A \mapsto$(multiplication by A on the right) is not a homomorphism. But $A \mapsto$(multiplication by A^{-1} on the right) is.*
- *If we take $G = \mathbb{Z}$ then each representation of G will be determined by its value at 1. Each is of the form $n \mapsto L^n$ for some $L \in GL(V)$, for some V.*
- *Write F^G for the set of all function $G \to F$. For $f \in F^G$ and $g \in G$ define $\rho(g)f$ to be the function*
$$\rho(g)f(x) = f(xg).$$
Then ρ is a representation.
- *same set-up as the previous. Define*
$$\lambda(g)f(x) = f(g^{-1}x).$$
Then λ is also a representation.
- *Take V to be a vector space and $\rho : G \to GL(V)$ a representation. Suppose that there exists W such that $\rho(g)w \in W$ for all $w \in W$. Then we obtain a homomorphism $\rho' : G \to GL(W)$. Such a representation is called a* **subrepresentation** *of ρ. The subspace W is said to be an* **invariant** *subspace.*
- *Suppose ρ is a representation of G on V and that W is an invariant subspace of V. Then*
$$\rho''(g) \,.\, (v + W) = \rho(g) \,.\, v + W$$
defines a representation of G on the quotient space V/W, which we call the quotient representation.
- *Suppose $\rho : G \to GL(V)$ is a representation and let V^* denote the set of all linear functions $V \to F$. When V^* is equipped with addition and scalar multiplication defined pointwise*
$$(\ell_1 + \ell_2)(v) := \ell_1(v) + \ell_2(v), \qquad (\alpha\ell)(v) = \alpha(\ell(v)), \qquad (\ell_1, \ell_2, \ell \in V^*,\ v \in V,\ \alpha \in F),$$
it becomes a vector space, called the **dual space** *and*
$$[\rho(g) \,.\, \ell](v) = \ell(\rho(g^{-1}) \,.\, v) \qquad (\ell \in V^*,\ v \in V,\ g \in G)$$
defines a representation $\rho : G \to GL(V^*)$ called the **dual representation.**

Exercise 1.0.10. *Define a function $T : V \to (V^*)^*$ by*
$$[T(v)](\ell) = \ell(v) \qquad (v \in V, \ell \in V^*).$$
Assume that V is finite dimensional. Check that T is an isomorphism of vector spaces. Then check that
$$\rho^{**}(g)T(v) = T(\rho(g)v), \qquad (\forall g \in G, v \in V).$$
(This second thing you are checking amounts to saying that T is also an isomorphism of representations.) Now assume that V has a basis with the same cardinality as $\mathbb{Z}$ and show that V^* has a basis with a properly larger cardinality.

Exercise justifies the usage of the term "dual." It also shows that in the finite dimensional case, the relationship between V and V^* is actually quite symmetrical. For this reason, one will often denote an element of V^* by v^* rather than ℓ and write $\langle v, v^* \rangle$ rather than $\ell(v)$. Finally, the last part shows that in the case of infinite dimensional representations, V^* is not a dual. That observation might be useful some day if you have occasion to consider infinite dimensional representations. It won't be of interest for us this semester, though.

1.1. ***G*-Modules.** There is an alternate language which is sometimes more convenient.

Definition 1.1.1. (*G*-module) *Let G be a group. A* ***G*-module** *is a vector space V (over some field F) which is equipped with an action of G by linear transformations, i.e., with a function $G \times V \to V$, written $(g, v) \mapsto g \cdot v$ such that the function $v \mapsto g \cdot v$ is linear for each $g \in G$.*

Exercise 1.1.2. *Let $\rho : G \to GL(V)$ be a representation. Define $g \cdot v = \rho(g)v$. Show that V is then a G-module. Then assume that V is a G-module and define $\rho : G \to V^V$ by $\rho(g)(v) = g \cdot v$. Show that the image of ρ is contained in $GL(V)$ and that ρ is a homomorphism. (Here V^V denotes the set of all functions $V \to V$.)*

(This amounts to verifying that "representation of G" and "G-module" are two ways of thinking about the same basic thing. The only difference is the emphasis.)

Exercise 1.1.3. *In an algebra textbook, look up the definition of a* **module** *over a ring R, as well as the definition the* **group ring** *(or* **group algebra***) $F[G]$ of G over F. Prove that a G-module, defined as above, is the same thing as a module over the ring $F[G]$.*

1.2. **A digression.** For each kind of structure you encounter in algebra, there is a corresponding concept of structure-preserving map, or function.

Examples 1.2.1.
- *Group – group homomorphism*
- *Ring – ring homomorphism*
- *Vector space – linear function*
- *Topological space – continuous function*

The corresponding notion for representations is the following.

Definition 1.2.2. (*G*-linear map) *Let $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ be representations, with V and W being vector spaces over the same field F. A function $T : V \to W$ is* ***G*-linear** *if it is linear*

$$\sigma(g) \,.\, T(v) = T(\rho(g) \,.\, v) \qquad (\forall g \in G,\ v \in V).$$

"Map" is basically just a synonym for "function." However, in Fulton & Harris, "vector space map" means linear function, "group map" means group homomorphism. And, generally a "map" which attached to some type of structure in that way is a structure-preserving function as discussed above.

2. Direct sum and tensor product

If $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ are representations, one defines

$$\rho \oplus \rho' : G \to GL(V \oplus V'), \qquad \rho \otimes \rho' : G \to GL(V \otimes V'), \qquad \text{by}$$

$$\rho \oplus \rho'(g) \,.\, (v, v') = (\rho(g) \,.\, v, \rho'(g) \,.\, v') \qquad \rho \oplus \rho'(g) \,.\, v \otimes v' = (\rho(g) \,.\, v) \otimes (\rho'(g) \,.\, v').$$

The formula on the right only defines the action of $\rho \oplus \rho'(g)$ on pure tensors. The action on other elements of the tensor product is determined by linearity.

Definition 2.0.3. *Let V, W be vector spaces over some field F. The set of all linear maps V to W will be denote* $\mathrm{Hom}(V, W)$ *or* $\mathrm{Hom}_F(V, W)$. *The space of all G-linear maps will be denoted* $\mathrm{Hom}_G(V, W)$, $\mathrm{Hom}_{G,F}(V, W)$, *or* $\mathrm{Hom}_{F,G}(V, W)$,

Proposition 2.0.4. *The function $T : V^* \otimes W \to \mathrm{Hom}(V, W)$ given on pure tensors by*

$$T(v^* \otimes w)(v) = \langle v, v^* \rangle w, \qquad (v \in V, v^* \in V^*,\ w \in W)$$

is an isomorphism, and identifies $\mathrm{Hom}_G(V, W)$ *with*

$$(V^* \otimes W)^G = \{u \in V^* \otimes W : \rho' \otimes \rho''(g) \,.\, u = u \; \forall g \in G\}.$$

Proof. Exercise □

Exercise 2.0.5. *Let X be a finite set and $\alpha : G \to S_X$ a homomorphism. Let V be a vector space with a basis $\{e_x : x \in X\}$ indexed by the elements of X and let $F^X = \{f : X \to F\}$ (all functions). Define $\rho : G \to GL(V)$ by*

$$\rho(g) \,.\, e_x = e_{\alpha(g) \,.\, x} \qquad (g \in G,\ x \in X)$$

(and by linearity),and define $\rho' : G \to GL(F^X)$ by $(\rho'(g) \,.\, f)(x) = f(\alpha(g^{-1}) \,.\, x)$. Define a function $T : F^X \to V$ by

$$T(f) = \sum_{x \in X} f(x) e_x.$$

Check that T is an isomorphism of G-modules, i.e., that it is an isomorphism of vector spaces and

$$T(\rho'(g) \,.\, f) = \rho(g) \,.\, [T(f)]. \qquad (\forall f \in F^X,\ g \in G)$$

3. Irreducibles and complete reducibility

Definition 3.0.6. (reducible/irreducible) *A representation $\rho : G \to GL(V)$ is* **reducible** *if it has a proper nontrivial invariant subspace, and* **irreducible** *if not.*

Obvious example: any one dimensional representation.

Exercise 3.0.7. *Assume $F = \mathbb{C}$. Show that no two dimensional representation of $\mathbb{Z}$ is reducible. (Hint: Let $A \in GL_2\mathbb{C}$ be the image of $1 \in \mathbb{Z}$. Let v be an eigenvector of A, and consider the span of v.) Show that there exist irreducible two dimensional representations of $\mathbb{Z}$ if $F = \mathbb{R}$. (Hint: show that a proper nontrivial invariant subspace would have to contain an eigenvector for A as above.)*

Definition 3.0.8. (decomposable/indecomposable) *A representation $\rho : G \to GL(V)$ is* **decomposable** *if there exist proper nontrivial invariant subspaces V, V' with $V = V \oplus V'$, and* **indecomposable** *if not.*

Clearly, decomposable $\implies$ reducible.

Exercises 3.0.9. *Assume $F = \mathbb{C}$.*

1. *Give a two dimensional representation ρ of $\mathbb{Z}$ which is reducible an indecomposable. (Hint: arrange for $\rho(1)$ to have only one one-dimensional space of eigenvectors.)*
2. *Show that a two dimensional representation of $\mathbb{Z}/3\mathbb{Z}$ is decomposable.*
3. *Now assume $F = \mathbb{R}$ and show that a two dimensional representation of $\mathbb{Z}/3\mathbb{Z}$ is either irreducible or decomposable.*

Theorem 3.0.10. *Suppose that G is a finite group. Then every reducible representation is decomposable. In fact, if $\rho : G \to GL(V)$ is a finite dimensional representation and V' is a proper nontrivial invariant subspace, then there exists another invariant subspace V'' with $V = V' \oplus V''$.*

Proof. Take U any complement of V' in V. Define $\pi_0 : V \to V'$ to be the projection corresponding to the direct sum decomposition $V = V' \oplus U$. Then define

$$\pi(v) = \sum_{g \in G} g.\pi_0(g^{-1}.v).$$

Then $\pi : V \to V'$ is $G-$linear. Let V'' be its kernel. $\square$

Lemma 3.0.11. (Schur's Lemma) *Assume $F = \mathbb{C}$. Let V, W be nontrivial irreducible G-modules. Then*

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} \{0\}, & V \ncong W, \\ \mathbb{C}, V \cong W. \end{cases}$$

Proof. It $T : V \to W$ is G-linear then its image and kernel are invariant subspaces. It follows that $V \ncong W \implies T \equiv 0$.

Next assume $V = W$. Then T has an eigenvector, and the eigenspace is invariant and nontrivial, whence everything.

Finally if $V \neq W$ but $V \cong W$ then for any two isomorphism $T_1, T_2 : V \to W$ the map $T_2^{-1} \circ T_1$ is an isomorphism $V \to V$, whence scalar, so that $T_1 = cT_2$ for some c. $\square$

Exercise 3.0.12. *Find an example with $F = \mathbb{R}$ where V is irreducible and* $\mathrm{Hom}_G(V, V) \ncong \mathbb{R}$.

Corollary 3.0.13. (complete reducibility) *Assume that $F = \mathbb{C}$ and G is finite. Then every G-module is a direct sum of irreducible G-submodules.*

4. Irreducible representations of finite abelian groups

From now on, we assume $F = \mathbb{C}$ and all representations are finite dimensional unless explicitly otherwise stated.

Theorem 4.0.14. *Let G be a finite abelian group and let V be an irreducible G-module. Then V is one-dimensional.*

Proof. Let $\rho : G \to GL(V)$ be the corresponding representation. Fix $g_0 \in G$. Put $A = \rho(g_0) \in GL(V)$. Then A has an eigenvector $v_0 \in V$. Let λ be the eigenvalue. Then the λ-eigenspace of A on V is nontrivial and invariant. So it is V. This works for all $g_0 \in G$, so there exists $\chi : G \to \mathbb{C}^\times$ such that $\rho(g) \,.\, v = \chi(g) \cdot v$ for all $g \in G$. But then the span of any nonzero $v \in V$ is nontrivial and invariant. $\square$

5. Irreducible Representations of S_3

Let S_3 be the symmetric group on 3 letters. We consider three representations of S_3.

(1) The **trivial representation** defined as action on a one dimensional space by $g \cdot v = v$.
(2) The representation $\sigma \cdot v = M(\sigma)v$ of S_3 on $\mathbb{C}^3$.
(3) The representation $\sigma.v = \det M(\sigma) \cdot v$ of S_3 on $\mathbb{C}$.

The first and third are one dimensional and hence irreducible. The second is not irreducible. Span ${}^t[1, 1, 1]$ and $\{{}^t[x, y, z] : x + y + z = 0\}$ are invariant. The first subrepresentation is isomorphic to the trivial representation. The second is irreducible.

Theorem 5.0.15. *Every irreducible representation of S_3 is isomorphic to one of the ones listed above.*

Proof. Let V be an irreducible S_3-modules and decompose V into eigenspaces for $(1, 2, 3)$. If $(1, 2, 3)$ acts trivially on V then the representation $\rho : G \to GL(V)$ factors through $\sigma \mapsto \det M(\sigma)$. If not, then one can check by inspection that it is isomorphic to the two dimensional representation of S_3 on $\{{}^t[x, y, z] : x + y + z = 0\}$. $\square$

6. Characters

Definition 6.0.16. (Character of a representation) *Let $\rho : G \to GL(V)$ be a representation. The character of ρ (or equivalently of the G-module V) is the function $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$. Here* Tr *denotes the trace. The character is also denoted χ_V.*

Examples 6.0.17.

- *Let $M : S_3 \to GL_3$ be the homomorphism defined by $M(\sigma)e_i = e_{\sigma(i)}$. Then $\chi_M(e) = 3, \chi_M((1,2)) = \chi_M((2,3)) = \chi_M((1,3)) = 1, \chi_M((1,2,3)) = \chi_M((1,3,2)) =$ 0.*
- *A one-dimensional representation is essentially equal to its character. (Since $GL_1(\mathbb{C}) = \mathbb{C}^\times \subset \mathbb{C}$.) In particular, the character of the trivial representation of S_3 (or any group) is the constant function 1.*
- *If G acts on X and we define a representation of G on a vector space V with basis $\{e_x : x \in X\}$ by $g \cdot e_x = e_{g \cdot x}$, then $\chi_V(g) = \#\{x \in X | g \cdot x = x\}$.*
- *We saw above that the action of S_3 on $\mathbb{C}^3$ is reducible. It is the direct sum of a subspace isomorphic to the trivial representation, and a two dimensional nontrivial subspace. The nontrivial component is spanned by*

  $$v_1 = e_1 + e^{2\pi i/3}e_2 + e^{4\pi i/3}e_3 \qquad \text{and} \qquad v_2 = e_1 + e^{4\pi i/3}e_2 + e^{2\pi i/3}e_3.$$

  Using this bases, the representation ρ gives matrices

  $$\rho(e) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \qquad \rho((1,2,3)) = \begin{pmatrix} e^{4\pi i} & \\ & e^{2\pi i} \end{pmatrix} \qquad \rho((1,3,2)) = \begin{pmatrix} e^{2\pi i} & \\ & e^{4\pi i} \end{pmatrix}$$

  $$\rho((1,2)) = \begin{pmatrix} & e^{2\pi i/3} \\ e^{4\pi i/3} & \end{pmatrix} \qquad \rho((1,3)) = \begin{pmatrix} & e^{4\pi i/3} \\ e^{2\pi i/3} & \end{pmatrix} \rho((2,3)) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

  So

  $$\chi_\rho(e) = 2, \chi_\rho((1,2,3)) = \chi_\rho((1,3,2)) = -1, \chi_\rho((1,2)) = \chi_\rho((1,3)) = \chi_\rho((2,3)) = 0.$$

Theorem 6.0.18. (Properties of the character)

(1) *The character of a representation depends only on the isomorphism class of that representation. That is, if V and W are isomorphic G-modules, then $\chi_V = \chi_W$.*
(2) *The character of a representation* **determines** *the isomorphism class of that representation. That is, if V and W are G-modules and $\chi_V = \chi_W$, then $V \cong W$.*
(3) *The character of any representation is a* **class function**, *i.e., it is constant on conjugacy classes in G. In other words*

$$\chi_\rho(ghg^{-1}) = \chi_\rho(h) \qquad (\forall g, h \in G),$$

for any representation ρ of any group G.
(4) $\chi_{V \oplus W} = \chi_V + \chi_W$
(5) $\chi_{V \otimes W} = \chi_V \chi_W$
(6) $\chi_{V^*} = \overline{\chi_V}$

Proof. (1) To compute the trace of $\rho(g)$ acting on V, we choose an ordered basis $B = (v_1, \ldots v_{\dim V})$ for V and write a matrix $[\rho(g)]_B$. If we have an isomorphism $\iota : V \to V'$ we can push the basis B through the isomorphism, obtaining an ordered basis $B' = (\iota(v_1), \ldots \iota(v_{\dim V}))$ for V'. Now check that $[\rho'(g)]_{B'} = [\rho(g)]_B$.
(2) We'll prove this last.
(3) Follows from the fact that $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$ for any matrices A and B.

(4) Let $B_V = (v_1, \dots, v_{\dim V})$ be an ordered basis for V and $B_W = (w_1, \dots, w_{\dim W})$ and ordered basis for W. Set $B_{V\oplus W} = (v_1, \dots, v_{\dim V}, w_1, \dots, w_{\dim W})$, which is an ordered basis for $V \oplus W$. Check that

$$[\rho_{V\oplus W}(g)]_{B_{V\oplus W}} = \begin{pmatrix} [\rho(g)]_{B_V} & \\ & [\rho'(g)]_{B_W} \end{pmatrix} \qquad \text{(a block matrix)}$$

for any $g \in G, \rho : G \to GL(V), \rho' : G \to GL(W)$.

(5) Similar to the previous.

(6) Let $B = (v_1, \dots v_{\dim V})$ be an ordered basis for V. The **dual basis** for V^* is the ordered basis $B^* = (v_1^*, \dots v_{\dim V}^*)$ such that $\langle v_i, v_j^* \rangle = \delta_{i,j}$ (Kronecker δ). Check that

$$\langle v, v^* \rangle = [v]_B \cdot [v^*]_{B^*} \qquad (\forall v \in V, v^* \in V^*).$$

Deduce that $[\rho^*(g)]_{B^*} = {}^t[\rho(g)]_B^{-1}$ for all $g \in G$. Since $\rho(g)$ is of finite order, its eigenvalues are all complex roots of unity, so

$$\mathrm{Tr}({}^t[\rho(g)]_B^{-1}) = \mathrm{Tr}([\rho(g)]_B^{-1}) = \overline{\mathrm{Tr}([\rho(g)]_B)}.$$

Now to prove (2), recall that the set $\mathrm{Hom}(V, W)$ of linear maps $V \to W$ can be identified with $V^* \otimes W$ and that then the set $\mathrm{Hom}_G(V, W)$ of G-module homomorphisms $V \to W$ is identified with $(V^* \otimes W)^G$. (The subspace of G-invariant elements.)

Lemma 6.0.19. *Take any G-module U and define $\pi(u) = \sum_{g\in G} g\,.\,u$. Then π maps U onto U^G and acts by identity on U^G. Further, $\mathrm{Tr}\,\pi = \sum_g \chi_U(g) = \dim U^G$.*

Proof. The first part is easy from the definitions. It's also immediate from the defs that $\mathrm{Tr}\,\pi = \sum_g \chi_U(g)$. To get $\mathrm{Tr}\,\pi \dim U^G$ start with a basis of U^G, enlarge it to a basis of U, and write the matrix of π with respect to that basis. ☐

Applying the lemma when $U = (V^* \otimes W)$ we get

$$\sum_g \overline{\chi_V}(g)\chi_W(g) = \begin{cases} 1, & V \cong W, \\ 0 & V \ncong W. \end{cases}$$

Then (2) is immediate. ☐

Proposition 6.0.20. *For $f, h : G \to \mathbb{C}$ define*

$$(f, h) = \sum_{g\in G} \overline{f(g)}h(g).$$

Then

(1) *For V, W irreducible G-modules*

$$(\chi_V, \chi_W) = \begin{cases} 1, & V \cong W, \\ 0, & V \ncong W. \end{cases}$$

(2) *For general V, W write*

$$V = \bigoplus_{i=1}^{r} V_i \qquad W = \bigoplus_{i=1}^{s} V_j (V_1, \dots, V_r, W_1, \dots W_s, \text{ irreducible}).$$

Then

$$(\chi_V, \chi_W) = \text{ the number of pairs } i, j \text{ such that } V_i \cong V_j.$$

(3) *If $V = \bigoplus_{i=1}^r V_i$ is any representation and W is irreducible then (χ_V, χ_W) is the number of indices i such that $V_i \cong W$.*

(4) *If* $V = \bigoplus_{i=1}^{r} V_i$ *is any representation and* W *is irreducible then*

$$\pi_W(v) := \sum_{g \in g} \overline{\chi_W}(g) g \,.\, v$$

defines a projection from V *onto the sum of the subspaces* V_i *with* $V_i \cong W$.

Proof. We proved the first part in the course of the last proof. The second two parts follow from the first. To prove the third part, take the trace of the operator π, acting on an irreducible component V_i, and you get (χ_W, χ_{V_i}). $\square$

Theorem 6.0.21. *Let* $V_1, \dots V_r$ *be a set of representatives for the isomorphism classes of irreducible representations of* G*. Then the right regular representation of* G *is isomorphic to the sum of* $\dim V_1$ *copies of* V_1, $\dim V_2$ *copies of* $V_2, \dots \dim V_r$ *copies of* V_r.

Proof. Check that $\chi_R(g) = |G| \delta_{g, 1_G}$. So $(\chi_{V_i}, \chi_R) = \chi_{V_i}(1_G) = \dim V_i$. $\square$

Theorem 6.0.22. *The number of isomorphism classes of irreducible representations of a group* G *is equal to the number of conjugacy classes in* G.

Proof. The mapping $V \to \chi_V$ sends G-modules to class functions $G \to \mathbb{C}$. Since χ_V depends only on the isomorphism class of V, we have a mapping from isomorphism classes to class functions. If $V \ncong W$ then $(\chi_V, \chi_W) = 0$. It follows that χ_V and χ_W are linearly independent functions. So

$$\begin{aligned}\#\{ \text{ isomorphism classes of irreducible representations of } G\} &= \#\{ \text{ distinct characters} : G \to \mathbb{C}\} \\ &\leq \dim\{ \text{ class functions } \} = \#\{ \text{ conjugacy classes}\}.\end{aligned}$$

To prove equality, we will show that the characters actually form an orthonormal basis for the set of all class functions.

To do this, we make the following definition.

Definition 6.0.23. *Let* G *be a finite group and* (ρ, V) *be a finite dimensional* G*-module. Take a class function* $f : G \to \mathbb{C}$*. We define the operator* $\rho(f) : V \to V$ *by*

$$\rho(f) \,.\, v := \sum_{g \in G} f(g) \rho(g) \,.\, v.$$

Lemma 6.0.24. *The operator* $\rho(f)$ *is a* G*-module map. That is* $\rho(g) \circ \rho(f) = \rho(f) \circ \rho(g)$ *for any class function* f *and any* $g \in G$.

Proof. For $v \in V$,

$$\rho(g) \circ \rho(f) \circ \rho(g^{-1}) \,.\, v = \sum_{g_1 \in G} f(g_1) \rho(g g_1 g^{-1}) \,.\, v = \sum_{g_1 \in G} f(g^{-1} g_2 g) \rho(g_2) \,.\, v = \sum_{g_1 \in G} f(g_2) \rho(g_2) \,.\, v = \rho(f) \,.\, v.$$

(The second to last equality is because f is a class function.) $\square$

Lemma 6.0.25.

$$\operatorname{Tr}(\rho(f)) = (\chi_{\rho^*}, f)$$

Proof. Both sides equal

$$\sum_{g \in G} f(g) \operatorname{Tr}(\rho(g)).$$

$\square$

Now suppose that f is a class function such that $(\chi_V, f) = 0$ for all irreducible V. Since $\rho(f)$ is an G-module map, it follows from Schur's lemma that $\rho(f)$ is a scalar multiple of the identity whenever V is irreducible. Since its trace is zero, evidently, the scalar is zero. It follows that $\rho(f)$ acts by zero on any representation– irreducible or reducible. So, it acts by zero on the regular representation, i.e.

$$\begin{aligned} R(f)\,.\,F &= 0 \quad \text{(the zero function)} && (\forall F : G \to \mathbb{C}) \\ R(f)\,.\,F(x) &= 0 && (\forall x \in G, F : G \to \mathbb{C}) \\ \sum_{g \in G} f(g) F(xg) &= 0 && (\forall x \in G, F : G \to \mathbb{C}). \end{aligned}$$

Take F to be the chracteristic function of the $\{1_G\}$, and you get $f(x^{-1}) = 0$. And since that holds for all x, the function f must be the zero function. □

References

[F-H] Fulton, William; Harris, Joe Representation theory. A first course. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991

E-mail address: jhundley@math.siu.edu

Math. Department, Mailcode 4408, Southern Illinois University Carbondale, 1245 Lincoln Drive Carbondale, IL 62901