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# Small Extensions of Witt Rings

Robert W. Fitzgerald

*Southern Illinois University Carbondale*, rfitzg@math.siu.edu

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# SMALL EXTENSIONS OF WITT RINGS

ROBERT W. FITZGERALD

Southern Illinois University

**We consider certain Witt ring extensions $S$ of a noetherian Witt ring $R$ obtained by adding one new generator. The conditions on the new generator are those known to hold when $R$ is the Witt ring of a field $F$, $S$ is the Witt ring of a field $K$ and $K/F$ is an odd degree extension. We show that if $R$ is of elementary type then so is $S$.**

The elementary type conjecture is a proposed classification of noetherian Witt rings. A potential source of counter-examples is as follows: start with a field $F$ where $WF$ is known (necessarily of elementary type) then look at noetherian $WK$ for extension fields $K$ of $F$. Jacob and Ware [3] have shown that $WK$ is again of elementary type when $[K : F] = 2$. Here we look at the simplest case of odd degree extensions, again showing $WK$ is of elementary type. We note that $WF$ is noetherian iff $G(F) \equiv F^{\cdot}/F^{\cdot 2}$ is finite. Also when $K/F$ has odd degree then $G(F) \cong F^{\cdot}K^{\cdot 2}/K^{\cdot 2}$ embeds into $G(K)$.

We will in fact work with abstract Witt rings $R$ (as defined by Marshall [4]) with associated group of one dimensional forms $G(R)$. The small extensions considered here are as follows. Let $H$ be a subgroup of $G(R)$. We say a Witt ring $S$ is an $H$-extension of $R$ if there exists an $\alpha \in G(S)$ such that:

(1) $G(S) = \{1, \alpha\}G(R)$, and
(2) For all $x \in G(R)$ we have:

$$D_S\langle 1, -x\rangle = \begin{cases} D_R\langle 1, -x\rangle, & \text{if } x \notin H \\ \{1, \alpha\}D_R\langle 1, -x\rangle & \text{if } x \in H \end{cases}$$
$$D_S\langle 1, -\alpha x\rangle = \{1, -\alpha x\}\left(D_R\langle 1, -x\rangle \cap H\right).$$

These conditions hold for $R = WF, S = WK$ when $K/F$ is an odd degree extension and $[G(K) : G(F)] = 2$ by [2,4.7] (we note that [2,4.7] should include the condition that $N_{K/F}(a) = 1$). No such field extensions are known. However, there are many examples of $H$-extensions of abstract Witt rings, which we determine inductively. This can be viewed as a first step in classifying extensions of noetherian Witt rings. It also helps the search for odd degree extensions $K/F$ with $[G(K) : G(F)] = 2$, while lessening the motivation for such a search.

For any group $H$, $H^{\cdot}$ denotes $H \setminus \{1\}$. The quaternionic mapping associated to $R$ will be denoted by $q$. For $x \in G(R)$, $Q(x) = \{q(x, y) : y \in G(R)\}$ and for a subgroup $H$,

$Q(H) = \{q(h, y) : h \in H, y \in G(R)\}$. The value set of $\langle 1, -x \rangle$ is $D\langle 1, -x \rangle = \{y \in G(R) : q(x, y) = 1\}$. We will often work with several Witt rings at once and write $q_R, Q_R(x)$ and $D_R\langle 1, -x \rangle$ to indicate these objects for $R$.

$R$ is of local type if $|q(G(R), G(R))| = 2$. We let $E_n$ denote the elementary 2-group of order $n$. The group ring $R[E_n]$ is again a Witt ring. An element $t \in G(R)$ is rigid if $D\langle 1, t \rangle = \{1, t\}$ and $t$ is birigid if both $t$ and $-t$ are rigid. The basic part of $R$, $B(R)$, consists of $\pm 1$ and all $x \in G(R)$ with either $x$ or $-x$ not rigid. $B(R)$ is a subgroup of $G(R)$ and $R = R_0[G(R)/B(R)]$, where $R_0$ is the Witt ring generated by $B(R)$. We express this last statement by writing $R_0 = W(B(R))$.

The product in the category of Witt rings is:

$$R_1 \sqcap R_2 = \{(r_1, r_2) : r_i \in R_i \quad \text{and} \quad \dim r_1 \equiv \dim r_2 \pmod 2\}$$

If $R = R_1 \sqcap R_2$ then $G(R) = G(R_1) \times G(R_2)$ and:

$$D_R\langle (1, 1), (x, y) \rangle = D_{R_1}\langle 1, x \rangle \times D_{R_2}\langle 1, y \rangle.$$

The radical of $R$ is $\mathrm{rad}(R) = \{x \in G(R) : D\langle 1, -x \rangle = G(R)\}$. We say $R$ is degenerate if $\mathrm{rad}(R) \neq 1$ and totally degenerate if $\mathrm{rad}(R) = G(R)$. $D_n$ denotes a totally degenerate Witt ring with square class group of order $2^n$. There are two possibilities for $D_n$ depending on whether -1 is a square or not. Specifically, $D_n$ is either a product of $n$ copies of $(\mathbb{Z}/2\mathbb{Z})[E_1]$ or $n$ copies of $\mathbb{Z}/4\mathbb{Z}$. If $R$ is degenerate then there exist uniquely determined $n$ and non-degenerate Witt ring $R_0$ such that $R = D_n \sqcap R_0$. $R_0$ is the non-degenerate part of $R$.

$R$ is of elementary type if it can be built from $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$ and Witt rings of local type by a succession of group ring extensions (for some $E_n$) and products. The elementary type conjecture is that every noetherian Witt ring is of elementary type.

## 1. Group ring extensions.

**Lemma 1.1.** *Let $S$ be an $H$-extension of $R$.*

(a) *If $H = 1$ then $S = R[E_1]$, with $E_1$ generated by $\alpha$.*
(b) *If $H = G(R)$ then $S = D_1 \sqcap R$, with $D_1$ generated by $\alpha$.*

*Proof.* Suppose first that $H = 1$. Then for all $g \in G(R)$ we have from the definition of an $H$-extension that $D_S\langle 1, -\alpha g \rangle = \{1, -\alpha g\}(D_R\langle 1, -g \rangle \cap H)$ and so $\alpha g$ is birigid. Thus $B(S) \subset G(R)$ and $G(S) = \{1, \alpha\}G(R)$. So by[4,5.19] $S = R[E_1]$, where $E_1$ is generated by $\alpha$.

Next suppose that $H = G(R)$. Then $D_S\langle 1, -\alpha \rangle = \{1, -\alpha\}(D_R\langle 1, -1 \rangle \cap H) = G(S)$. Hence $\alpha \in \mathrm{rad}(S)$. Then by [4,pp. 105–106] $S = D_1 \sqcap R$, where $D_1$ is generated by $\alpha$. $\square$

When $H \neq 1$, which we will often assume in light of (1.1), we use the following notation

(recall that $B(R)$ is the basic part of $R$):

$$T = \bigcup_{h \in H^\cdot} D_R\langle 1, -h \rangle$$

$$T_0 = \bigcup_{h \in H^\cdot} (D_R\langle 1, -h \rangle \setminus \{-h\})$$

$$B(H) = H \cap B(R)$$

$$BT = \bigcup_{h \in B(H)^\cdot} D_R\langle 1, -h \rangle$$

$$BT_0 = \bigcup_{h \in B(H)^\cdot} (D_R\langle 1, -h \rangle \setminus \{-h\})$$

**Lemma 1.2.** *Let $S$ be an $H$-extension of $R$ with $|H| > 1$. Then:*

(a) $\pm T_0 \subset B(R) \subset \pm T$,
(b) $B(S) = \pm\{1, \alpha\}T$.

*Proof.* First note that $D_S\langle 1, -\alpha \rangle = \{1, -\alpha\}H$, and $|H| > 1$ imply $\alpha \in B(S)$. If $x \in G(R) \setminus \pm T$ then $D_R\langle 1, \pm x \rangle \cap H = \{1\}$ so by the definition of $H$-extensions, $D_S\langle 1, \pm\alpha x \rangle = \{1, \pm\alpha x\}$. Hence $\alpha x \notin B(S)$, and as $\alpha \in B(S)$, $x \notin B(S)$. That is, $x$ is birigid in $S$, and so also in $R$. Thus $x \notin B(R)$ and we have:

$$B(R) \subset \pm T,$$
$$B(S) \subset \pm\{1, \alpha\}T$$

Let $x \in T_0, x \neq -1$ so that for some $h \in H^\cdot, x \in D_R\langle 1, -h \rangle$ and $x \neq -h$. Then $D_R\langle 1, -x \rangle$ contains $-x, h$ which are distinct and not equal to 1. So $-x \in B(R)$ and also $x \in B(R)$. If $x = -1$ then again $x \in B(R)$. Thus $T_0 \subset B(R)$, and so $\pm T_0 \subset B(R)$ completing the proof of (a).

If $x \in T$ with $x \in D_R\langle 1, -h \rangle, h \in H^\cdot$ then $D_S\langle 1, -x \rangle = \{1, -\alpha x\}(D_R\langle 1, -x \rangle \cap H)$ contains $\{1, -\alpha x, h, -\alpha x h\}$. Thus $-\alpha x \in B(S)$. Again $\alpha \in B(S)$ so $x \in B(S)$. This shows $\pm\{1, \alpha\}T \subset B(S)$, completing the proof of (b). $\square$

**Lemma 1.3.** *If $S$ is an $H$-extension of $R$ with $|H| > 1$ then $B(R) = \pm BT$ and either:*

(a) $B(R) = \pm B(H)$ and $B(S) = \pm\{1, \alpha\}H$,   *or*
(b) $B(R) = \pm T$ and $B(S) = \{1, \alpha\}B(R)$.

*Proof.* If $h \in H \setminus B(H)$ then $D_R\langle 1, -h \rangle = \{1, -h\}$. So:

(1.4) $$\pm T = \pm BT \cup \pm(H \setminus B(H)).$$

Now $B(R) \subset \pm T$ by (1.2)(a) and $B(R) \cap \pm(H \setminus B(H)) = \emptyset$ so $B(R) \subset \pm BT$. Conversely,

$$\pm BT = \pm(BT \cup B(H)) \subset \pm T_0 \cup \pm B(H) \subset B(R),$$

which proves the first statement.

Now (1.4) gives:
$$\pm T = \pm BT \cup \pm H = B(R) \cup \pm H,$$

and (1.2)(b) gives:
$$B(S) = \pm\{1,\alpha\}T = \{1,\alpha\}B(R) \cup \pm\{1,\alpha\}H.$$

This expresses the group $B(S)$ as the union of two subgroups, hence either:

   (i) $\{1,\alpha\}B(R) \subset \pm\{1,\alpha\}H,$    or

  (ii) $\pm\{1,\alpha\}H \subset \{1,\alpha\}B(R).$

In case (i) $B(S) = \pm\{1,\alpha\}H$ and $B(R) \subset \pm\{1,\alpha\}H \cap G(R) = \pm H$. Hence $B(R) = \pm B(H)$.
In case (ii) $B(S) = \{1,\alpha\}B(R)$ and $H \subset B(R)$. Then $H = B(H)$, $BT = T$ and by the first statement $B(R) = \pm T$. $\square$

Recall that any Witt ring $R$ can be written as $R_0[G(R)/B(R)]$, where $R_0 = W(B(R))$, the Witt ring generated by $B(R)$. See [4,Chapter 5, Section 7] for details.

**Proposition 1.5.** *Let $R = R_0[E_n]$, with $R_0$ basic. Let $S$ be an $H$-extension of $R$. Then:*

  (a) *If $H = 1$ then $S = R_0[E_{n+1}]$.*

  (b) *If $|H| > 1$ and $H \subset G(R_0)$ then $S = S_0[E_n]$, for some Witt ring $S_0$ that is an $H$-extension of $R_0$ (with the same $\alpha$).*

  (c) *If $H \not\subset G(R_0)$ and $-1 \in H$ then $S = (D_1 \sqcap R_0[H/B(H)])\,[G(R)/H]$, and $G(R_0) \subset H$.*

  (d) *If $H \not\subset G(R_0)$ and $-1 \notin H$ then $S = (\mathbb{Z} \sqcap R_0[H/B(H)])\,[G(R)/\pm H]$ and $G(R_0) \subset \pm H$.*

*Proof.* If $H = 1$ then $S = R[E_1]$ by (1.1), which gives (a). So assume $|H| > 1$. Further suppose that $H \subset G(R_0)$ so that $B(H) = H \cap B(R) = H \cap G(R_0) = H$. Then $-H^{\cdot} \subset T \subset G(R_0)$. Hence $\pm H \subset \pm T \subset G(R_0)$. Thus if (1.3)(a) holds, so that $G(R_0) = B(R) = \pm B(H) = \pm H$, then $B(R) = \pm T$ also. So we are always in case (b) of (1.3). Then, since $G(S)/B(S) \cong G(R)/B(R)$, we have $S = S_0[E_n]$, where $S_0 = W(B(S))$. From $H \subset B(R)$ we have that $S_0$ is an $H$-extension of $R_0$.

Next suppose that $H \not\subset G(R_0)$. We still have that $G(R_0) = B(R)$. If $B(R) = \pm T$ then $H \subset B(R)$, contrary to our assumption. Thus we are in Case (a) of (1.3). First say that $-1 \in H$. Note that $-1 \in H \cap B(R) = B(H)$ also. Then $B(R) = B(H)$ and $B(S) = \{1,\alpha\}H$. Thus $S = S_0[G(R)/H]$, for $S_0 = W(\{1,\alpha\}H)$, since $G(S)/B(S) \cong G(R)/H$.

Now $D_{S_0}\langle 1,-\alpha\rangle = D_S\langle 1,-a\rangle \cap \{1,\alpha\}H = \{1,\alpha\}H$. Hence $\alpha \in \mathrm{rad}(S_0)$. Write $S_0 = D_1 \sqcap S_1$, for some Witt ring $S_1$ and with $D_1$, generated by $\alpha$, being $\mathbb{Z}_2[E_1]$ or $\mathbb{Z}_4$ using [4, p. 104]. Note that $S_1 = W(H)$.

If $h \in H \setminus B(H)$ then $D_R\langle 1,-h\rangle = \{1,-h\}$ and $D_S\langle 1,-h\rangle = \{1,\alpha,-h,-\alpha h\}$ so that $D_{S_1}\langle 1,-h\rangle = D_S\langle 1,-h\rangle \cap H = \{1,-h\}$. Similarly, $D_{S_1}\langle h\rangle = \{1,h\}$. And if $h \in B(H)$ then $D_{S_1}\langle 1,-h\rangle = \{1,\alpha\}D_R\langle 1,-h\rangle \cap H = D_R\langle 1,-h\rangle \cap H = D_R\langle 1,-h\rangle \cap B(H) = D_{R_0}\langle 1,-h\rangle$. Thus $S_1 = R_0[H/B(H)]$.

We still suppose $H \not\subset G(R_0)$, so that we are in Case (a) of (1.3), and now say that $-1 \notin H$. Then $S = S_0[G(R)/\pm H]$ as $G(S)/B(S) = \{1,\alpha\}G(R)/\pm\{1,\alpha\}H \cong G(R)/\pm H$. Here $S_0 = W(\pm\{1,\alpha\}H)$.

Now $D_{S_0}\langle 1, -\alpha \rangle = \{1, \alpha\}H$ has index two in $G(S_0) = \pm\{1, \alpha\}H$. Further $\alpha \notin D_S\langle 1, -\alpha \rangle$ else $-1 \in \{1, -\alpha\}H$ and $-1 \in H$. Thus we have an orthogonal decomposition in the sense of [1]:

$$(1.6) \qquad\qquad G(S_0) = \{1, \alpha\} \perp D_S\langle 1, -\alpha \rangle.$$

Set $S_1 = W(\{1, \alpha\})$ and $S_2 = W(D_S\langle 1, -\alpha \rangle)$. Note that $S_1 = \mathbb{Z}$ as $-1 \notin D_S\langle 1, -\alpha \rangle$. If $h \in \pm H \setminus \pm B(H)$ then $D_R\langle 1, \pm h \rangle = \{1, \pm h\}, D_S\langle 1, -h \rangle = \{1, \alpha, -h, -\alpha h\}$ and $D_S\langle 1, h \rangle = \{1, h\}$. Thus we have $D_{S_1}\langle 1, \pm h \rangle = D_S\langle 1, \pm h \rangle \cap \pm H = \{1, \pm h\}$. So $S_2 = S_3[H/B(H)]$, for some Witt ring $S_3$. $S_2$ is indeed a group ring as $H \not\subset G(R_0)$ implies $H \neq B(H)$.

We wish to apply [1,3.4] and deduce that the decomposition (1.6) yields a product of Witt rings. First we need to handle the case where $S_2$ is decomposable, that is, $S_2 = \mathbb{Z}[E_1]$. In this case $|G(S_2)| = |D_S\langle 1, -\alpha \rangle| = 4$ so that $H = \{1, t\}$, for some $t \notin G(R_0)$ and $G(R_0) = \{\pm 1\}$ as $G(R_0) \subset \pm H$. Now $D_S\langle 1, -t \rangle = \{1, \alpha, -t, -\alpha t\}$ so we consider instead the orthogonal decomposition:

$$\{1, t\} \perp D_S\langle 1, -t \rangle.$$

Now $Q_S(\{1, t\}) = \{1, q(t, -1)\}$ and $Q_S(D_S\langle 1, -t \rangle) = \{1, q(\alpha, -1), q(-t, -1), q(-\alpha t, -1)\}$, using $q(\alpha, -t) = q(\alpha, -1)$ and $q(t, -\alpha t) = 1$. Then $Q_S(\{1, t\}) \cap Q_S(D_S\langle 1, -t \rangle) = 1$. Thus:

$$S_0 = W(\{1, t\}) \sqcap W(D_S\langle 1, -t \rangle).$$

Now $W(\{1, t\}) = \mathbb{Z}$ as $-1 \notin D_S\langle 1, -t \rangle$ and $W(D_S\langle 1, -t \rangle) = R_0[\{1, \alpha\}]$ since $\alpha$ is birigid in $qfst$ and $G(R_0) = \{\pm 1\}$. This gives the desired result of (d).

We now apply [1,3.4] and obtain that either (1.6) yoilds a product or $Q_S(\{1, \alpha\}) = Q_S(D_S\langle 1, -\alpha \rangle)$. But if $t \in H \setminus G(R_0)$ then $q(t, t) = q(t, -1)$ and $q(\alpha, \alpha) = q(\alpha, -1)$ are distinct, since $-1 \notin D_S\langle 1, -\alpha t \rangle$. Hence the decomposition (1.6) in fact yields the product $S_0 = S_1 \sqcap S_2$. We have already seen that $S_1 = \mathbb{Z}$ and that $S_2 = S_3[H/B(H)]$. If $h \in \pm B(H)$ then $D_{S_3}\langle 1, -h \rangle = D_{S_1}\langle 1, -h \rangle = D_R\langle 1, -h \rangle = D_{R_0}\langle 1, -h \rangle$. So $S_3 = R_0$. $\square$

## 2. Local type rings.

**Notation.** For a subset $A \subset G(R)$ set:

$$C_R(A) = \bigcap_{a \in A} D_R\langle 1, -a \rangle.$$

**Lemma 2.1.** *Suppose that $S$ is an $H$-extension of $R$. Suppose $k \in C_R(H) \setminus H$. Then $Q_R(H) \cap Q_R(k) = 1$.*

*Proof.* Let $\rho \in Q_R(H) \cap Q_R(k)$ so that $\rho = q(h, x) = q(k, y)$ with $h \in H$, and $x, y \in G(R)$. Since $H \subset D_S\langle 1, -\alpha \rangle$ we have that $q(k, y) = q(\alpha x, h)$. By linkage there exists a $t \in G(S)$ such that:

$$q(k, y) = q(k, t) = q(\alpha x, t) = q(\alpha x, h).$$

The first equality gives $ty \in D_S\langle 1, -k \rangle = D_R\langle 1, -k \rangle \subset G(R)$, since $k \notin H$. Hence $t \in G(R)$. The second equality gives:

$$t \in D_S\langle 1, -\alpha xk \rangle \cap G(R) = D_R\langle 1, -xk \rangle \cap H.$$

This implies $t \in D_R\langle 1, -x \rangle$ since $H \subset D_R\langle 1, -k \rangle$. The third equality gives:

$$ht \in D_S\langle 1, -\alpha x \rangle \cap G(R) = D_R\langle 1, -x \rangle \cap H.$$

Hence $h \in D_R\langle 1, -x \rangle$ and $\rho = q(x, h) = 1$. $\square$

The small Witt rings of local type will often be treated separately. The only local type Witt ring with two generators is $\mathbb{Z}$. There are two Witt rings of local type on four generators and both are group rings. If $L$ is local type and $|G(L)| \geq 8$ then $L$ is not a group ring. See [4,Chapter 5, Section 3] for details.

**Lemma 2.2.** *Suppose $R = L \sqcap R_2$, with $L$ a Witt ring of local type and $|G(L)| \geq 8$. Let $\pi_1$ be the projection map of $G(R)$ onto $G(L)$. Let $S$ be an $H$-extension of $R$. Then $\pi_1(H) = 1$ or $G(L)$.*

*Proof.* Set $B = \pi_1(H)$ and write $Q(L) = \{1, \rho\}$. Suppose that $B \neq 1$. If $(u, v) \in H$ with $u \neq 1$ then pick $r \in G(L) \setminus D_L\langle 1, -u \rangle$. We get $q((u, v), (r, 1)) = (\rho, 1)$ and so $(\rho, 1) \in Q(H)$.

Now $H \subset B \times G(R_2)$ so that $C_L(B) \times 1 = C_R(B \times G(R_2)) \subset C_R(H)$. If $C_L(B) = 1$ then $B = G(L)$ and we are done. So suppose there exists $1 \neq z \in C_L(B)$. Set $h = (z, 1) \in C_R(H)$. Then $Q(h) = \{1, (\rho, 1)\} \subset Q(H)$. By (2.1) we must have $h \in H$ and so $C_L(B) \times 1 \subset H$. In particular, $C_L(B) \subset B$.

Continue to let $h = (z, 1)$ where $1 \neq z \in C_L(B)$. We claim there exist an $h_1 \in H$ and an $x \in D_R\langle 1, -h \rangle$ such that $q(x, h_1) = (\rho, 1)$. Suppose not. We consider any $x = (a, 1)$ with $a \in D_L\langle 1, -z \rangle$. Then $q(x, (u, v)) = (\rho, 1)$ unless $a \in D_L\langle 1, -u \rangle$. Thus $D_L\langle 1, -z \rangle \subset D_L\langle 1, -u \rangle$ for all $(u, v) \in H$, that is, $D_L\langle 1, -z \rangle \subset C_L(B)$. Then $C_L(B) = D_L\langle 1, -z \rangle$ and so $B = \{1, z\}$. But $C_L(B) \subset B$ so that $D_L\langle 1, -z \rangle \subset \{1, z\}$ and $|G(L)| = 4$, a case we are excluding.

Thus there does exist an $h_1 \in H$ and an $x \in D_R\langle 1, -h \rangle$ such that $q(x, h_1) = (\rho, 1)$. Then in $S$ we have $q(\alpha x, h_1) = (\rho, 1)$ and so $Q_S(h) \subset Q_S(\alpha x)$. We obtain:

$$|Q_S(\alpha x) \cap Q_S(h)| = 2.$$

On the other hand:

$$D_S\langle 1, -\alpha x \rangle \cap D_S\langle 1, -h \rangle = \{1, -\alpha x\} \left( D_R\langle 1, -x \rangle \cap H \right) \cap \{1, \alpha\} D_R\langle 1, -h \rangle.$$

Here $D_R\langle 1, -x \rangle \cap H \subset D_R\langle 1, -h \rangle$ as $h \in C_R(H)$. Also, by construction, $x$ is an element of $D_R\langle 1, -h \rangle$ and $h \in C_R(H) \subset D_R\langle 1, -h \rangle$. So $-1, x \in D_R\langle 1, -h \rangle$ and $-\alpha x \in \alpha D_R\langle 1, -h \rangle$. Thus $D_S\langle 1, -\alpha x \rangle \subset D_S\langle 1, -h \rangle$. By [4,5.2]:

$$|Q_S(\alpha x) \cap Q_S(h)| = \frac{|D_S\langle 1, -\alpha xh \rangle|}{|D_S\langle 1, -\alpha x \rangle \cap D_S\langle 1, -h \rangle|} = \frac{2|D_R\langle 1, -xh \rangle \cap H|}{2|D_R\langle 1, -x \rangle \cap H|},$$

and $D_R\langle 1, -xh \rangle \cap H = D_R\langle 1, -x \rangle \cap H$ as $h \in C_R(H)$ implies $H \subset D_R\langle 1, -h \rangle$. So $|Q_S(\alpha x) \cap Q_S(h)| = 1$, a contradiction. Hence $\pi_1(H) = G(L)$. $\square$

**Corollary 2.3.** *Suppose $R$ is of local type with $|G(R)| \neq 4$. If $S$ is an $H$-extension of $R$ then either:*

(a) $H = 1$ and $S = R[E_1]$, with $E_1$ generated by $\alpha$,    or

(b) $H = G(R)$ and $S = D_1 \sqcap R$, with $D_1$ generated by $\alpha$.

*Proof.* If $|G(R)| = 2$ then it is clear that $H = 1$ or $H = G(R)$. If $|G(R)| \geq 8$ the take $R_2 = 1$ in (2.2) to get $H = 1$ or $G(R)$. Now apply (1.1). $\square$

## 3. Products: General Lemmas.

We start with a lemma that may be of general interest.

**Lemma 3.1.** *Let $R$ be a Witt ring of elementary type. Let $K$ be a proper subgroup of $G(R)$ and let $y \in G(R)$. If*

$$G(R) = \bigcup_{k \in K} D\langle 1, -yk \rangle,$$

*then $y \in \operatorname{rad}(R) \cdot K$.*

*Proof.* We first prove the result for non-degenerate $R$ where we must show $y \in K$. It suffices to prove this for subgroups $K$ of index two. Namely, if $K_0$ is any subgroup satisfying the hypothesis let $\mathcal{A}$ denote the set of subgroups $K$ of index two that contain $K_0$. Then for any $K \in \mathcal{A}$ :

$$G(R) = \bigcup_{k \in K_0} D\langle 1, -yk \rangle \subset \bigcup_{k \in K} D\langle 1, -yk \rangle.$$

Assuming the result holds for subgroups of index two, we obtain $y \in K$. Then $y \in \bigcap_{K \in \mathcal{A}} K = K_0$, as desired.

So suppose $[G(R) : K] = 2$. We work by induction on $|G(R)|$. We need to prove that $y \in K$ when $R$ is of local type, a group ring or a product. First suppose $R$ is of local type. Then $K = D\langle 1, -a \rangle$, for some $a \in G(R)$. We have:

$$G(R) = \bigcup_{k \in D\langle 1, -a \rangle} D\langle 1, -yk \rangle = D\langle 1, -y, ay \rangle.$$

Multiplying by $-a$ gives $G(R) = D\langle\langle -a, -y \rangle\rangle'$, the pure part of the Pfister form $\langle\langle -a, -y \rangle\rangle$. In particular, $-1 \in D\langle\langle -a, -y \rangle\rangle'$ so $\langle\langle -a, -y \rangle\rangle = 0$ and $y \in D\langle 1, -a \rangle = K$.

Next let $R = R_0[E_1]$, with $E_1 = \{1, t\}$ and $|G(R_0)| \geq 2$ (if $G(R_0) = 1$ then $R$ is degenerate). Suppose $y \notin K$. We claim $G(R_0) \subset K$. Choose any $g \in G(R_0)$. Then there exist $k_1, k_2 \in K$ with $-gt \in D\langle 1, -yk_1 \rangle$ and $-t \in D\langle 1, -yk_2 \rangle$. We get $gt = yk_1$ and $t = yk_2$ since $y \notin K$. Hence $g = k_1 k_2 \in K$. This proves the claim. Both $G(R_0)$ and $K$ have index two so $K = G(R_0)$. From $t = yk_2$ we have $y \in tG(R_0)$. Pick $g \in G(R_0)^{\cdot}$. Then $g \in D\langle 1, -yk_3 \rangle$ for some $k_3 \in K$. But $yk_3 \in yK = tG(R_0)$, so this is impossible. The contradiction implies $y \in K$.

Lastly, say $R = R_1 \sqcap R_2$. Write $y = (y_1, y_2)$. Now $K \cap (G(R_1) \times 1)$ is a subgroup of index at most two in $G(R_1) \times 1$. Let $K_1$ be its projection into $G(R_1)$. Similarly, let $K_2$

be the projection of $K \cap (1 \times G(R_2))$ into $G(R_2)$. Then $[G(R_i) : K_i] \leq 2$, for $i = 1, 2$. If $K_2 = G(R_2)$ then:

$$\bigcup_{k \in K} D_R \langle 1, -yk \rangle = \bigcup_{k_1 \in K_1} D \langle 1, -y_1 k_1 \rangle \times G(R_2)$$

so that $G(R_1) = \bigcup_{K_1} D \langle 1, -y_1 k_1 \rangle$. By induction then $y_1 \in K_1$ and hence $y = (y_1, y_2) \in K_1 \times G(R_2) = K$. In the same way, if $K_1 = G(R_1)$ then $y \in K$ as desired. So we may assume that $[G(R_i) : K_i] = 2$ for $i = 1, 2$. Write $K = \{1, \gamma\} K_1 \times K_2$. We have:

$$G(R) = \bigcup_{k_1 \in K_1, k_2 \in K_2} \left( D \langle 1, -y_1 k_1 \rangle \times D \langle 1, -y_2 k_2 \rangle \cup D \langle 1, -y_1 \gamma_1 k_1 \rangle \times D \langle 1, -y_2 \gamma_2 k_2 \rangle \right),$$

where $\gamma = (\gamma_1, \gamma_2)$.

Suppose $y_1 \in K_1$. If $G(R_1) = \cup D \langle 1, -y_1 \gamma_1 k_1 \rangle$ then by induction we have $y_1 \gamma_1 \in K_1$ and so $\gamma_1 \in K_1$. Then $K = K_1 \times G(R_2)$ and $K_2 = G(R_2)$ a case we have already covered. We may thus assume there exists a $g_1 \in G(R_1) \setminus \cup D \langle 1, -y_1 \gamma_1 k_1 \rangle$. Then $g_1 \times G(R_2) \subset \cup (D \langle 1, -y_1 k_1 \rangle \times D \langle 1, -y_2 k_2 \rangle)$ and so $G(R_2) = \cup D \langle 1, -y_2 k_2 \rangle$. By induction $y_2 \in K_2$ and $y = (y_1, y_2) \in K_1 \times K_2 \subset K$, and we are done.

We may thus assume $y_1 \notin K_1$. Similarly, $y_2 \notin K_2$. Pick, for $i = 1, 2$, a $g_i \in G(R_i) \setminus \cup D \langle 1, -y_i k_i \rangle$, which is possible by induction. Then $g_1 \times G(R_2) \subset \cup (D \langle 1, -y_1 \gamma_1 k_1 \rangle \times D \langle 1, -y_2 \gamma_2 k_2 \rangle)$ and so $G(R_2) = \cup D \langle 1, -y_2 \gamma_2 k_2 \rangle$. By induction once more, we have $y_2 \gamma_2 \in K_2$. Similarly, $y_1 \gamma_1 \in K_1$. Then $y \in (\gamma_1, \gamma_2)(K_1 \times K_2) \subset K$, as desired. This proves the result for non-degenerate $R$.

Now suppose $R$ is degenerate. Write $R = D \sqcap R_2$, with $\mathrm{rad}(R) = G(D) \times 1$ and $R_2$ non-degenerate. Let $\pi_2$ be the projection of $G(R)$ onto $G(R_2)$. Set $K_2 = \pi_2(K)$ and write $y = (y_1, y_2)$, with $y_1 \in G(D)$ and $y_2 \in G(R_2)$. Our assumption is:

$$G(R) = G(D) \times G(R_2) = \bigcup_{(k_1, k_2) \in K} D \langle (1, 1), -(y_1 k_1, y_2 k_2) \rangle$$

$$= G(D) \times \left( \bigcup_{k_2 \in K_2} D_{R_2} \langle 1, -y_2 k_2 \rangle \right).$$

From the non-degenerate case we get $y_2 \in K_2 = \pi_2(K)$. Thus there exists a $d \in G(D)$ such that $(d, y_2) \in K$. Hence $y = (y_1, y_2) = (dy_1, 1)(d, y_2) \in \mathrm{rad}(R) \cdot K$.   $\square$

Our key reduction lemma follows.

**Lemma 3.2.** *Let $R = R_1 \sqcap R_2$ and suppose $S$ is an $H$-extension of $R$. If $H = H_1 \times G(R_2)$ then there exists a Witt ring $T$ that is an $H_1$-extension of $R_1$ such that $S \cong T \sqcap R_2$.*

*Proof.* We first construct $T$. Let $G(T)$ be a group containing $G(R_1)$ as a subgroup of index 2; write $G(T) = \{1, \beta\} G(R_1)$. Let $\varphi : G(T) \to \{1, \alpha\}(G(R_1) \times 1)$ be the isomorphism sending $g_1 \mapsto (g_1, 1)$ and $\beta g_1 \mapsto \alpha(g_1, 1)$, where $g_1 \in G(R_1)$. For $z \in G(T)$ define:

$$D_T \langle 1, -z \rangle = \varphi^{-1}(D_S \langle 1, -\varphi(z) \rangle \cap \{1, \alpha\}(G(R_1) \times 1)).$$

We check that $T$ is an $H_1$-extension of $R_1$. If $z \in G(R_1) \setminus H_1$ then:

$$D_T\langle 1, -z \rangle = \varphi^{-1}((D_{R_1}\langle 1, -z \rangle \times G(R_2)) \cap \{1, \alpha\}(G(R_1) \times 1))$$
$$= \varphi^{-1}(D_{R_1}\langle 1, -z \rangle \times 1)$$
$$= D_{R_1}\langle 1, -z \rangle.$$

If $z \in H_1$ then:

$$D_T\langle 1, -z \rangle = \varphi^{-1}(\{1, \alpha\}(D_{R_1}\langle 1, -z \rangle \times G(R_2)) \cap \{1, \alpha\}(G(R_1) \times 1))$$
$$= \varphi^{-1}(\{1, \alpha\}(D_{R_1}\langle 1, -z \rangle \times 1))$$
$$= \{1, \beta\}D_{R_1}\langle 1, -z \rangle.$$

Lastly, if $z \in G(R_1)$ then:

$$D_T\langle 1, -\beta z \rangle = \varphi^{-1}(D_S\langle 1, -\alpha(z, 1) \rangle \cap \{1, \alpha\}(G(R_1) \times 1)).$$

Now:

$$D_S\langle 1, -\alpha(z, 1) \rangle = \{1, -\alpha(z, 1)\}((D_{R_1}\langle 1, -z \rangle \times G(R_2)) \cap (H_1 \times G(R_2)))$$
$$= \{1, -\alpha(z, 1)\}((D_{R_1}\langle 1, -z \rangle \cap H_1) \times G(R_2))$$

Since $(1, -1) \in D_{R_1}\langle 1, -z \rangle \cap H_1 \times G(R_2)$ we have:

$$D_S\langle 1, -\alpha(z, 1) \rangle = \{1, \alpha(-z, 1)\}((D_{R_1}\langle 1, -z \rangle \cap H_1) \times G(R_2)).$$

Hence:

$$D_T\langle 1, -\beta z \rangle = \varphi^{-1}(\{1, \alpha(-z, 1)\}(D_{R_1}\langle 1, -z \rangle \cap H_1) \times G(R_2)) \cap \{1, \alpha(-z, 1)\}(G(R_1) \times 1))$$
$$= \varphi^{-1}(\{1, \alpha(-z, 1)\}(D_{R_1}\langle 1, -z \rangle \cap H_1) \times 1)$$
$$= \{1, -\beta z\}(D_{R_1}\langle 1, -z \rangle \cap H_1).$$

We begin the verification that $(G(T), D_T)$ is linked, so that $T$ is indeed a Witt ring. Let $t = \alpha^{\epsilon_1}(u, v) \in G(S)$ and let $\beta^{\epsilon_2}x, \beta^{\epsilon_3}y \in G(T)$ with each $\epsilon_i = 0$ or 1.

**Claim**. If $t \in \varphi(\beta^{\epsilon_3}y)D_S\langle 1, -\varphi(\beta^{\epsilon_2}x) \rangle$ then $\beta^{\epsilon_1}u \in \beta^{\epsilon_3}yD_T\langle 1, -\beta^{\epsilon_2}x \rangle$.

We first assume that $\epsilon_1 = 0$. We have four cases:

*Case 1:* $\epsilon_2 = 0, \epsilon_3 = 0$. Here $(uy, v) \in D_S\langle 1, -(x, 1) \rangle$, hence $uy \in D_{R_1}\langle 1, -x \rangle \subset D_T\langle 1, -x \rangle$.

*Case 2:* $\epsilon_2 = 0, \epsilon_3 = 1$. Here $\alpha(uy, v) \in D_S\langle 1, -(x, 1) \rangle$. We must have that $x \in H_1$ so $\alpha(uy, v) \in \{1, \alpha\}(D_{R_1}\langle 1, -x \rangle \times G(R_2))$. Then $uy \in D_{R_1}\langle 1, -x \rangle$. Thus $\beta uy \in D_T\langle 1, -x \rangle = \{1, \beta\}D_{R_1}\langle 1, -x \rangle$.

*Case 3:* $\epsilon_2 = 1, \epsilon_3 = 0$. Here:

$$(uy, v) \in \{1, -\alpha(x, 1)\}((D_{R_1}\langle 1, -x \rangle \times G(R_2)) \cap H)$$
$$= \{1, -\alpha(x, 1)\}((D_{R_1}\langle 1, -x \rangle \cap H_1) \times G(R_2))$$

Thus $uy \in D_{R_1}\langle 1, -x \rangle \cap H_1$. We obtain $uy \in D_T\langle 1, -\beta x \rangle = \{1, -\beta x\}(D_{R_1}\langle 1, -x \rangle \cap H_1)$.

*Case 4*: $\epsilon_2 = 1, \epsilon_3 = 1$. Here $\alpha(uy, v) \in \{1, -\alpha(x, 1)\}((D_{R_1}\langle 1, -x \rangle \cap H_1) \times G(R_2))$ so that $(-xuy, v) \in (D_{R_1}\langle 1, -x \rangle \cap H_1) \times G(R_2)$. Thus $-xuy \in D_{R_1}\langle 1, -x \rangle \cap H_1$ and $\beta uy \in D_T\langle 1, -\beta x \rangle = \{1, -\beta x\}(D_{R_1}\langle 1, -x \rangle \cap H_1)$.

The four cases with $\epsilon_1 = 1$ are identical to the four above cases. For example, if $\epsilon_1 = 1, \epsilon_2 = 0, \epsilon_3 = 0$ then we have $\alpha(uy, v) \in D_S\langle 1, -(x, 1) \rangle$, which is *Case 2* above. Thus the **Claim** is proven.

We can now check linkage in $T$. Let $x, y, z, w \in G(T)$ and suppose:

$$xD_T\langle 1, -y \rangle \cap D_T\langle 1, -yz \rangle \cap wD_T\langle 1, -z \rangle \neq \emptyset.$$

Apply $\varphi$ to get:

$$\varphi(x)D_S\langle 1, -\varphi(y) \rangle \cap D_S\langle 1, -\varphi(yz) \rangle \cap \varphi(w)D_S\langle 1, -\varphi(z) \rangle \neq \emptyset.$$

By linkage on $S$, there exists a $t \in G(S)$ in:

$$\varphi(y)D_S\langle 1, -\varphi(x) \rangle \cap D_S\langle 1, -\varphi(xw) \rangle \cap \varphi(z)D_S\langle 1, -\varphi(w) \rangle.$$

Now apply the **Claim** :

$$yD_T\langle 1, -x \rangle \cap D_T\langle 1, -xw \rangle \cap zD_T\langle 1, -w \rangle \neq \emptyset,$$

as desired.

Lastly, set $W = T \sqcap R_2$. Then $G(W) = (\{1, \beta\}G(R_1)) \times G(R_2)$. Set $\gamma = (\beta, 1)$ so that $G(W) = \{1, \gamma\}(G(R_1) \times G(R_2)) = \{1, \gamma\}G(R)$. We will show $W$ is an $H$-extension of $R$, via $\gamma$, and hence that $S \cong W$.

First let $h = (h_1, g_2)) \in H$, where $h_1 \in H_1 \subset G(R_1)$ and $g_2 \in G(R_2)$. Then:

$$\begin{aligned} D_W\langle 1, -h \rangle &= D_T\langle 1, -h_1 \rangle \times D_{R_2}\langle 1, -g_2 \rangle \\ &= (\{1, \beta\}D_{R_1}\langle 1, -h_1 \rangle) \times D_{R_2}\langle 1, -g_2 \rangle \\ &= \{1, \gamma\}D_R\langle 1, -h \rangle. \end{aligned}$$

Next let $g = (g_1, g_2) \in G(R) \setminus H$, with $g_1 \in G(R_1) \setminus H_1$ and $g_2 \in G(R_2)$. Then:

$$\begin{aligned} D_W\langle 1, -g \rangle &= D_T\langle 1, -g_1 \rangle \times D_{R_2}\langle 1, -g_2 \rangle \\ &= D_{R_1}\langle 1, -g_1 \rangle \times D_{R_2}\langle 1, -g_2 \rangle = D_R\langle 1, -g \rangle. \end{aligned}$$

Lastly, let $g = (g_1, g_2) \in G(R)$, with $g_1 \in G(R_1)$ and $g_2 \in G(R_2)$. Then:

$$\begin{aligned} D_W\langle 1, -\gamma g \rangle &= D_W\langle 1, -(\beta g_1, g_2) \rangle \\ &= D_{R_1}\langle 1, -\beta g_1 \rangle \times D_{R_2}\langle 1, -g_2 \rangle \\ &= \left(\{1, -\beta g_1\}(D_{R_1}\langle 1, -g_1 \rangle \cap H_1)\right) \times D_{R_2}\langle 1, -g_2 \rangle \\ &= (D_{R_1}\langle 1, -g_1 \rangle \cap H_1) \times D_{R_2}\langle 1, -g_2 \rangle \\ &\quad \cup \gamma\left(-g_1(D_{R_1}\langle 1, -g_1 \rangle \cap H_1) \times -g_2 D_{R_2}\langle 1, -g_2 \rangle\right) \\ &= \{1, -\gamma(g_1, g_2)\}(D_R\langle 1, -(g_1, g_2) \rangle \cap H) \\ &= \{1, -\gamma g\}(D_R\langle 1, -g \rangle \cap H). \end{aligned}$$

Thus $W = T \sqcap R_2$ is an $H$-extension of $R$ and so is isomorphic to $S$.  $\square$

Our last general lemma is the most technical, but it also does most of the work.

**Lemma 3.3.** *Let $u \in G(R)$ and $h \in H$. Then:*

$$D_R\langle 1, uh, -h\rangle \setminus uH \subset \bigcup_{t \in h(D\langle 1, -u\rangle \cap H)} D_R\langle 1, -t\rangle.$$

*Proof.* Let $w \in D_R\langle 1, uh, -h\rangle \setminus uH$. Then $w \in D_R\langle 1, -hv\rangle$ for some $v \in D_R\langle 1, -u\rangle$ and $uw \notin H$. We have:

$$q(\alpha u, h) = q(u, h) = q(u, vh) = q(uw, vh).$$

Thus, by linkage, there exists a $t \in G(S)$ such that:

$$q(\alpha u, h) = q(\alpha u, t) = q(uw, t) = q(uw, vh).$$

The third equality gives $vht \in D_S\langle 1, -uw\rangle$. Since $uw \notin H$ this implies $t \in G(R)$. Then the first two equalities give:

$$ht \in D_S\langle 1, -\alpha u\rangle \cap G(R) = D_R\langle 1, -u\rangle \cap H,$$
$$t \in D_S\langle 1, -\alpha w\rangle \cap G(R) = D_R\langle 1, -w\rangle \cap H.$$

Hence $w \in D_R\langle 1, -t\rangle$ where $t \in h(D_R\langle 1, -u\rangle \cap H)$.    $\square$

## 4. Products: Degenerate Witt rings.

If $R$ is a degenerate Witt ring then $R = D \sqcap R_2$, for some Witt ring $R_2$ and where $G(D) = \{1, d\}$, with $D_D\langle 1, 1\rangle = D_D\langle 1, d\rangle = \{1, d\}$. We will often use the fact that if $(u, v) \in G(R)$ then $D_R\langle 1, -(u, v)\rangle = D_R\langle 1, -(du, v)\rangle$.

**Lemma 4.1.** *Suppose $R = D \sqcap R_2$ is degenerate. Let $\pi_1$ be the projection of $G(R)$ onto $G(D)$. Let $S$ be an $H$-extension of $R$. Then either $G(D) \times 1 \subset H$ or $S$ is isomorphic to an $H_0$-extension of $R$, for some subgroup $H_0 \subset G(R)$ with $\pi_1(H_0) = 1$.*

*Proof.* Suppose $(d, 1) \notin H$ and that $\pi_1(H) \neq 1$, so that $(d, y) \in H$ for some $y \in G(R_2)$. Let $G_2$ be the subgroup of $G(R_2)$ such that $1 \times G_2 = H \cap (1 \times G(R_2))$. Then $H = (1 \times G_2) \cup (d \times yG_2)$. Set $H_0 = 1 \times \{1, y\}G_2$, and note that $\pi_1(H_0) = 1$.

Let $\beta^2 = 1$ and set $G(S_0) = \{1, \beta\}G(R_2)$. Define $S_0$-value set s so that $S_0$ is an $H_0$-extension of $R$. We wish to show $S \cong S_0$. Extend $G_2$ to a subgroup $K$ of index two in $G(R_2)$ that does not contain $y$. Define $\varphi : G(S) \to G(S_0)$ by $\alpha \mapsto \beta$ and for $(u, v) \in G(R)$:

$$\varphi(u, v) = \begin{cases} (u, v), & \text{if } v \in K \\ (du, v), & \text{if } v \notin K. \end{cases}$$

It is quickly checked that $\varphi$ is an isomorphism. We will show:

$$(4.2) \qquad\qquad \varphi(D_S\langle 1, -s\rangle) = D_{S_0}\langle 1, -\varphi(s)\rangle,$$

for all $s \in G(S)$. This shows both that $S_0$ is a Witt ring and that $S \cong S_0$.

**Claim**. If $(u, v) \in G(R)$ then $\varphi(D_R\langle 1, -(u,v)\rangle) = D_R\langle 1, -(u,v)\rangle$.

$$\begin{aligned}
D_R\langle 1, -(u,v)\rangle &= \{1, d\} \times D_{R_2}\langle 1, -v\rangle \\
&= 1 \times (D_{R_2}\langle 1, -v\rangle \cap K) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap K) \\
&\cup 1 \times (D_{R_2}\langle 1, -v\rangle \cap yK) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap yK).
\end{aligned}$$

Thus:

$$\begin{aligned}
\varphi(D_R\langle 1, -(u,v)\rangle) &= 1 \times (D_{R_2}\langle 1, -v\rangle \cap K) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap K) \\
&\cup d \times (D_{R_2}\langle 1, -v\rangle \cap yK) \cup 1 \times (D_{R_2}\langle 1, -v\rangle \cap yK) \\
&= D_R\langle 1, -(u,v)\rangle,
\end{aligned}$$

proving the **Claim**.

We now check (4.2) in various cases. First suppose $s = (u,v) \in G(R)$, with $v \in K$. Then $s \in H$ iff $u = 1$ and $v \in G_2$. We have $\varphi(s) = s$ and $\varphi(s) \in H_0$ iff $u = 1$ and $v \in G_2$ iff $s \in H$. $D_S\langle 1, -s\rangle = \{1, \alpha\}D_R\langle 1, -s\rangle$ or $D_R\langle 1, -s\rangle$ depending on whether or not $s \in H$. So by the **Claim**, $\varphi(D_S\langle 1, -s\rangle) = \{1, \beta\}D_R\langle 1, -s\rangle$ or $D_R\langle 1, -s\rangle$ depending on whether or not $\varphi(s) \in H_0$. Thus $\varphi(D_S\langle 1, -s\rangle) = D_{S_0}\langle 1, -\varphi(s)\rangle$.

Next suppose that $s = (u,v) \in G(R)$ with $v \in yK$. Then $s \in H$ iff $u = d$ and $v \in yG_2$. We have $\varphi(s) = (du, v)$ so that $\varphi(s) \in H_0$ iff $u = d$ and $v \in yG_2$ iff $s \in H$. Again using the **Claim**:

$$D_S\langle 1, -s\rangle = \begin{cases} \{1, \alpha\}D_R\langle 1, -s\rangle, & \text{if } s \in H \\ D_R\langle 1, -s\rangle, & \text{if } s \notin H. \end{cases}$$

Thus:

$$\varphi(D_S\langle 1, -s\rangle) = \begin{cases} \{1, \beta\}D_R\langle 1, -s\rangle, & \text{if } \varphi(s) \in H_0 \\ D_R\langle 1, -s\rangle & \text{if } \varphi(s) \notin H_0. \end{cases}$$

Now $D_R\langle 1, -s\rangle = D_R\langle 1, -(u,v)\rangle = D_R\langle 1, -(du,v)\rangle = D_R\langle 1, -\varphi(s)\rangle$. Hence we have as desired that $\varphi(D_S\langle 1, -s\rangle) = D_{S_0}\langle 1, -\varphi(s)\rangle$.

Now suppose $s = \alpha(u,v) \in \alpha G(R)$. Then:

$$\begin{aligned}
D_S\langle 1, -s\rangle &= \{1, -\alpha(u,v)\}(D_R\langle 1, -(u,v)\rangle \cap H) \\
&= \{1, -\alpha(u,v)\}[(\{1, d\} \times D_{R_2}\langle 1, -v\rangle) \cap (1 \times G_2 \cup d \times yG_2)] \\
&= \{1, -\alpha(u,v)\}[1 \times (D_{R_2}\langle 1, -v\rangle \cap G_2) \cup d \times (D_{R_2}\langle 1, -v\rangle \cap yG_2)]
\end{aligned}$$

Now:

$$\begin{aligned}
\varphi((1 \times &(D_{R_2}\langle 1, -v\rangle \cap G_2)) \cup (d \times (D_{R_2}\langle 1, -v\rangle \cap yG_2))) \\
&= 1 \times (D_{R_2}\langle 1, -v\rangle \cap G_2) \cup 1 \times (D_{R_2}\langle 1, -v\rangle \cap yG_2) \\
&= 1 \times (D_{R_2}\langle 1, -v\rangle \cap \{1, y\}G_2) \\
&= D_R\langle 1, -(u,v)\rangle \cap H_0.
\end{aligned}$$

Thus if $v \in K$ then:

$$\varphi(D_S\langle 1, -s\rangle) = \varphi(D_S\langle 1, -\alpha(u, v)\rangle)$$
$$= \{1, -\beta(u, v)\}(D_R\langle 1, -(u, v)\rangle \cap H_0) = D_{S_0}\langle 1, -\varphi(s)\rangle,$$

verifying (4.2) in this case.

Lastly, if $v \in yK$ then:

$$\varphi(D_S\langle 1, -s\rangle) = \varphi(D_S\langle 1, -\alpha(u, v)\rangle)$$
$$= \{1, -\beta(du, v)\}(D_R\langle 1, -(u, v)\rangle \cap H_0)$$
$$= \{1, -\beta(du, v)\}(D_R\langle 1, -(du, v)\rangle \cap H_0)$$
$$= D_{S_0}\langle 1, -\varphi(s)\rangle.$$

Thus (4.2) holds in all cases. $\square$

## 5. Products: Local type factors.

Lemma (3.3) looks simpler when one factor has local type.

**Lemma 5.1.** *Suppose* $R = L \sqcap R_2$, *with* $L$ *of local type. Suppose* $S$ *is an* $H$-*extension of* $R$. *Let* $h = (h_1, h_2) \in H$ *and* $u = (u_1, u_2) \in G(R)$ *such that* $u_1 \notin D_L\langle 1, -h_1\rangle$ *while* $u_2 \in D_{R_2}\langle 1, -h_2\rangle$. *Then:*

$$G(R) = uH \cup (u_1 \times G(R_2)) \cup \bigcup_{t \in h(D_R\langle 1, -u\rangle \cap H)} D_R\langle 1, -t\rangle.$$

*Proof.* Write $Q(L) = \{1, \rho\}$. Then $\langle\langle -u, -h\rangle\rangle = (\rho, 1)$. Hence $-D_R\langle\langle -u, -h\rangle\rangle' = \{(x, y) \in G(R) : x \neq 1\}$. Now $-u \cdot \langle -u, -h, uh\rangle \simeq \langle 1, uh, -h\rangle$. Thus $D_R\langle 1, uh, -h\rangle = \{(x, y) \in G(R) : x \neq u_1\}$. Apply (3.3). $\square$

**Lemma 5.2.** *Suppose* $R = L \sqcap R_2$, *with* $L$ *of local type. Let* $\pi_1$ *be the projection of* $G(R)$ *onto* $G(L)$. *Let* $S$ *be an* $H$-*extension of* $R$ *and suppose that* $\pi_1(H) = G(L)$. *Let* $u = (u_1, u_2) \in G(R)$.

    (a) *If* $|G(L)| \geq 4$ *then* $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_L\langle 1, -u_1\rangle$.
    (b) *If* $L = \mathbb{Z}$ *and* $\pi_1(D_R\langle 1, -u\rangle \cap H) \neq D_L\langle 1, -u_1\rangle$ *for some* $u$, *then* $H = 1 \times H_2 \cup -1 \times -H_2$, *where* $1 \times H_2 = H \cap (1 \times G(R_2))$ *and* $H_2$ *is an ordering on* $R_2$.

*Proof.* Suppose that $\pi_1(D_R\langle 1, -u\rangle \cap H) = K < D_L\langle 1, -u_1\rangle$.
**Claim.** *If* $v \in G(L) \setminus K$ *then* $(v, -1) \in H$.
Since $\pi_1(H) = G(L)$ there exists a $w \in G(R_2)$ such that $(v, w) = h \in H$. Now:

$$D_R\langle 1, uh, -h\rangle = -u(D_L\langle\langle -u, -v\rangle\rangle' \times D_{R_2}\langle\langle -u_2, -w\rangle\rangle')$$
$$\supset u_1 G(L)^{\cdot} \times u_2 T,$$

where $T = -D_{R_2}\langle\langle -u_2, -w\rangle\rangle'$. Also:

$$\bigcup_{t \in h(D_R\langle 1, -u\rangle \cap H)} D_R\langle 1, -t\rangle \subset \left(\bigcup_{k \in K} D_L\langle 1, -vk\rangle\right) \times G(R_2).$$

Now $v \notin K$ implies $G(L) \neq \cup D_L\langle 1, -vk\rangle$ by (3.1). Choose a $g \in G(L) \setminus \cup D_L\langle 1, -vk\rangle$.

We check that we may assume $g \neq u_1$. If $v \in D_L\langle 1, -u_1\rangle$ then we have $u_1 D_L\langle 1, -v\rangle = D_L\langle 1, -v\rangle \subset \cup D_L\langle 1, -vk\rangle$ and so no $g \in G(L) \setminus \cup D_L\langle 1, -vk\rangle$ is equal to $u_1$. If instead $v \notin D_L\langle 1, -u_1\rangle$ then, as $|K| < |D_L\langle 1, -u_1\rangle|$, there exists a $w \neq u_1$ such that $K \subset D_L\langle 1, -u_1\rangle \cap D_L\langle 1, -w\rangle$. If $v \notin D_L\langle 1, -w\rangle$ then $w$ is not in any $D_L\langle 1, -vk\rangle$, for $k \in K$, and we may take $g = w$. If $v \in D_L\langle 1, -w\rangle$ then $v \notin D_L\langle 1, -u_1 w\rangle$ and we may take $g = u_1 w$.

We thus have $g \in u_1 G(L)^{\cdot} \setminus \cup D_L\langle 1, -vk\rangle$. So $g \times u_2 T \subset uH$ by (3.3). Hence $u_1 g \times T \subset H$. Then $(u_1 g, u_2), (u_1 g, -u_2 w) \in H$ and so $(1, -w) \in H$. We obtain that $(v, -1) = (v, w)(1, -w) \in H$ and the **Claim** is proven.

Now suppose $u_1 \neq 1$. Let $x \in G(L)^{\cdot}$. Since $|K| < |D_L\langle 1, -u_1\rangle|$ we have $|K| \leq \frac{1}{4}|G(L)|$. So we can choose $v \in G(L) \setminus \{1, x\}K$. Then $(v, -1)$ and $(vx, -1)$ are in $H$ by the **Claim**. Hence $(x, 1) \in H$. This shows that $G(L) \times 1 \subset H$. But then $D_L\langle 1, -u_1\rangle \times 1 \subset D_R\langle 1, -u\rangle \cap H$ and $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_L\langle 1, -u_1\rangle$, as desired.

Next suppose $u_1 = 1$ and $|G(L)| \geq 4$. We show $\pi_1(D_R\langle 1, -u\rangle \cap H) = G(L)$. Pick any $a \in G(L)$ and pick a $b \in G(L)^{\cdot}$ such that $a \in D_L\langle 1, -b\rangle$. This is possible since $|G(L)| \geq 4$ implies there are at least two $b$'s such that $a \in D_L\langle 1, -b\rangle$. So there is such a $b$ not equal to 1. Then, by the above paragraph, $\pi_1(D_R\langle 1, -(b, u_2)\rangle \cap H) = D_L\langle 1, -b\rangle$ contains $a$. Thus there exists a $k \in D_{R_2}\langle 1, -u_2\rangle$ such that $(a, k) \in H$. Thus $(a, k) \in D_R\langle 1, -(u_1, u_2)\rangle$, as $u_1 = 1$, and so $a \in \pi_1(D_R\langle 1, -u\rangle \cap H)$.

Lastly, suppose $u_1 = 1$ and $L = \mathbb{Z}$. Here $K = \{1\}$ and $v = -1$. The **Claim** shows that $(-1, -1) \in H$. Now $H_2 = H \cap (1 \times G(R_2))$ has index 2 in $H$ since $1 \times G(R_2)$ has index 2 in $G(R)$ and $H \not\subset 1 \times G(R_2)$. Hence $H = 1 \times H_2 \cup -1 \times -H_2$. The last paragraph of the proof of the **Claim** gives $u_1 g \times T \subset H$, where $g \neq u_1$. Thus $g = -1$ and after multiplying by $-1 \in H$ we get:

$$(5.3) \qquad\qquad 1 \times D_{R_2}\langle\langle -u_2, -w\rangle\rangle' \subset H.$$

This holds for all $w \in G(R_2)$ such that $(-1, w) \in H$, that is, for all $w \in -H_2$. Thus for any $h_2 \in H_2$:

$$-u_2 D_{R_2}\langle 1, h_2\rangle \subset D_{R_2}\langle -u_2, h_2, -u_2 h_2\rangle \subset H_2$$
$$D_{R_2}\langle 1, h_2\rangle \subset H_2.$$

Thus $H_2$ is a preordering. Also (5.3) holds for any $u_2 \in G(R_2)$ with $\pi_1(D_R\langle 1, -(1, u_2)\rangle) = 1$. That is, $D_R\langle 1, -(1, u_2)\rangle \cap (-1 \times -H_2) = \emptyset$ or equivalently, $u_2$ is not in $D_{R_2}\langle 1, h_2\rangle$ for any $h_2 \in H_2$. For such a $u_2$, (5.3) implies $-u_2 \in H_2$. Hence:

$$G(R_2) = -H_2 \cup \bigcup_{h_2 \in H_2} D_{R_2}\langle 1, h_2\rangle.$$

But $H_2$ is a preordering so that $\cup D_{R_2}\langle 1, h_2\rangle \subset H_2$. Thus $G(R_2) = -H_2 \cup H_2$, $H_2$ has index 2 in $G(R_2)$ and so $H_2$ is an ordering. $\square$

**Notation.** Suppose $R = R_1 \sqcap R_2$ and that $H$ is a subgroup of $G(R)$. For $x \in G(R_1)$ set $F(x) = \{y \in G(R_2) : (x, y) \in H\}$.

**Lemma 5.4.** *Let $R = R_1 \sqcap R_2$ and let $S$ be an $H$-extension of $R$. Let $\pi_1$ be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) = G(R_1)$. Then for all $x \in G(R_1)$:*

(a) $F(x)$ *is non-empty.*
(b) $F(1)$ *is a subgroup of $G(R_1)$.*
(c) $F(x)$ *is a coset of $F(1)$.*

*Proof.* No $F(x)$ is empty since $\pi_1(H) = G(R_1)$. Clearly $F(1)$ is a subgroup. Fix $y_0 \in F(x)$. If $y \in F(1)$ then $(1, y), (x, y_0) \in H$ implies $(x, yy_0) \in H$ and so $yy_0 \in F(x)$. This says $y_0 F(1) \subset F(x)$.

Now let $y \in F(x)$. Then $(x, y_0), (x, y) \in H$ so that $(1, yy_0) \in H$. Hence $yy_0 \in F(1)$ and we have the reverse inclusion $F(x) \subset y_0 F(1)$. □

**Lemma 5.5.** *Let $R = R_1 \sqcap R_2$ be of elementary type. Let $S$ be an $H$-extension of $R$. Let $\pi_1$ be the projection of $G(R)$ onto $G(R_1)$. Suppose the following:*

(1) $\pi_1(H) = G(R_1)$.
(2) $F(a) \cap \mathrm{rad}(R_2) \subset \{1\}$, *for all $a \in G(R_1)$.*
(3) *For all $u = (u_1, u_2) \in G(R)$ we have $\pi_1(D_R\langle 1, -u\rangle \cap H) = D_{R_1}\langle 1, -u_1\rangle$.*

*Then $H = G(R_1) \times H_2$, for some subgroup $H_2 \subset G(R_2)$.*

*Proof.* Let $a \in G(R_1)$. We will first show that:

$$(5.6) \qquad\qquad G(R_2) = \bigcup_{k \in F(a)} D_{R_2}\langle 1, -k\rangle.$$

Pick any $b \in D_{R_1}\langle 1, -a\rangle$ and any $g \in G(R_2)$. Then $a \in D_{R_1}\langle 1, -b\rangle = \pi_1(D_R\langle 1, -(b, g)\rangle \cap H)$ by assumption (3). Hence there exists a $k \in D_{R_2}\langle 1, -g\rangle$ with $(a, k) \in H$. That is, $g \in D_{R_2}\langle 1, -k\rangle$ for some $k \in F(a)$, proving (5.6).

Write $F(a) = yF(1)$ as in (5.4). Then (5.6) becomes:

$$G(R_2) = \bigcup_{k \in F(1)} D_{R_2}\langle 1, -yk\rangle.$$

Thus $y \in \mathrm{rad}(R_2) \cdot F(1)$ by (3.1). That is, there exists a $d \in \mathrm{rad}(R_2)$ such that $d \in yF(1) = F(a)$. By assumption (2) then $d = 1$. Hence $y \in F(1)$ and so $F(a) = F(1)$. By assumption (1) we have $(a, m) \in H$ for some $m \in G(R_2)$. Then $m \in F(a) = F(1)$ so that $(1, m) \in H$ also. So $(a, 1) = (a, m)(1, m) \in H$. Hence $G(R_1) \times 1 \subset H$ and $H = G(R_1) \times F(1)$. □

We first complete the case of a local factor $L$ with $|G(L)| \geq 8$.

**Corollary 5.7.** *Let $R = L \sqcap R_2$, with $R_2$ of elementary type, $L$ of local type and $|G(L)| \geq 8$. Let $S$ be an $H$-extension of $R$. Suppose $F(a) \cap \mathrm{rad}(R_2) \subset \{1\}$ for all $a \in G(L)$. Then either $H = 1 \times H_2$ or $H = G(L) \times H_2$ for some subgroup $H_2 \subset G(R_2)$.*

*Proof.* Again let $\pi_1$ denote the projection of $G(R)$ onto $G(L)$. We know that $\pi_1(H) = 1$ or $G(L)$, by (2.2). If $\pi_1(H) = 1$ then clearly $H = 1 \times H_2$ for some subgroup $H_2$. So suppose that $\pi_1(H) = G(L)$, the first hypothesis of (5.5). We are assuming the second hypothesis as well. And (5.2) shows the third hypothesis of (5.5) holds. Hence $H = G(L) \times H_2$, for some subgroup $H_2$. □

The argument for $R = \mathbb{Z} \sqcap R_2$ is different.

**Lemma 5.8.** *Let $R$ be a real Witt ring of elementary type. Let $P \subset G(R)$ be an ordering. Suppose that for all $x \in P$ that:*

$$P = \bigcup_{k \in D\langle 1, -x \rangle \cap P} D\langle 1, xk \rangle.$$

*Then $R = \mathbb{Z} \sqcap R_2$, for some Witt ring $R_2$.*

*Proof.* Suppose $\mathbb{Z}$ is not a factor of $R$. Then $R$ has a group ring factor that is real. Thus $R = R_0[E_1] \sqcap R_2$, for some Witt rings $R_0, R_2$, and we may assume $P = P_0\{1, t\} \times G(R_2)$, where $P_0 \subset G(R_0)$ is an ordering on $R_0$ and $E_1 = \{1, t\}$. Then take $x = (t, 1)$. We have that $D\langle 1, -x \rangle = \{1, -t\} \times G(R_2)$ and $D\langle 1, -x \rangle \cap P = 1 \times G(R_2)$. Thus:

$$P = \bigcup_{k \in D\langle 1, -x \rangle \cap P} D\langle 1, xk \rangle = \bigcup_{g_2 \in G(R_2)} D\langle (1, 1), (t, g_2) \rangle$$
$$= \{1, t\} \times G(R_2)$$

Hence $P_0 = 1$ and $R_0 = \mathbb{Z}$, giving a contradiction.  $\square$

**Lemma 5.9.** *Let $R = \mathbb{Z} \sqcap R_2$ and suppose $S$ is an $H$-extension of $R$. Then one of the following occurs.*

    (a) $H = 1 \times H_2$ *for some subgroup $H_2 \subset G(R_2)$.*
    (b) $R = \mathbb{Z} \sqcap R_3$, *for some Witt ring $R_3$, and $\{\pm 1\} \times 1 \subset H$.*
    (c) $R = \mathbb{Z} \sqcap \mathbb{Z} \sqcap R_3$, *for some Witt ring $R_3$, and $(1, 1) \times G(R_3) \subset H$.*

*Proof.* Again let $\pi_1$ be the projection of $G(R)$ onto $G(\mathbb{Z}) = \{\pm 1\}$. If $\pi_1(H) = 1$ the we are in case (a). Thus we may assume that $\pi_1(H) = G(\mathbb{Z})$. If for every $u \in G(R)$ we have that $\pi_1(D_R\langle 1, -u \rangle \cap H) = D\langle 1, -\pi_1(u) \rangle$ then (5.5) implies we are in case (b). So suppose this fails for some $u \in G(R)$. Then by (5.2) $H = 1 \times H_2 \cup (-1 \times -H_2)$, for some ordering $H_2$ of $G(R_2)$. We will first show that for every $h_2 \in H_2$ that:

$$H_2 = \bigcup_{k \in D_{R_2}\langle 1, -h_2 \rangle \cap H_2} D_{R_2}\langle 1, h_2 k \rangle.$$

Consider $\varphi = \langle (1, 1), (1, h_2), (1, -1) \rangle \in S$. We compute its value set two ways. First:

$$D_S\langle (1, 1), (1, h_2), (1, -1) \rangle = \bigcup_{\beta \in D_S\langle (1,1), (1, -h_2) \rangle} D_S\langle (1, 1), \beta(1, h_2) \rangle.$$

Now $(-1, h_2) \notin H$ so $D_S\langle (1, 1), (1, -h_2) \rangle = 1 \times D_{R_2}\langle 1, -h_2 \rangle$. For $\varphi$ to represent an element of $\alpha(1 \times G(R_2))$ we must have $\beta \in -H = H$. That is, $\beta = (1, \beta_2)$ with $\beta_2 \in H_2$. Thus:

$$D_S(\varphi) \cap \alpha(1 \times G(R_2)) = \alpha \cdot \bigcup_{\beta_2 \in D_{R_2}\langle 1, -h_2 \rangle \cap H_2} (1 \times D_{R_2}\langle 1, \beta_2 h_2 \rangle).$$

Next:

$$D_S(\varphi) = (1, h_2) \cdot \bigcup_{\gamma \in D_S\langle(1,1),(1,-1)\rangle} D_S\langle(1,1), \gamma(1, h_2)\rangle.$$

For any $x \in H_2$ take $\gamma = (1, x) \in D_S\langle(1,1), (1, -1)\rangle$. Then since $(-1, -xh_2) \in H$:

$$\alpha(1, x) \in (1, h_2)D_S\langle(1,1), -(-1, -xh_2)\rangle$$
$$= (1, h_2) \cdot \{1, \alpha\}(1 \times D_{R_2}\langle1, xh_2\rangle)$$

Thus $D_S(\varphi) \cap \alpha(1 \times G(R_2)) = \alpha(1 \times H_2)$. The two computations of $D_S(\varphi)$ thus yield $H_2 = \cup D_{R_2}\langle1, \beta h_2\rangle$, over $\beta \in D_{R_2}\langle1, -h_2\rangle \cap H_2$.

We may now apply (5.8) to obtain $R_2 = \mathbb{Z} \sqcap R_3$, for some Witt ring $R_3$. Let $H_3 \subset G(R_3)$ be the subgroup such that $H_2 \cap (1 \times G(R_3)) = 1 \times H_3$. We note that both $H_2$ and $1 \times G(R_3)$ have index two in $G(R_2)$.

If $H_2 = 1 \times G(R_3)$ then $(1,1) \times G(R_3) \subset H$ and we are in case (c). So suppose $H_2 \neq 1 \times G(R_3)$. Then $1 \times H_3$ has index two in $H_2$ and $H_3$ has index two in $G(R_3)$. Write $H_2 = 1 \times H_3 \cup (-1 \times zH_3)$, for some $z \in G(R_3)$. Then:

$$(5.10) \qquad H = [(1,1) \times H_3] \cup [(1,-1) \times zH_3] \cup [(-1,-1) \times -H_3] \cup [(-1,1) \times -zH_3].$$

Now $[G(R_3) : H_3] = 2$ implies at least one of the cosets $zH_3, -H_3, -zH_3$ equals $H_3$. Say $zH_3 = H_3$. Then the second term of (5.10) shows $(1,1,1), (1,-1,1) \in H$. Set $R_4$ equal to the product of the first copy of $\mathbb{Z}$ and $R_3$. Then $R = \mathbb{Z} \sqcap R_4$ and $\{\pm1\} \times 1 \subset H$. We are thus in case (b). Next say $-H_3 = H_3$. Then $(1,-1) \in 1 \times H_3 \subset H_2$. Since $H_2$ is an ordering we have $D_{R_2}\langle(1,1),(1,-1)\rangle \subset H_2$. But the $1 \times G(R_3) \subset H_2$, a case we have already considered. Lastly, suppose $-zH_3 = H_3$. Then the fourth term of (5.10) shows $(1,1,1), (-1,1,1) \in H$. This is case (b) again. $\square$

## 6. Products: Group ring factors.

**Lemma 6.1.** *Let* $R = R_1 \sqcap R_2$, *with* $R_1 = R_0[E_1]$ *and* $E_1$ *generated by* $t$. *Let* $S$ *be an* $H$-*extension of* $R$. *Let* $\pi_1$ *be the projection of* $G(R)$ *onto* $G(R_1)$ *and suppose* $\pi_1(H) \not\subset G(R_0)$. *Then either* $\pi_1(H) = G(R_1)$ *or* $1 \times G(R_2) \subset H$.

*Proof.* From $\pi_1(H) \not\subset G(R_0)$ we may assume $h = (t, g_2) \in H$, for some $g_2 \in G(R_2)$. Suppose $\pi_1(H) \neq G(R_1)$, Choose $-g_1 \in G(R_1) \setminus \pi_1(H)$. Then $-g_1 t \notin \pi_1(H)$. Set $u = (g_1 t, 1)$ and note that $\pi_1(D_R\langle1, -u\rangle \cap H) = 1$. Now:

$$D_R\langle1, uh, -h\rangle = D_{R_1}\langle1, g_1, -t\rangle \times D_{R_2}\langle1, g_2, -g_2\rangle$$
$$\supset g_1 \times G(R_2).$$

Also:

$$\bigcup_{k \in h(D_R\langle1,-u\rangle \cap H)} D_R\langle1, -k\rangle \subset D_{R_1}\langle1, -t\rangle \times G(R_2) = \{1, -t\} \times G(R_2).$$

Hence by (3.3), $g_1 \times G(R_2) \subset uH$. Multiplying by $u$ gives $t \times G(R_2) \subset H$. Thus $1 \times G(R_2) \subset H$. $\square$

**Lemma 6.2.** *Let $R = R_1 \sqcap R_2$, with $R_1 = R_0[E_1]$ and $E_1$ generated by $t$. Let $S$ be an $H$-extension of $R$. Let $\pi_1$ be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) = G(R_1)$. If $u_1 \in G(R_0)^{\cdot}$ and $u = (u_1, u_2)$ then $\pi_1(D_R\langle 1, -u \rangle \cap H) = D_{R_1}\langle 1, -u_1 \rangle$.*

*Proof.* Set $K = \pi_1(D_R\langle 1, -u \rangle \cap H)$ and suppose $K < D_{R_1}\langle 1, -u_1 \rangle$. Let $g \in G(R_0)$. Then $(gt, g_2) \in H$ for some $g_2 \in G(R_2)$, since $\pi_1(H) = G(R_1)$. Now $D_R\langle 1, uh, -h \rangle$ contains $-gtD_{R_1}\langle 1, -u_1 \rangle \times u_2 T$, where $T = -D_{R_2}\langle\langle -u_2, -g_2 \rangle\rangle'$. Also:

$$\bigcup_{w \in h(D_R\langle 1, -u \rangle \cap H)} D_R\langle 1, -w \rangle \subset \bigcup_{k \in K} D_{R_1}\langle 1, -kgt \rangle \times G(R_2)$$

$$= (\{1\} \cup -gtK) \times G(R_2).$$

Hence by (3.3), if $y \in D_{R_1}\langle 1, -u_1 \rangle \setminus K$ then:

$$-gty \times u_2 T \subset uH$$
$$-gtu_1 y \times T \subset H.$$

Now $u_2$ and $-u_2 g_2$ are in $T$ so $(-gtu_1 y, u_2)$ and $(-gtu_1 y, -u_2 g_2)$ are in $H$. Thus $(1, -g_2) \in H$ and as result $(gt, -1) \in H$.

This holds for all $g \in G(R_0)$ so we have that $tG(R_0) \times -1 \subset H$. Thus $G(R_0) \times 1 \subset H$. But the $D_{R_1}\langle 1, -u_1 \rangle \times 1 \subset D_R\langle 1, -u \rangle \cap H$ and $\pi_1(D_R\langle 1, -u \rangle \cap H) = D_{R_1}\langle 1, -u_1 \rangle$, a contradiction. $\square$

**Lemma 6.3.** *Let $R = R_1 \sqcap R_2$, with $R_1 = R_0[E_1]$ non-degenerate and $E_1$ generated by $t$. Let $S$ be an $H$-extension of $R$. Let $\pi_1$ be the projection of $G(R)$ onto $G(R_1)$ and suppose $\pi_1(H) = G(R_1)$. Then $G(R_1) \times 1 \subset H$.*

*Proof.* We will first show $F(g) = F(1)$ for all $g \in G(R_0)$. Let $g \in G(R_0)^{\cdot}$. Pick $u_1 \in G(R_0)^{\cdot}$ such that $g \in D_{R_1}\langle 1, -u_1 \rangle$. For all $u_2 \in G(R_2)$, since $g \in D_{R_1}\langle 1, -u_1 \rangle = \pi_1(D_R\langle 1, -(u_1, u_2)\rangle \cap H)$, there exists a $k \in G(R_2)$ with $(g, k) \in H$ and $k \in D_{R_2}\langle 1, -u_2 \rangle$. That is, $G(R_2) = \cup_{k \in F(g)} D_{R_2}\langle 1, -k \rangle$. By (3.1) and (5.4), $F(g) = F(1)$.

We next show $G(R_0) \times 1 \subset H$. Continue to let $g \in G(R_0)^{\cdot}$. Now, as $g \in \pi_1(H) = G(R_1)$, we have $(g, m) \in H$ for some $m \in G(R_2)$. Then $m \in F(g) = F(1)$ so $(1, m) \in H$ and hence $(g, 1) = (g, m)(1, m) \in H$. This shows $G(R_0) \subset H$.

We will be done if we show $F(t) = F(1)$. Then, if $(t, k) \in H$ we get $(1, k)$ and hence $(t, 1)$ are in $H$. Apply the previous paragraph to get $G(R_1) \times 1 = \{1, t\}G(R_0) \times 1 \subset H$.

So suppose $F(t) \neq F(1)$. We have by (3.1):

$$G(R_2) \neq \bigcup_{k \in F(t)} D_{R_2}\langle 1, -k \rangle.$$

Pick $u_2 \in G(R_2) \setminus \cup D_{R_2}\langle 1, -k \rangle$. Set $u = (-t, u_2)$. Then, as there is no $k$ with $(t, k) \in H$ and $k \in D_{R_2}\langle 1, -k \rangle$, we have $\pi_1(D_R\langle 1, -u \rangle \cap H) = 1$. Pick any $g \in G(R_0)^{\cdot}$ ( we note that $|G(R_0)| > 1$, else $R_1$ is degenerate). Pick any $g_2 \in F(g)$ and set $h = (g, g_2) \in H$. Now:

$$D_R\langle 1, uh, -h \rangle = D_{R_1}\langle 1, -gt, -g \rangle \times D_{R_2}\langle 1, u_2 g_2, -g_2 \rangle$$

$$\supset -gt \times D_{R_2}\langle 1, u_2 g_2, -g_2 \rangle.$$

Also:

$$\bigcup_{w \in h(D_R\langle 1, -u \rangle \cap H)} D_R \langle 1, -w \rangle \subset D_{R_1}\langle 1, -g \rangle \times G(R_2).$$

Hence, by (3.3),

$$-gt \times D_{R_2}\langle 1, u_2 g_2, -g_2 \rangle \subset uH$$
$$-t \times D_{R_2}\langle g_2, u_2, -1 \rangle \subset H.$$

In particular, $(-t, g_2) \in H$. Since $(-1, 1) \in G(R_0) \times 1 \subset H$ , we get $(t, g_2) \in H$. But then $g_2 \in F(t) \cap F(g)$, which equals $F(t) \cap F(1)$ by previous work. $F(t)$ is a coset of $F(1)$, by (5.4), so in fact $F(t) = F(1)$ as desired. $\square$

## 7. The Main Theorem.

**Theorem 7.1.** *Let $R$ be a Witt ring of elementary type. If $S$ is an $H$-extension of $R$, for some subgroup $H \subset G(R)$, then $S$ is also of elementary type.*

*Proof.* We argue by induction on $|G(R)|$. If $|G(R)| \leq 2$ then either $H = 1$ or $H = G(R)$ and we are done by (1.1). Suppose $|G(R)| > 2$. If $R = A \sqcap B$ then $\pi_A$ will denote the projection of $G(R)$ onto $G(A)$.

Now suppose $R$ is degenerate. Write $R = D_k \sqcap R_2$, where $R_2$ is non-degenerate and $G(D_k) \times 1 = \mathrm{rad}(R)$. If some $d \in \mathrm{rad}(R) \cap H$ then write $R = D_1 \sqcap R_3$, where $d$ generates $D_1$. We have $G(D_1) \times 1 \subset H$ and so $H = G(D_1) \times H_3$, for some subgroup $H_3 \subset G(R_3)$. Then (3.2) implies $S = D_1 \sqcap S_3$, where $S_3$ is an $H_3$-extension of $R_3$. By induction, $S_3$ is of elementary type and so $S$ is also. We may thus assume $\mathrm{rad}(R) \cap H = 1$. If $\pi_{D_k}(H) \cap G(D_k) \neq 1$ then by (4.1) we may replace $H$, without affecting $S$, by another subgroup $H_0$ such that $\pi_{D_k}(H_0) \cap G(D_k) = 1$. We assume this has already been done so that $\pi_{D_k}(H) \cap G(D_k) = 1$. We note this also holds trivially if $R$ is non-degenerate and $k = 0$.

Next suppose $R$ has a local type factor $L$ with $|G(L)| \geq 8$. Write $R = L \sqcap R_4$. $D_k$ is a factor of $R_4$. We check the hypothesis of (5.7). Let $a \in G(L)$ and suppose $x \in F(a) \cap \mathrm{rad}(R_4)$. This means $(a, x) \in H$ and so $x \in \pi_{D_k}(H) = 1$. Thus $F(a) \cap \mathrm{rad}(R_4) \subset \{1\}$, as desired. Apply (5.6) to get that either $G(L) \times 1 \subset H$ or $\pi_L(H) = 1$. In the first case, $H = G(L) \times H_4$, for some subgroup $H_4$ of $G(R_4)$. Then (3.2) implies $S = L \sqcap S_4$, where $S_4$ is an $H_4$-extension of $R_4$. By induction, $S_4$ is of elementary type and so $S$ is also. We may thus assume we are in the second case: $\pi_L(H) = 1$.

Now suppose the local factor is $L_1 = \mathbb{Z}$. There are three cases according to (5.9). In case (b) we can write $R = \mathbb{Z} \sqcap R_5$, with $\{\pm 1\} \times 1 \subset H$. Then $H = G(\mathbb{Z}) \times H_5$, for some subgroup $H_5 \subset G(R_5)$. Applying (3.2) again gives $S = \mathbb{Z} \sqcap S_5$, where $S_5$ is an $H_5$-extension of $R_5$. Induction again shows $S$ is of elementary type. In case (c) we can write $R = \mathbb{Z} \sqcap \mathbb{Z} \sqcap R_6$, with $(1, 1) \times G(R_6) \subset H$. Once again (3.2) yields $S = S_0 \sqcap R_6$, where $S_0$ is an $H_0$-extension of $\mathbb{Z} \sqcap \mathbb{Z}$, for some subgroup $H_0$. Since $\mathbb{Z} \sqcap \mathbb{Z} \cong \mathbb{Z}[E_1]$, (1.5) shows $S_0$ is of elementary type (in (1.5)(b) we have $R_0 = \mathbb{Z}$ so that its extension is of elementary type as $|G(R_0)| = 2$). Thus $S$ is also of elementary type. We may thus assume we are in case (a) of (5.9), namely, that $\pi_{L_1}(H) = 1$. This is the same conclusion as when the local factor has at least 8 square classes.

The only local type factors we have omitted are those with 4 square classes and these Witt rings are group rings. We are thus in the following position: $R = Y \sqcap W_1 \sqcap \ldots \sqcap W_n$, where $Y$ is a product of $D_k$ with $k \geq 0$, and local type rings $L$ with $|G(L)| \neq 4$ and each $W_i$ is a non-degenerate group ring. (It is possible that $Y = 1$.) We also have $\pi_Y(H) = 1$, so that if $n = 0$ then $H = 1$ and we are done by (1.1). So suppose $n \geq 1$. Write $W_i = V_i[E_1]$. We first suppose that $\pi_{W_i}(H) \not\subset G(V_i)$ for some $i$. Write $R = W_i \sqcap R_7$. There are two possibilities according to (6.1).

The first possibility is that $1 \times G(R_7) \subset H$. Write $H = H_0 \times G(R_7)$, for some subgroup $H_0$ of $G(W_i)$. Then (3.2) gives that $S = S_0 \sqcap R_7$, for $S_0$, some $H_0$-extension of $W_i$. If $G(R_7) \neq 1$ then we are done by induction. We drop the subscript $i$ and suppose then that $R = W = W_0[E_n]$, where $W_0$ is basic and $n \geq 1$. In cases (a),(c),(d) we have $S$ is of elementary type. In case (b) $S = S_0[E_n]$, where $S_0$ is an $H$-extension of $W_0$, and so agsin $S$ is of elementary type by induction.

The second possibility in (6.1) is that $\pi_{W_i}(H) = G(W_i)$. Then by (6.3) we have $H = G(W_i) \times H_7$, where $H_7$ is a subgroup of $R_7$. Apply (3.2) once again to get that $S = W_i \sqcap S_7$, where $S_7$ is an $H_7$-extension of $R_7$. Induction gives that $S$ is of elementary type. This completes the result when (6.1) applies, that is, when $\pi_{W_i}(H) \not\subset G(V_i)$, for some $i$.

We may thus assume we have $R = Y \sqcap W_1 \sqcap \ldots \sqcap W_n$, with $\pi_Y(H) = 1$ and every $\pi_{W_i}(H) \subset V_i$. Choose $t_i \notin V_i$, for each $i$ and set $g = (1, t_1, \ldots, t_n) \in G(R)$. Then:

$$g \notin \pm \bigcup_{h \in H^{\cdot}} D_R\langle 1, -h \rangle,$$

as for any $h \in H^{\cdot}$ has a coordinate in $G(V_i)^{\cdot}$. By (1.2), $g \notin B(R)$. Thus $R$ is itself a group ring, a case we covered two paragraphs ago.   $\square$

The previous sections can be used to determine the possible $H$-extensions of a given ring $R$. As an example, consider $R = (D_1 \sqcap L_3)[E_2]$. Let $d$ generate $D_1$, $-1, a, b$ generate $L_3$ and $s, t$ generate $E_2$. Thus $G(R) = \mathrm{gp}(d, -1, a, b, s, t)$, where $\mathrm{gp}(A)$ denotes the group generated by $A$. $G(R)$ has 2825 subgroups, 47 of which will yield $H$-extensions. Up to isomorphism, there are exactly 8 $H$-extensions of $R$. Below we list the 8 extensions $S$ along with one choice for the corresponding subgroup $H$.

1. $(D_1 \sqcap L_3)[E_3]$                                                 1

2. $(D_1 \sqcap (D_1 \sqcap L_3)[E_1])[E_1]$                     $\mathrm{gp}(d, -1, a, b, s)$

3. $D_1 \sqcap (D_1 \sqcap L_3)[E_2]$                              $G(R)$

4. $(\mathbb{Z} \sqcap (D_1 \sqcap L_3)[E_1])[E_1]$              $\mathrm{gp}(d, a, b, s)$

5. $\mathbb{Z} \sqcap (D_1 \sqcap L_3)[E_2]$                        $\mathrm{gp}(d, a, b, s, t)$

6. $(D_1 \sqcap L_3[E_1])[E_2]$                                      $\mathrm{gp}(d)$

7. $(D_2 \sqcap L_3)[E_2]$                                              $\mathrm{gp}(d, -1, a, b)$

8. $(D_1[E_1] \sqcap L_3)[E_2]$                                      $\mathrm{gp}(-1, a, b)$

We give a brief sketch of how this list was derived. Begin by running through the cases of (1.5). In (a) $H = 1$ and $S$ is (1) by (1.1). In (c), $G(R_0)$ is a proper subgroup of $H$,

so $S$ is (2) or (3), depending on whether or not $H = G(R)$. In (d) $G(R_0) \subset \pm H, -1 \notin H$ and $H \not\subset G(R_0)$. Thus $H$ looks like a subgroup $K$ of index 2 in $G(R_0)$ that does not contain -1, together with one or more elements from $\{t, s, ts\}$. $S$ is (4) if $|H \cap E_2| = 2$ and (5) if $|H \cap E_2| = 4$. In (1.5)(b) $S = S_0[E_2]$, where $S_0$ is an $H$-extension of $R_0$. Now $R_0 = D_1 \sqcap L_3$. By (4.1) we can assume that either $H = G(D_1) \times H_2$, or that $H = 1 \times H_2$, for some $H_2 \subset G(L_3)$. Now $H_2 = 1$ or $H_2 = G(L_3)$ by (2.3) and (5.7). Since we have already done the case $H = 1$ this gives three choices: $G(D_1) \times 1, G(D_1) \times G(L_3)$ and $1 \times G(L_3)$. The corresponding $S$ is (6), (7) and (8), respectively.

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Carbondale, IL 62901-4408
*E-mail address*: rfitzg@math.siu.edu