Octonions

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Octonions

The simple Lie groups over $\mathbb{C}$ come in four infinite families plus five exceptional groups. All five of the exceptional groups, G_2, F_4 and E_6, E_7, E_8, are constructed from the octonions. That is our goal.

Here an *algebra* is a vector space over $\mathbb{R}$ with a bilinear multiplication (it need not be commutative or associative) and a unity 1. An *involution* on an algebra A is a linear map $^{-}\colon A \to A$ such that

$$\bar{\bar{x}} = x \qquad \text{and} \qquad \overline{xy} = \bar{y}\bar{x}.$$

We further always assume that $x + \bar{x}, x\bar{x} \in \mathbb{R} \cdot 1$. The associated norm is $N(x) = x\bar{x}$. A is a *composition algebra* if $N(xy) = N(x)N(y)$ and N is anisotropic (that is, $N(x) = 0$ implies $x = 0$). For x in an algebra with an involution, we set $Re(x) = \frac{1}{2}(x + \bar{x})$, the real part of x. We call x purely imaginary if $Re(x) = 0$, or equivalently, $\bar{x} = -x$.

1. The Cayley-Dickson Process.

Let A be an algebra with an involution. Set $CD(A) = A + Av$, where v is a new symbol and

$$(a + bv)(c + dv) = (ac - \bar{d}b) + (da + b\bar{c})v.$$

Note that $v^2 = -1$. Define $\overline{a + vb} = \bar{a} - bv$.

We first check that this is an involution. Clearly $\bar{\bar{x}} = x$. For $x = a + bv$ and $y = c + dv$ we have:

$$\begin{aligned}
\bar{y}\bar{x} &= (\bar{c} - dv)(\bar{a} - bv) \\
&= (\bar{c}\bar{a} - (\overline{-b})(-d)) + ((-b)\bar{c} + (-d)\bar{\bar{a}})v \\
&= (\bar{c}\bar{a} - \bar{b}d) - (b\bar{c} + da)v \\
\overline{xy} &= \overline{(ac - \bar{d}b) + (da + b\bar{c})v} \\
&= (\overline{ac} - \overline{\bar{d}b}) - (da + b\bar{c})v \\
&= (\bar{c}\bar{a} - \bar{b}d) - (da + b\bar{c})v.
\end{aligned}$$

Note that $Re(a + bv) = Re(a) \in \mathbb{R}$ and $N(a + bv) = N(a) + N(b) \in \mathbb{R}$.

Start with $A = \mathbb{R}$ and the trivial involution $\bar{x} = x$. Then $CD(\mathbb{R}) = \mathbb{C}$, namely replacing v by i, we have $CD(\mathbb{R}) = \mathbb{R} + i\mathbb{R}$ with $(a + ib)(c + id) = (ac - db) + i(ad + bc)$. The involution on $CD(\mathbb{R})$ is the usual complex conjugation and $N(a + ib) = a^2 + b^2$. Next $CD(\mathbb{C}) = \mathbb{H}$, the quaternions, since if we replace v by j we have:

$$j^2 = -1 \qquad ji = -ij,$$

(taking $a = 0, b = 1, c = i, d = 0$ in the formula). Again we get the usual involution on $\mathbb{H}$ and the norm is the sum of four squares. One can prove $\mathbb{H}$ is associative (because $\mathbb{C}$ is commutative).

The *octonions* are defined to be $\mathbb{O} = CD(\mathbb{H})$. We want to prove the basic properties of $\mathbb{O}$, in particular that it is a composition algebra, and to indicate why the process is not repeated.

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First, we will use ℓ in place of v. Thus $\mathbb{O}$ is 8-dimensional, with a basis of $1, i, j, k = ij, \ell, i\ell, j\ell$, and $k\ell$. The real part is the coefficient of 1 and the norm is the sum of the coefficients squared. Let $a, b \in \mathbb{H}$ and set $x = a, y = b\ell$. Then

$$x(y\ell) = a(b\ell \cdot \ell) = a(-b) = -ab$$
$$(xy)\ell = (a \cdot b\ell)\ell = (ba\ell)\ell = -ba.$$

Thus, since $\mathbb{H}$ is not commutative, $\mathbb{O}$ is not associative.

Lemma 1. *Let A be an algebra with an involution. Let $x = a + vb, y = c + vd$ be elements of $CD(A)$.*

(1) $Re(xy) = Re(yx)$.
(2) $x\bar{x} = N(x) = \bar{x}x$.

Proof. (1) For purely imaginary x, y:

$$\overline{xy} = \bar{y}\bar{x} = (-y)(-x) = yx.$$

So $Re(xy) = Re(\overline{xy}) = Re(yx)$. Still assuming x, y are purely imaginary and letting $e, f \in \mathbb{R}$ gives:

$$\begin{aligned} Re(e + x)(f + y)) &= Re(ef + ey + xf + xy) \\ &= ef + Re(xy) \\ &= fe + Re(yx) \\ &= Re(f + y)(e + x). \end{aligned}$$

(2) $\bar{x}x = (\bar{a} - vb)(a + vb) = \bar{a}a + \bar{b}b = N(x) = x\bar{x}$. □

Lemma 2. *Let A be an algebra with an involution. Then A is associative iff*

$$\bar{x}(xy) = (\bar{x}x)y \qquad \text{and} \qquad (xy)\bar{y} = x(y\bar{y})$$

holds for all $x, y \in CD(A)$.

Proof. Let $x = a + bv$ and $y = c + dv$. Then

$$\begin{aligned} \bar{x}(xy) &= (\bar{a} - bv)[(ac - v\bar{d}b) + (da + b\bar{c})v] \\ &= [\bar{a}(ac - \bar{d}b) - (\overline{da + b\bar{c}})(-b)] + [(da + b\bar{c})\bar{a} - b(\overline{ac - \bar{d}b})]v \\ &= [\bar{a}(ac) - \bar{a}(\bar{d}b) + (\bar{a}\bar{d})b + (c\bar{b})b] + [(da)\bar{a} + (b\bar{c})\bar{a} - b(\bar{c}\bar{a}) + b(\bar{b}d)]v. \end{aligned}$$

Hence if A is associative, $\bar{x}(xy) = N(x)(c + dv) = N(x)y = (\bar{x}x)y$, by Lemma 1 (2). For the second equation, replace x by $\bar{y}$ and y by $\bar{x}$ and conjugate. Conversely, if $\bar{x}(xy) = (\bar{x}x)y$ then $(\bar{a}\bar{d})b = \bar{a}(\bar{d}b)$ and A is associative. □

The *associator* of x, y, z is $[x, y, z] = (xy)z - x(yz)$. An algebra is *alternative* if $[x, y, z]$ is an alternating function. That is, if $\sigma \in S_3$ then we require that

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = sgn(\sigma)[x_1, x_2, x_3],$$

for all $x_1, x_2, x_3 \in A$.

Lemma 3. *Let $x, y, z \in CD(A)$.*

(1) *$[x, y, z]$ changes sign when the involution is applied to one variable.*
(2) *If $CD(A)$ is alternative then $[x, y, z]$ is purely imaginary.*

Proof. (1) Write $x = e + x_0$ with $e = Re(x)$ and x_0 purely imaginary. Then:

$$\begin{aligned}
[x, y, z] &= ((e + x_0)y)z - (e + x_0)(yz) \\
&= (ey)z + (x_0y)z - e(yz) - x_0(yx) \\
&= [x_0, y, z]. \\
[\bar{x}, y, z] &= [-x_0, y, z] = -[x_0, y, z] = -[x, y, z].
\end{aligned}$$

(2) We have:

$$\begin{aligned}
\overline{[x, y, z]} &= \overline{(xy)z - x(yz)} \\
&= \bar{z}(\bar{y}\bar{x}) - (\bar{z}\bar{y})\bar{x} \\
&= -[\bar{z}, \bar{y}, \bar{x}] \\
&= [z, y, x] \qquad \text{by (1)} \\
&= -[x, y, z],
\end{aligned}$$

by the alternative property. Hence $[x, y, z]$ is purely imaginary. $\square$

Theorem 4. *Let A be an algebra with involution. Then $CD(A)$ is alternative iff A is associative. In particular, $\mathbb{O}$ is alternative.*

Proof. We only prove ($\Leftarrow$). Let $x, y, z, w \in CD(A)$. By Lemma 2, $[\bar{x}, x, y] = 0 = [x, y, \bar{y}]$. Lemma 3 (1) gives $[x, x, y] = 0 = [x, y, y]$. Hence:

$$[w + x, w + x, y] = 0 = [x, y + z, y + z].$$

In particular:

$$\begin{aligned}
(w + x)^2y - (w + x)((w + x)y) &= 0 \\
w^2y + (wx)y + (xw)y + x^2y &= w(wy) + w(xy) + x(wy) + x(xy) \\
[w, w, y] + [w, x, y] + [x, w, y] + [x, x, y] &= 0 \\
[w, x, y] &= -[x, w, y].
\end{aligned}$$

Similarly, $[x, y, z] = -[x, z, y]$. Since (1 2) and (2 3) generate S_3, we have that $CD(A)$ is alternative. $\square$

This is why the Cayley-Dickson process is not repeated. $CD(\mathbb{O})$ is an algebra with an involution but it is not alternative, that is, there is no useful version of an associative law.

Corollary 5. *Let $x, y, z \in \mathbb{O}$. The real parts of $(xy)z, (yz)x, (zx)y, x(yz), y(zx)$ and $z(xy)$ are the same.*

Proof. Theorem 4 and Lemma 3 (2) give that the real part of $[x, y, z]$ is zero. Hence $Re(x(yz)) = Re((xy)z)$. Lemma 1 (1) gives $Re(x(yz)) = Re((yz)x)$. Repeat the process: $Re([y, z, x]) = 0$ so $Re((yz)x) = Re(y(zx))$ and so on. $\square$

Corollary 6. $\mathbb{O}$ *is a composition algebra.*

Proof. We have for $x, y \in \mathbb{O}$:

$$\begin{aligned} N(xy) &= Re((\overline{xy})(xy)) \\ &= Re((\bar{y}\bar{x})(xy)) \\ &= Re(\bar{y}(\bar{x}(xy))) && \text{by Corollary 5} \\ &= Re(\bar{y}((\bar{x}x)y)) && \text{by Lemma 2} \\ &= N(x)Re(\bar{y}y) = N(x)N(y). \end{aligned}$$

□

The Cayley-Dickson process will not yield more composition algebras. In fact, there are no others. Hurwitz (1898) proved that the only composition algebras over $\mathbb{R}$ are: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. His proof was via quadratic forms. First, as $N(1) = 1$ and $N(x) = 0$ only for $x = 0$, there is a basis $\{v_i\}$ of a composition algebra A such that

$$N(\sum x_i v_i) = \sum x_i^2.$$

The formula $N(x)N(y) = N(xy)$ then gives a sum of squares identity of the form:

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) = z_1^2 + z_2^2 + \cdots + z_n^2 \quad z_i \in \mathbb{R}[X, Y]$$

Hurwitz proved that there is such an identity iff $n = 1, 2, 4$ or 8.

We can now give the first construction: $G_2 = \mathrm{Aut}(\mathbb{O})$. Since $\mathbb{O}$ is 8-dimensional, each element of G_2 can be viewed as an 8×8 matrix (an 8-dimensional representation). An automorphism will fix the purely imaginary octonions, yielding a 7-dimensional representation.

2. Digression: Arbitrary fields.

Let F be a field, A an algebra over F with an involution, and let $\lambda \in F^*$. Then $CD(A, \lambda) = A + Av$ where

$$(a + bv)(c + dv) = (ac + \lambda d\bar{b}) + (\bar{a}d + cb)v.$$

The previous construction was with $\lambda = -1$.

Then $CD(F, -\lambda_1) = F(\sqrt{-\lambda_1})$ and the norm of $x_1 + x_2 v$ is:

$$N_1(x_1, x_2) = x_1^2 + \lambda_1 x_2^2.$$

$CD(F(\sqrt{-\lambda_1}), -\lambda_2) = (-\lambda_1, -\lambda_2)_F$. This has a basis $1, e_2, e_3, e_4$ with

$$e_2^2 = -\lambda_1 \quad e_3^2 = -\lambda_2 \quad e_2 e_3 = e_4 = -e_3 e_2.$$

The norm of $\sum x_i e_i$ is:

$$N_2(x_1, x_2, x_3, x_4) = N_1(x_1, x_2) + \lambda_2 N_1(x_3, x_4) = x_1^2 + \lambda_1 x_2^2 + \lambda_2 x_3^2 + \lambda_1 \lambda_2 x_4^2.$$

Lastly, $CD((-\lambda_1, -\lambda_2)_F, -\lambda_3)$ is a generalized octonion with norm form

$$N_2(x_1, x_2, x_3, x_4) + \lambda_3 N_2(x_5, x_6, x_7, x_8).$$

Quadratic forms of these shapes are known as 1-fold, 2-fold, 3-fold Pfister forms, respectively. Then Hurwitz's Theorem extends: Let q be a quadratic form in n variables. There is an identity

$$q(x_1, x_2, \ldots, x_n) q(y_1, y_2, \ldots, y_n) = q(z_1, z_2, \ldots, z_n) \quad z_i \in F[X, Y]$$

iff q is equivalent to a k-fold Pfister form for $k = 0, 1, 2$ or 3. In particular, the only composition algebras over F are: F, $F(\sqrt{-\lambda})$ and the generalized quaternions and octonions.

Pfister's connection is by his result (1966): There is such an identity but with $z_i \in F(X, Y)$ iff q is equivalent to a k-fold Pfister form for some k. In particular, n is a power of 2.

3. Jordan algebras.

A *Jordan algebra* is an algebra where the multiplication satisfies

(JA1) $$ab = ba$$

(JA2) $$a^2(ab) = a(a^2 b).$$

Jordan algebras were introduced in 1932 as part of an algebraic formalism for quantum mechanics. They can be thought of as similar to Lie algebras, with (JA1) in place of anti-commuting and (JA2) in place of Jacobi's identity. Further, Lie algebras can be constructed from an associative algebra A by setting $[a, b] = ab - ba$. Jordan algebras can be constructed from the *anticommutator*

$$a \circ b = \tfrac{1}{2}(ab + ba).$$

Then clearly (JA1) holds. Note that $a \circ a = a^2$. For (JA2):

$$\begin{aligned}
a^2 \circ (a \circ b) &= a^2 \circ \tfrac{1}{2}(ab + ba) \\
&= \tfrac{1}{4}[a^2(ab + ba) + (ab + ba)a^2] \\
&= \tfrac{1}{4}(a^3 b + a^2 ba + aba^2 + ba^3) \\
a \circ (a^2 \circ b) &= a \circ \tfrac{1}{2}(a^2 b + ba^2) \\
&= \tfrac{1}{4}[a(a^2 b + ba^2) + (a^2 b + ba^2)a] \\
&= \tfrac{1}{4}(a^3 b + aba^2 + a^2 ba + ba^3) \\
&= a^2 \circ (a \circ b).
\end{aligned}$$

Thus $(A, \circ)$ is a Jordan algebra. Note that $(A, \circ)$ is not associative in general: $a \circ (b \circ b) - (a \circ b) \circ b = (ab - ba)b - b(ab - ba)$, which fails unless A is commutative.

There is one important difference between Lie algebras and Jordan algebras. Every Lie algebra is a Lie subalgebra of an associative algebra with the standard Lie bracket–the Poincare-Birkoff-Witt theorem. However, not every Jordan algebra comes from the anticommutator. Call a Jordan algebra *special* if it is a Jordan subalgebra of an associative algebra with the circle product. Non-special Jordan algebras exist and are called *exceptional*. An example is:

$$\begin{aligned} H_3(\mathbb{O}) &= \{A \in M_3(\mathbb{O}) : \bar{A}^t = A\} \\ &= \left\{ \begin{pmatrix} \lambda_1 & x_2 & x_3 \\ \bar{x_2} & \lambda_2 & x_4 \\ \bar{x_3} & \bar{x_4} & \lambda_3 \end{pmatrix} : \lambda_i \in \mathbb{R}, x_i \in \mathbb{O} \right\}, \end{aligned}$$

$H_3(\mathbb{O})$ is thus 27-dimensional. The operation is again the anticommutator (but not on an associative algebra). The basic facts are:

Theorem 7. *$H_3(\mathbb{O})$ is an exceptional Jordan algebra. Any irreducible, exceptional Jordan algebra (over $\mathbb{R}$) is isomorphic to $H_3(\mathbb{O})$.*

This is exceedingly difficult to prove. $H_3(\mathbb{O})$ was shown to be a Jordan algebra by Jordan-von Neumann-Wigner in 1934. Albert, still in 1934, showed $H_3(\mathbb{O})$ was exceptional. Zelmanov (1978) proved the rest. Here we will show why $H_3(\mathbb{O})$ is a Jordan algebra and why $H_n(\mathbb{O})$, $n \geq 4$, is not. (When $n = 1, 2$ the hermitian matrices do form a Jordan algebra but they are special.)

Lemma 8. *$H_n(\mathbb{O})$ is closed under the anticommutator $\circ$.*

Proof. Let $A = (x_{ij})$ and $B = (y_{ij})$ be in $H_n(\mathbb{O})$. Note $\overline{x_{ij}} = x_{ji}$. Then:

$$\begin{aligned} 2\overline{(A \circ B)_{ij}} &= \sum_{k=1}^{n} \overline{x_{ik}y_{kj} + y_{ik}x_{kj}} \\ &= \sum_{k=1}^{n} \overline{y_{kj}}\,\overline{x_{ik}} + \overline{x_{kj}}\,\overline{y_{ik}} \\ &= \sum_{k=1}^{n} y_{jk}x_{ki} + x_{jk}y_{ki} \\ &= (BA + AB)_{ji} = 2(A \circ B)_{ji}. \end{aligned}$$

Thus $\overline{A \circ B}^t = A \circ B$. $\square$

Proposition 8. *$H_n(\mathbb{O})$, $n \geq 3$ is a Jordan algebra iff $n = 3$.*

Proof. (JA1) clearly holds. (JA2), $a^2(ab) = a(a^2b)$, is linear in b and so suffices to check this on vectors in the obvious basis, namely E_{ii} and $B = yE_{ij} + \bar{y}E_{ji}$, $y \in \mathbb{O}$. We only do the more difficult second case. To save on notation, we take $i = 1, j = 2$. Let $A = (x_{ij}) \in H_n(\mathbb{O})$ and write $A^2 = (z_{ij})$. We want to check if

$$A^2(AB + BA) + (AB + BA)A^2 = A(A^2B + BA^2) + (A^2B + BA^2)A.$$

Now

$$AB+BA=\begin{pmatrix} x_{12}\bar{y} & x_{11}y & 0 & \dots & 0 \\ x_{22}\bar{y} & x_{21}y & 0 & \dots & 0 \\ x_{32}\bar{y} & x_{31}y & 0 & \dots & 0 \\ \vdots & \vdots & 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} yx_{21} & yx_{22} & yx_{23} & \dots \\ \bar{y}x_{11} & \bar{y}x_{12} & \bar{y}x_{13} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$= \begin{pmatrix} x_{12}\bar{y}+yx_{21} & x_{11}y+yx_{22} & yx_{23} & yx_{24} & \dots \\ .x_{22}\bar{y}+\bar{y}x_{11} & x_{21}y+\bar{y}x_{12} & \bar{y}x_{13} & \bar{y}x_{14} & \dots \\ x_{32}\bar{y} & x_{31}y & 0 & 0 & \dots \\ x_{42}\bar{y} & x_{41}y & 0 & 0 & \dots \\ \vdots & \vdots & 0 & 0 & \dots \end{pmatrix}$$

The identity holds at the entries (ij) with $i \le 2$ or $j \le 2$. For $i, j \ge 3$, the entries are:

$$\begin{aligned} [A^2(AB+BA)]_{ij} &= z_{i1}(yx_{2j}) + z_{i2}(\bar{y}x_{1j}) \\ [(AB+BA)A^2]_{ij} &= (x_{i2}\bar{y})z_{1j} + (x_{1i}y)z_{2j} \\ [A(A^2B+BA^2)]_{ij} &= x_{i1}(yz_{2j}) + x_{i2}(\bar{y}z_{1j}) \\ [(A^2B+BA^2)A]_{ij} &= (z_{i2}\bar{y})x_{1j} + (z_{i1}y)x_{2j}. \end{aligned}$$

Subtracting the second two from the sum of the first two gives that (JA2) holds iff

$$\begin{aligned} -[z_{i1}, y, x_{2j}] - [z_{i2}, \bar{y}, x_{1j}] + [x_{i2}, \bar{y}, z_{1j}] + [x_{i1}, y, z_{2j}] &= 0 \\ -[\overline{z_{1i}}, y, x_{2j}] - [\overline{z_{2i}}, \bar{y}, x_{1j}] + [\overline{x_{2i}}, \bar{y}, z_{1j}] + [\overline{x_{1i}}, y, z_{2j}] &= 0 \quad \text{(hermitian)} \\ [z_{1i}, y, x_{2j}] - [z_{2i}, \bar{y}, x_{1j}] + [x_{2i}, y, z_{1j}] - [x_{1i}, y, z_{2j}] &= 0 \quad \text{Lemma 3.} \end{aligned}$$

First suppose that $n = 3$. Then $i = j = 3$. Then $H_3(\mathbb{O})$ is a Jordan algebra iff

$$[z_{13}, y, x_{23}] - [z_{23}, y, x_{13}] + [x_{23}, y, z_{13}] - [x_{13}, y, z_{23}] = 0,$$

which holds by the alternative property of $\mathbb{O}$.

Next suppose $n \ge 4$. Take $i = 3$ and $j = 4$. Then $H_n(\mathbb{O})$ is a Jordan algebra iff:

$$[z_{13}, y, x_{24}] - [z_{23}, y, x_{14}] + [x_{23}, y, z_{14}] - [x_{13}, y, z_{24}] = 0,$$

But take

$$A = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & c & 0 \\ \bar{a} & \bar{c} & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

We have:

$$x_{14} = x_{24} = 0 \quad x_{13} = z_{14} = a \quad x_{23} = z_{24} = c.$$

So (JA2) holds iff $[c, y, a] - [a, y, c] = 0$ iff $2[c, y, a] = 0$ by the alternative property. But if this held for all c, y, a then $\mathbb{O}$ would be associative, which it is not. $\square$

We can give the second construction: $F_4 = \mathrm{Aut}(H_3(\mathbb{O}))$. These can be viewed as 27×27 matrices. As automorphisms fix the matrices of trace 0, there is a 26-dimensional representation.

4. Constructing the exceptional Lie algebras.

A *derivation* of an algebra A is a linear map $D : A \to A$ such that

$$D(xy) = xD(y) + D(x)y,$$

for all $x, y \in A$ (i.e. it satisfies the product rule). For an example, if A is associative and $a \in A$ then $D_a(x) = ax - xa$ is a derivation:

$$\begin{aligned} xD_a(y) + D_a(x)y &= x(ay - ya) + (ax - xa)y \\ &= axy - xya = D_a(xy). \end{aligned}$$

For another example, when A is only assumed to be alternative, define $\ell_a(x) = ax$ and $r_a(x) = xa$. Then

$$D_{a,b} = [\ell_a, \ell_b] + [\ell_a, r_b] + [r_a, r_b]$$

is a derivation (with usual Lie brackets). It is extremely tedious to check this.

Our construction is due to Tits (1966). Let A be one of the composition algebras and let $B = H_3(\mathbb{O})$, the exceptional Jordan algebra. Let A_0 be the elements of A with real part zero. Let B_0 be the elements of B of trace zero. Define new products:

$$\begin{aligned} &\text{for } A_0 \qquad a \star b = ab - Re(ab) \\ &\text{for } B_0 \qquad x \star y = xy - \tfrac{1}{3}\mathrm{Tr}(xy). \end{aligned}$$

Set

$$\mathfrak{L}(A, B) = \mathrm{Der}(A) \oplus (A_0 \otimes B_0) \oplus \mathrm{Der}(B),$$

with the product, for $a, b \in A_0$, $x, y \in B_0$, $D_1 \in \mathrm{Der}(A)$, $D_2 \in \mathrm{Der}(B)$,

$$\begin{aligned} &\text{usual Lie bracket on} \quad \mathrm{Der}(A) \text{ and } \mathrm{Der}(B) \\ &[a \otimes x, D_1 + D_2] = D_1(a) \otimes x + a \otimes D_2(x) \\ &[a \otimes x, b \otimes y] = \tfrac{1}{12}\mathrm{Tr}(xy)D_{a,b} + (a \star b) \otimes (x \star y) + \mathrm{Re}(ab)[r_x, r_y] \end{aligned}$$

Then $\mathfrak{L}(A, B)$ is a Lie algebra. We have:

$$\begin{aligned} \mathfrak{g}_2 &= \mathrm{Der}(\mathbb{O}) \\ \mathfrak{f}_4 &= \mathrm{Der}(H_3(\mathbb{O})) = \mathfrak{L}(\mathbb{R}, H_3(\mathbb{O})) \\ \mathfrak{e}_6 &= \mathfrak{L}(\mathbb{C}, H_3(\mathbb{O})) \\ \mathfrak{e}_7 &= \mathfrak{L}(\mathbb{H}, H_3(\mathbb{O})) \\ \mathfrak{e}_8 &= \mathfrak{L}(\mathbb{O}, H_3(\mathbb{O})). \end{aligned}$$

One can also do this by fixing $A = \mathbb{O}$ and letting B vary. The full table is known as Freudenthal's Magic Square:

$$\mathfrak{L}(A, H_3(B)) =$$

$A\backslash B$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{c}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{a}_2$	$\mathfrak{a}_2 \oplus \mathfrak{a}_2$	$\mathfrak{a}_5$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{c}_3$	$\mathfrak{a}_5$	$\mathfrak{d}_6$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$