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HIGHLY DEGENERATE QUADRATIC FORMS
OVER FINITE FIELDS OF CHARACTERISTIC 2

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Abstract. Let $K/F$ be an extension of finite fields of characteristic two. We consider quadratic forms written as the trace of $xR(x)$, where $R(x)$ is a linearized polynomial. We show all quadratic forms can be so written, in an essentially unique way. We classify those $R$, with coefficients 0 or 1, where the form has a codimension 2 radical. This is applied to maximal Artin-Schreier curves and factorizations of linearized polynomials.

Let $q$ be a 2-power, $q = 2^t$. Set $F = GF(q)$ and let $K = GF(q^k)$ be an extension. Let

$$R(x) = \sum_{j=0}^{h} \epsilon_j x^{q^j},$$

with each $\epsilon_j \in K$. We consider the quadratic forms $Q^K_R : K \to F$ given by $Q^K_R(x) = \text{tr}_{K/F}(xR(x))$.

These trace forms have appeared in a variety of contexts. They have been used to compute weight enumerators of certain binary codes [1,2], to construct curves with many rational points and the associated trace codes [8], as part of an authentication scheme [3], and to construct certain binary sequences in [5] and [4].

In each of these applications one wants the number of solutions (in $K$) to $Q^K_R(x) = 0$, denoted by $N(Q^K_R)$. This is easily worked out (see [7, 6.26,6.32]) in terms of the standard classification of quadratic forms:

$$N(Q^K_R) = \frac{1}{q}(q^k + \Lambda(Q^K_R)(q-1)\sqrt{q^{k+w}}).$$

where $w$ is the dimension of the radical, $v = (k-w)/2$ and

$$\Lambda(Q^K_R) = \begin{cases} 
0, & \text{if } Q^K_R \simeq z^2 + \sum_{i=1}^{w} x_i y_i \\
1, & \text{if } Q^K_R \simeq \sum_{i=1}^{v} x_i y_i \\
-1, & \text{if } Q^K_R \simeq x_1^2 + sy_1^2 + \sum_{i=1}^{w} x_i y_i.
\end{cases}$$

Here $s$ is any element of $F$ with $\text{tr}_{F/GF(2)}(s) = 1$.

Key words and phrases. quadratic form, trace, linearized polynomial.
However, there is no simple way to determine the dimension of the radical or the invariant $\Lambda$. The one general result is due to Klapper [6] which only covers the case when $R$ consists of a single term. In roughly half the applications ([1,2,8]) one wants highly degenerate forms, which give large $N(Q_R^K)$ when $\Lambda = 1$. We restrict to those $R$ with all coefficients $\epsilon_i \in GF(2)$ as is the case in each of the cited papers except [8]. Our main result is to determine all such $R$, and all extensions $K$, such that the radical of $Q_K^R$ has codimension (namely $2v$) at most 2. We compute the invariant $\Lambda$ in each case.

We first show that every quadratic form $Q : K \to F$ can be written as $Q_R^K$ in an essentially unique way. Thus our result is more general than it appears. We apply our main result to obtain a classification of those $R$ such that the number of points on the Artin-Schreier curve $y^q + y = xR(x)$ equals the Hasse-Weil bound. We also obtain results on the factors of self-reciprocal linearized polynomials.

1. Quadratic forms.

A quadratic form $Q : K \to F$ is a map such that

1. $Q(ax) = a^2Q(x)$ for all $a \in F$ and $x \in K$, and
2. $B(x, y) := Q(x + y) + Q(x) + Q(y)$ is a bilinear map $K \times K \to F$.

The radical of $Q$ is

$$rad Q = \{x \in K : B(x, y) = 0 \text{ for all } y \in K\}.$$  

The codimension of the radical, $k - \dim rad Q$ is always even.

To simplify notation, we write simply tr for $tr_{K/F}$. We will write $Tr_K$ for the absolute trace $tr_{K/GF(2)}$.

**Proposition 1.1.** Let $Q : K \to F$ be a quadratic form. Let $m = \lfloor k/2 \rfloor$. Let $h = \frac{1}{2} codim rad Q$.

1. There exist $c, a_1, b_1, \ldots, a_h, b_h \in K$, independent over $F$, such that

$$Q(x) = \begin{cases} tr(cx)^2 + \sum_{i=1}^{h} tr(a_i x) tr(b_i x), & \text{if } \Lambda(Q) = 0 \\ \sum_{i=1}^{h} tr(a_i x) tr(b_i x), & \text{if } \Lambda(Q) = 1 \\ tr(a_1 x)^2 + tr(b_1 x)^2 + \sum_{i=1}^{h} tr(a_i x) tr(b_i x), & \text{if } \Lambda(Q) = -1. \end{cases}$$

2. There exist $\epsilon_0, \epsilon_1, \ldots, \epsilon_m \in K$ such that

$$Q(x) = tr\left( x \cdot \sum_{i=0}^{m} \epsilon_i x^{q^i} \right).$$

**Proof.** (1) Suppose $\Lambda(Q) = 1$. Pick a basis of $K$ over $F$ and let $M$ be the matrix of $Q$ with respect to this basis. We apply the classification of quadratic forms. Let $H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and let $N$ be the $k \times k$ matrix with $h$ copies of $H$ on the diagonal and the rest zero. Then
there exists an invertible \( k \times k \) matrix \( P \) over \( F \) such that
\[
M = P^t N P
\]
\[
Q(X) = X^t P^t N P X
\]
\[
Q(X) = \sum_{i=1}^{h} (r_{2i-1}X)(r_{2i}X),
\]
where \( r_j \) is the \( j \)th row of \( P \). As map from \( K \rightarrow F \), rather than from \( F^k \rightarrow F \), each \( r_jX \) is linear and so equal to \( \text{tr}(d_jx) \) for some \( d_j \in K \). The rows of \( P \) are independent over \( F \) so the \( d_j \) are also. This gives the desired representation of \( Q \).

The cases when \( \Lambda(Q) = 0 \) or \(-1\) are similar except the first copy of \( H \) in \( N \) is replaced by
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\text{ or } \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix},
\]
respectively.

(2) This proof is taken from [9,3.2, 5.1]. Our only change is to correct a slight error in the \( i = 1 \) term and to include the cases of \( \Lambda(Q) = 0, -1 \).

\[
\text{tr}(ax)\text{tr}(bx) = \text{tr}(\text{tr}(ax)bx)
\]
\[
= \text{tr}\left(\sum_{i=0}^{k-1} (ax)^i (bx)\right).
\]

Now
\[
\text{tr}(a^{q^i}b) = \text{tr}(ab^{q^{k-i}})
\]
so that
\[
\text{tr}(ax)\text{tr}(bx) = \begin{cases}
\text{tr}(abx^2 + \sum_{i=1}^{m} (a^{q^i}b + ab^{q^i})x^{q^i}), & \text{if } k = 2m + 1 \text{ is odd} \\
\text{tr}(abx^2 + \sum_{i=1}^{m} (a^{q^i}b + ab^{q^i})x^{q^i} + a^{q^m}bx^{q^m}), & \text{if } k = 2m \text{ is even}
\end{cases}
\]
In either case,
\[
\text{tr}(ax)\text{tr}(bx) = \text{tr}\left(x \cdot \sum_{j=0}^{m} \epsilon_j x^{q^i+1}\right),
\]
for some \( \epsilon_j \in K \). Lastly,
\[
\text{tr}(cx)^2 = \text{tr}((cx)^2) = \text{tr}(x \cdot c^2x).
\]
Thus (1) implies (2). \( \square \)

The first representation of (1.1) is not unique. For instance,
\[
\text{tr}(ax)^2 + \text{tr}(ax)\text{tr}(bx) + \text{tr}(bx)^2 = \text{tr}(ax)^2 + \text{tr}(ax)\text{tr}((a + b)x) + \text{tr}((a + b)x)^2.
\]
However, for the second representation we have:
Theorem 1.2. Let $Q : K \rightarrow F$ be a quadratic form and let $m = \lfloor k/2 \rfloor$. Then there exist unique $\epsilon_i \in K$, $0 \leq i \leq m$, such that

$$Q(x) = \text{tr} \left( x \cdot \sum_{i=0}^{m} \epsilon_i x^{q^i} \right),$$

except when $k$ is even in which case $\epsilon_m$ is only unique modulo $GF(q^m)$.

Proof. We count. If we fix a basis of $K$ over $F$ then each quadratic form is represented uniquely by an upper triangular matrix. Hence there are $q^k(m+1)/2$ many quadratic forms.

Suppose $k = 2m + 1$. The number of $R(x) = \sum_{i=0}^{m} \epsilon_i x^{q^i}$ is $(q^k)^{m+1} = q^{k(k+1)/2}$. Since this is the number of quadratic forms, (1.1) implies the representation $Q(x) = \text{tr}(xR(x))$ is unique.

Suppose $k = 2m$. Note that

$$(x^{q^m+1})^m = x^{q^m+1} \in GF(2^m)$$

for all $x \in K$. Thus if $\epsilon \in GF(q^m)$ then $\text{tr}(x \cdot \epsilon x^{q^m}) = 0$ for all $x \in K$. The number of $R(x) = \sum_{i=0}^{m} \epsilon_i x^{q^i}$, with $\epsilon_0, \ldots, \epsilon_{m-1} \in K$ and $\epsilon_m \in K/GF(q^m)$ is

$$(q^k)^m \cdot (q^m) = q^{m^2+m} = q^{k(k+1)/2}.$$ 

As before, this shows the representation of $Q(x)$ as $\text{tr}(xR(x))$ is unique (taking $\epsilon_m$ modulo $GF(q^m)$). \(\square\)

Throughout the remainder of the paper we assume

$$R(x) = \sum_{j=0}^{h} \epsilon_j x^{q^j} \quad \text{with } \epsilon_j \in GF(2),$$

where $h = \lfloor (k-1)/2 \rfloor$. Here we have dropped the $\epsilon_m$ term when $k = 2m$ as $\epsilon_m = 0$ or 1, both of which are in $GF(2^m)$.

Corollary 1.3. Let $R = \sum_{i=0}^{h} \epsilon_i x^{q^i}$, where each $\epsilon_i \in GF(2)$ and $h = \lfloor (k-1)/2 \rfloor$. Then $Q^K_R$ has radical of codimension 2 iff there exist independent $a, b, c \in K$ such that

(Ei) \[ a^{q^i} b + ab^{q^i} = \epsilon_i \quad \text{for } 1 \leq i \leq h \]

and

(E0) \[ \epsilon_0 = \begin{cases} 
 c^2 + ab, & \text{if } \Lambda(Q^K_R) = 0 \\
 ab, & \text{if } \Lambda(Q^K_R) = 1 \\
 a^2 + ab + sb^2, & \text{if } \Lambda(Q^K_R) = -1,
\end{cases} \]

plus, if $k = 2m$,

(Em) \[ a^{q^m} b \in GF(q^m). \]
Again here $s \in F$ is an element with $\text{Tr}_F(s) = 1$.

**Proof.** We have by (1.1)(1) that the quadratic forms with radical of codimension 2 are

$$\text{tr}(cx)^2 + \text{tr}(ax)\text{tr}(bx) \quad \text{tr}(ax)\text{tr}(bx) \quad \text{tr}(ax)^2 + \text{tr}(ax)\text{tr}(bx) + \text{str}(bx)^2,$$

where the invariants are $0$, $1$, $-1$ respectively (see [9, 3.1]). The computation of $\text{tr}(ax)\text{tr}(bx)$ in (1.1)(2) gives the equations (Ei) for $1 \leq i \leq h$. (Em) follows as $\text{tr}(a^m bx^{m+1}) = 0$ iff $a^m b \in GF(q^m)$, by (1.2). And $\text{tr}(cx)^2 = \text{tr}((cx)^2) = \text{tr}(c^2 x \cdot x)$ yields the three forms of (E0). □

2. The Main Theorem.

We begin with three lemmas needed to solve the equations (Ei).

**Lemma 2.1.** Suppose $y^2 = y + z$. Then

$$y^{2^i} = y + z + z^2 + z^4 + \cdots + z^{2^{i-1}}.$$

**Proof.** Induction. □

The following identity is well-known and may be derived in many ways. For instance, one may take Waring’s identity, expressing the sum of two $n$th powers in terms of a Dickson polynomial, modulo 2. We use instead a simple induction argument.

**Lemma 2.2.** Let $u = x + y$ and $v = xy$. Then

$$x^{2^n+1} + y^{2^n+1} = u^{2^n+1} + \sum_{i=0}^{n-1} u^{2^{n+1-2^{i-1}}} v^{2^i}.$$

**Proof.** By induction,

$$x^{2^n+1} + y^{2^n+1} = (x^{2^n} + y^{2^n})(x^{2^n} + y^{2^n+1}) = x^{2^n} y^{2^n+1} + x^{2^n+1} y^{2^n} = u^{2^n} + \sum_{i=0}^{n-1} u^{2^n+1-2^{i-1}} v^{2^i} + u v^{2^n} = u^{2^{n+1}+1} + \sum_{i=0}^{n} u^{2^{n+1}+1-2^{i-1}} v^{2^i},$$

as desired. □

The following highly technical lemma is need to compute the invariant $\Lambda$ in one case.
Lemma 2.3. Let $v = 2^{3^r}$ and let
\[ g_v(x) = x^{v+1}(1 + x^{-2} + x^{-4} + \cdots + x^{-v}) + 1. \]

Let $\delta$ be a root of $g_v(x)$ in some extension of $F$. Then
\begin{align*}
(1) & \quad \delta \in GF(v^3) \setminus GF(v) \\
(2) & \quad \delta^{2^{v^2}} + \delta^{v^2+1} + \delta^2 = 1 \\
(3) & \quad \delta^{v^2+1} + \delta^{2^v} = 1 \\
(4) & \quad \delta^2 + \delta^{v^2+v} = 1.
\end{align*}

Proof. We have
\[
1 + \delta^{-2} + \delta^{-4} + \cdots + \delta^{-v} = \delta^{-(v+1)}.
\]
Add this to its square to get
\[
\delta^{-2} + \delta^{-2v} = \delta^{-(v+1)} + \delta^{-2(v+1)}.
\]
Multiply by $\delta^{2(v+1)}$ to get (2).

Re-write (2) by dividing by $\delta^2$
\[
(5) \quad \delta^{2(v-1)} + \delta^{v-1} + (1 + \delta^{-2}) = 0.
\]
This has the form $y^2 + y + z = 0$ with $y = \delta^{v-1}$ and $z = 1 + \delta^{-2}$. By (1.4)
\[
\delta^{(v-1)v} = \delta^{v-1} + z + z^2 + \cdots + z^{v/2}.
\]
As $v$ is an odd power of 2 there are an odd number of $z^i$ terms. So
\[
\delta^{(v-1)v} = \delta^{v-1} + 1 + \delta^{-2} + \delta^{-4} + \cdots + \delta^{-v}
\]
\[
= \delta^{v-1} + \delta^{-(v+1)},
\]
by the original equation. Then
\[
\delta^{v^2+1} + \delta^{2^v} = \delta^{v+1}(\delta^{v^2-v} + \delta^{v-1}) = \delta^{2^v}\delta^{-2v} = 1,
\]
giving (3).

Now multiply (5) by $\delta^{v-1}$ to get
\[
\delta^{3(v-1)} = \delta^{2(v-1)} + \delta^{v-1} + \delta^{v-3}
\]
\[
= 1 + \delta^{-2} + \delta^{v-3},
\]
using (5). Multiply by $\delta^{v+3}$ to get $\delta^{4v} = \delta^{v+3} + \delta^{v+1} + \delta^{2v}$. Apply (2), divide by $\delta^2$ and apply (2) again:
\begin{align*}
\delta^{4v} + \delta^{v+3} & = \delta^2 + 1 \\
\delta^{4v-2} + \delta^{v+1} & = 1 + \delta^{-2} \\
\delta^{4v-2} + \delta^{-2} & = \delta^{2v} + \delta^2.
\end{align*}
Next divide (3) by $\delta$

(7)  \hspace{1cm} \delta^{v^2} + \delta^{2v-1} = \delta^{-1}.

Square (7) and apply (6)

\[
\delta^{2v^2} + \delta^{4v-2} = \delta^{-2}
\]

(8)  \hspace{1cm} \delta^{2v^2} = \delta^{2v} + \delta^2.

Now raise (7) to the $v$th power

\[
\delta^{v^3} = \delta^{2v^2-v} + \delta^{-v} = \delta^{-v}(\delta^{2v^2} + 1) = \delta^{-v}(\delta^{2v} + \delta^2 + 1) \quad \text{by (8)}
\]

\[
= \delta^{-v}\delta^{v+1} = \delta,
\]

using (2). Hence $\delta \in GF(v^3)$, giving (1). If $\delta \in GF(v)$ then $\delta^{v^2+1} = \delta^2$ and $\delta^{2v} = \delta^2$ also which contradicts (3). Thus $\delta \notin GF(v)$.

Lastly, re-write (3)

\[
1 = \delta^{v^2+1} + \delta^{2v} = \delta^{v^3+v^2} + \delta^{2v} = (\delta^{v^2+v} + \delta^2)^v.
\]

Hence $\delta^{v^2+v} + \delta^2 = 1$, giving (4). \qed

Set

\[
Ad(x) = \sum_{\substack{j=1 \atop d|j}}^{h} x^{q^j}
\]

**Theorem 2.4.** Let $R = \sum_{i=0}^{h} \varepsilon_i x^{q^i}$, where each $\varepsilon_i \in GF(2)$ and $h = \lfloor (k - 1)/2 \rfloor$. Then $Q_K^R$ has radical of codimension 2 iff

- (1) $3|k$ and $R = A_3$ or $x + A_3$, or
- (2) $4|k$ and $R = A_2$ or $x + A_2$.

The classification in these cases (assuming the restriction on $k$) is

\[
\Lambda(Q_{A_2}^K) = -1
\]

\[
\Lambda(Q_{x+A_2}^K) = \begin{cases} 1, & \text{if } t \text{ is odd} \\ -1, & \text{if } t \text{ is even} \end{cases}
\]

\[
\Lambda(Q_{A_3}^K) = 0
\]

\[
\lambda(Q_{x+A_3}^K) = \begin{cases} 1, & \text{if } t \text{ is even} \\ -1, & \text{if } t \text{ is odd} \end{cases}
\]
Recall that $q = 2^t$.

Proof. In the first half of the proof we find all extensions $K$, all independent $a, b, c \in K$, and all $\epsilon_i, i \geq 1$, that satisfy the equations (Ei), for $i \geq 1$, and (Em). We will see that $R$ must be $A2, x + A2, A3$ or $x + A3$ with the desired restrictions on $k$.

Set $u = a^{q-1} + b^{q-1}$ and $v = ab$. Then (E1) is $uv = \epsilon_1$. If $\epsilon_1 = 0$ then either $a = 0$, $b = 0$ or $a^{q-1} = b^{q-1}$ (and so $a = \lambda b$ for some $\lambda \in F$), contradicting the independence of $a, b$ over $F$. Thus $\epsilon_1 = 1$ and $u = 1/v$.

Now (E2) is

$$ab((a^{q-1})^{q+1} + (b^{q-1})^{q+1}) = \epsilon_2$$

$$v \left[ u^{q+1} + \sum_{i=0}^{t-1} u^{q+1-2^i} (v^{q-1})^{2^i} \right] = \epsilon_2,$$

using (2.2). Replacing $u$ by $1/v$ and multiplying by $v^q$ yields

$$(2.5) \sum_{i=0}^{t-1} v^{2^i(q+1)} = \epsilon_2 v^q.$$

We first treat the case of $\epsilon_2 = 0$. Set $w = v^{q+1}$. Then, by (2.5), $w^{q/2} + \cdots + w + 1 = 0$. Hence $w^{q/2} = w, w \in F$ and $Tr_F(w) = 1$.

Now the $(q + 1)$st roots of $w \in F$ lie in $L = GF(q^2)$ since if $z$ generates $GF(q^2)^*$ then $z^{q+1}$ generates $GF(q)^*$. Thus $v = ab \in L$. As $\epsilon_2 = 0$ we have $(a/b)^{q^2 - 1} = 1$ and so $a/b \in L$ also. Thus $a, b \in L$. Now if $a, b \in F$ then they are dependent over $F$. Hence at least one of $a, b$ is in $L \setminus F$. Say $a \in L \setminus F$. So if $a \in K$ then $2|k$.

By construction, $\epsilon_1 = 1$ and $\epsilon_2 = 0$. As $a \in L$ we have

$$a^{q^i} = \begin{cases} a, & \text{if } i \text{ is even} \\ a^q, & \text{if } i \text{ is odd} \end{cases}$$

and similarly for $b$. Hence for $i \geq 3$

$$\epsilon_i = a^{q^i} b + ab^{q^i} = \begin{cases} \epsilon_1, & \text{if } i \text{ is odd} \\ \epsilon_2, & \text{if } i \text{ is even} \end{cases}$$

Thus $R = A2$ or $x + A2$.

Lastly, we know $k = 2m$ is even so we check (Em). If $m$ is odd then

$$a^{q^m} b = a^q b = a^{q-1} v = a^{q-1}/u \in L \setminus F,$$

so that $a^{q^m} b \notin GF(q^m)$. And if $m$ is even then $a^{q^m} b = ab \in GF(q^2) \subset GF(q^m)$. Thus to have a solution in $K$ we require that $m$ be even, that is, that $4|k$.

We now treat the case of $\epsilon_2 = 1$. From (2.5) we have:

$$v^{(q+1)q/2} + v^{(q+1)q/4} + \cdots + v^{q+1} + 1 = v^q.$$
Squaring this gives

\[ v^{(q+1)q} = v^{(q+1)q/2} + \ldots + v^{(q+1)^2} + 1 + v^{2q} \]

\[ = v^{q+1} + v^q + v^{2q}, \]

by (2.5). Divide this by \( v^q \) and then raise to the \( q \)th power:

\[ (2.6) \]

\[ v^{q^2} = 1 + v + v^q \]

\[ v^{q^3} = 1 + v^q + v^{q^2} = v. \]

Thus \( v \in E \equiv GF(q^3) \) and \( \text{tr}_{E/F}(v) = 1 \).

Now \( va^{q-1} = a^q b \) and \( vb^{q-1} = ab^q \) sum to 1 and their product is \( a^{q+1}b^{q+1} = v^{q+1} \). Thus \( va^{q-1} \) and \( vb^{q-1} \) are roots of \( y^2 + y + v^{q+1} \in E[y] \). Now

\[ \text{Tr}_E(v^{q+1}) = \sum_{i=0}^{t-1} v^{2^i(q+1)} + \sum_{i=0}^{t-1} v^{2^i(q+1)q} + \sum_{i=0}^{t-1} v^{2^i(q+1)q^2} \]

\[ = (1 + v^q) + (1 + v^q)^q + (1 + v^q)^{q^2} \quad \text{by (2.5)} \]

\[ = 1 + v^q + v^{q^2} + v^{q^3} = 0, \]

by (2.7). Thus \( y^2 + y + v^{q+1} \) has its roots in \( E \), by [7, 3.79]. So \( a^{q-1} \) and \( b^{q-1} \) are in \( E \).

Next, by (2.1)

\[ y^q = y + v^{q+1} + v^{2(q+1)} + \ldots + v^{(q+1)q/2} \]

\[ y^q = y + 1 + v^q \quad \text{by (2.5)} \]

\[ y^{q^2} = y + v^q + v^{q^2} \]

\[ y^{q^2+q} = y^2 + y^q(1 + v^{q^2}) + v^q + v^{q^2} + v^{2q} + v^{q^2+q} \]

\[ = yv^{q^2} + v^{q^2} \quad \text{by (2.7)} \]

\[ y^{q^2+q+1} = yv^{q^2} + v^{q^2+q+1} + yv^{q^2} = v^{q^2+q+1}. \]

Hence, dividing by \( v^{q^2+q+1} \) yields \( a^{q^3-1} = 1 = b^{q^3-1} \). Thus \( a, b \in E \). In particular,

\[ \epsilon_3 = a^{q^3}b + ab^{q^3} = ab + ab = 0. \]

By construction \( \epsilon_1 = 1 = \epsilon_2 \). And

\[ a^{q^i} = \begin{cases} 
  a, & \text{if } i \equiv 0 \pmod{3} \\
  a^q, & \text{if } i \equiv 1 \pmod{3} \\
  a^{q^2}, & \text{if } i \equiv 2 \pmod{3}.
\end{cases} \]

Thus for \( i \geq 3 \), \( \epsilon_i = \epsilon_j \) where \( j \in \{1, 2, 3\} \) and \( i \equiv j \pmod{3} \). Hence \( R = A_3 \) or \( x + A_3 \).
Again, if \( a, b \in F \) then they are dependent over \( F \). Hence at least one of \( a, b \) is in \( E \setminus F \). Say \( a \in E \setminus F \). So if \( a \in K \) then \( 3 \mid k \). Finally, if \( k = 2m \) is even we must check (Em). But \( a^{q^m} b \in E = GF(q^3) \subset GF(q^m) \), as \( 3 \mid k \), so (Em) is satisfied. This completes the first half of the proof.

In the second half of the proof we show that each of \( R = A2, x + A2, A3 \) and \( x + A3 \) does give a quadratic form with radical of codimension 2 (assuming the restrictions on \( k \)) and compute their invariants. We do this by finding explicit solutions to the equations (Ei). There are six cases.

First consider \( Q_{A2}^K \) when \( 4 \mid k \). Fix an \( s \in F \) with \( \text{Tr}_F(s) = 1 \). Then \( y^2 + y + s \in F[y] \) is irreducible. Let \( \alpha \in GF(q^2) \subset K \) be a root. Let \( \beta \) be a primitive element of \( GF(q^2) \). Set \( b = \beta^{q-1} \) and \( a = \alpha b \). These are independent over \( F \) as \( \alpha \notin F \). We compute

\[
\begin{align*}
\text{(E0)} & \quad a^2 + ab + sb^2 = b^2((a/b)^2 + (a/b) + s) = 0 \\
\text{(E1)} & \quad a^q b + ab^q = (\alpha^q + \alpha)b^{q+1} = \text{Tr}_F(s)\delta^{q+1} = 1 \\
\text{(E2)} & \quad a^{q^2} b + ab^{q^2} = ab + ab = 0.
\end{align*}
\]

Also \( ab \in GF(q^2) \) implies \( \epsilon_{i+2} = \epsilon_i \) for \( i \geq 1 \). If \( k = 2m \) then \( a^{q^m} b \in GF(q^2) \subset GF(q^m) \) as \( m \) is even, so that (Em) is satisfied. Hence

\[
\text{tr}(ax)^2 + \text{tr}(ax)\text{tr}(bx) + \text{str}(bx)^2 = Q_{A2}^K(x).
\]

By (1.3) \( Q_{A2}^K \) has radical with codimension 2 and invariant \(-1\).

Next consider \( Q_{x+A2}^K \) when \( 4 \mid k \) and \( q = 2^t \) with \( t \) odd. Let \( \beta \in GF(q^2) \subset K \) be primitive. As \( t \) is odd, \( 3 \mid (q+1) \). Set

\[
a = \beta^{(q-2)(q+1)/3} \quad b = \beta^{2(q+1)/3}.
\]

Note that \( a \) and \( b \) are independent over \( F \) as \( (b/a)^{q-1} = \beta^{(q-2)/3} \neq 1 \) so that \( b/a \notin F \). We compute

\[
\begin{align*}
\text{(E0)} & \quad ab = \beta^{(q-2)(q+1)}b^{q+1} = \beta^{q-1} = 1 \\
\text{(E1)} & \quad a^q b + ab^q = a^{q-1} + b^{q-1} = \beta^{(q-2)/3} + \beta^{2(q-2)/3} = 1 \\
\text{(E2)} & \quad a^{q^2} b + ab^{q^2} = ab + ab = 0.
\end{align*}
\]

As in the previous case \( \epsilon_{i+2} = \epsilon_i \) for \( i \geq 1 \) and (Em) is satisfied. Hence \( \text{tr}(ax)\text{tr}(bx) = Q_{x+A2}^K(x) \) is, by (1.3), a form of codimension 2 radical and invariant 1.

Next consider \( Q_{x+A2}^K \) when \( 4 \mid k \) and \( t \) even. Fix \( s \in F \) with \( \text{Tr}_F(s) = 1 \). Then \( \text{Tr}_F(s + 1) = \text{Tr}_F(s) = 1 \) as \( t \) is even. Thus \( x^2 + x + s + 1 \) is irreducible over \( F \). Let \( \alpha \in GF(q^2) \subset K \) be a root. Set \( a = \alpha \) and \( b = 1 \); they are independent over \( F \) as \( \alpha \notin F \). Then

\[
\begin{align*}
\text{(E0)} & \quad a^2 + ab + sb^2 = a^2 + \alpha + s = 1 \\
\text{(E1)} & \quad a^q b + ab^q = a^q + \alpha = \text{Tr}_F(s + 1) = 1 \\
\text{(E2)} & \quad a^{q^2} b + ab^{q^2} = ab + ab = 0.
\end{align*}
\]
Again $\epsilon_{i+2} = \epsilon_i$ for $i \geq 1$ and (Em) is satisfied. Hence $\text{tr}(ax)^2 + \text{tr}(ax)\text{tr}(bx) + \text{str}(bx)^2 = Q^K_{x+1}(x)$ is, by (1.3), a form of codimension 2 radical and invariant $-1$.

We now consider $Q^K_{43}$ when $3\nmid k$. Let $3^r$ be the highest power of 3 dividing $t$ so that $t = 3^rt_0$ with $(3, t_0) = 1$. Set $q = 3^r$ so that $q = v^{t_0}$. Let $\delta$ be a root of the polynomial of (2.3). Then $\delta \in GF(v^3) \subset GF(q^3) \subset K$ by (2.3)(1). Set $a = \delta^v$, $b = \delta$ and $c = \delta^v + \delta + 1$. We first check that $a, b, c$ are independent over $F = GF(q)$. If not then 1 is in the $F$-span of $a$ and $b$. Hence $a = gb + h$ for some $g, h \in F$ and so $\delta^v = g\delta + h$. We plug into (2.3)(2):

$$
\delta^{2v} + \delta^{v+1} + \delta^2 = 1
$$

(2.8)

$$
h^2 + h\delta + (1 + g + g^2)\delta^2 = 1.
$$

Now $\delta \notin GF(v)$, by (2.3)(1), and so has degree 3 over $GF(v)$. As $(3, t_0) = 1$, $\delta$ has degree 3 over $F = GF(v^{t_0})$ as well. Thus $1, \delta, \delta^2$ are independent over $F$. Then (2.8) gives $h^2 = 1$ and $h = 0$, a contradiction. Thus $a, b, c$ are independent over $F$.

We compute (E0)

$$
c^2 + ab = 1 + \delta^2 + \delta^{2v} + \delta^{v+1} = 0 \quad \text{by (2.3)(2)}.
$$

For the other equations, first suppose $t_0 \equiv 1 \pmod{3}$. Then $\delta^q = \delta^{v^{t_0}} = \delta^v$ as $\delta^{v^3} = \delta$. Similarly, $\delta^{v^2} = \delta^{v^{2t_0}} = \delta^{v^2}$. Then

(E1) \hspace{1cm} a^q b + ab^q = \delta^{v^2 + 1} + \delta^{2v} = 1 \quad \text{by (2.3)(3)}

(E2) \hspace{1cm} a^q^2 b + ab^q^2 = \delta^{v^3 + 1} + \delta^{v^2 + v} = \delta^2 + \delta^{v^2 + v} = 1 \quad \text{by (2.3)(4)}

(E3) \hspace{1cm} a^q^3 b + ab^q^3 = ab + ab = 0.

When $t_0 = 2 \pmod{3}$ then $\delta^q = \delta^{v^2}$ and $\delta^{v^2} = \delta^v$. Then

(E1) \hspace{1cm} a^q b + ab^q = \delta^{v^3 + 1} + \delta^{v^2 + v} = 1

(E2) \hspace{1cm} a^q^2 b + ab^q^2 = \delta^{v^2 + 1} + \delta^{2v} = 1

(E3) \hspace{1cm} a^q^3 b + ab^q^3 = ab + ab = 0.

Also $\epsilon_{i+3} = \epsilon_i$ for $i \geq 1$ and if $k = 2m$ then $a^q^m b \in GF(q^3) \subset GF(q^m)$ so that (Em) holds. Hence $Q^K_{43}$ has radical of codimension 2 and invariant 0.

Next consider $Q^K_{x+4}$ when $3 | k$ and $t$ is odd. Since $t$ is odd we can pick $s = 1$ as our element of $F$ with absolute trace 1. Let $v, \delta, a$ and $b$ be as in the previous case. We know $a, b$ are independent over $F$ and $\epsilon_1 = 1 = \epsilon_2$, $\epsilon_3 = 0$, $\epsilon_{i+3} = \epsilon_i$ for $i \geq 1$ and that (Em) holds. We need only check (E0):

$$
a^2 + ab + b^2 = \delta^{2v} + \delta^{v+1} + \delta^2 = 1
$$

by (2.3)(2). Hence $Q^K_{x+4}$ has radical of codimension 2 and invariant $-1$. 

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Lastly, we consider $Q_{x+A3}$ when $3|k$ and $t$ is even. Then $GF(q^3) \subset K$; let $\gamma$ be a primitive element of $GF(q^3)$. As $t$ is even, $3$ divides $q^2 + q + 1$. Set $\varphi = \gamma^{(q^2+q+1)/3}$. Then $\varphi$ has order $3(q-1)$ so that $\varphi^{2(q-1)} + \varphi^{-1} + 1 = 0$. Set $a = \varphi^{-2}$ and $b = \varphi^2$. They are independent over $F$ as $(b/a)^{q-1} = \varphi^{4(q-1)} = \varphi^{q-1} \neq 1$ so that $b/a \notin F$. We compute

\begin{align*}
(E0) \quad ab &= 1 \\
(E1) \quad a^q b + ab^q &= a^{q-1} + b^{q-1} = \varphi^{q-1} + \varphi^{2(q-1)} = 1 \\
(E2) \quad a^{q^2} b + ab^{q^2} &= (\varphi^{q-1})^{q+1} + (\varphi^{q-1})^{q+1} = \varphi^{q-1} + \varphi^{2(q-1)} = 1 \\
(E3) \quad a^{q^3} b + ab^{q^3} &= ab + ab = 0.
\end{align*}

Also $\epsilon_{i+3} = \epsilon_i$ for $i \geq 1$ and if $k = 2m$ then $a^{q^m} b \in GF(q^3) \subset GF(q^m)$ so that (Em) holds. Hence $Q_{x+A3}^K$ has radical of codimension 2 and invariant 1. \hfill \Box

3. Artin-Schreier curves with many rational points.

We again consider polynomials

$$R(x) = \sum_{i=0}^{h} \epsilon_i x^{q^i},$$

with each $\epsilon_i \in GF(2) = F$ and $h = \lfloor k - 1/2 \rfloor$. The Artin-Schreier curve is

$$C_R : y^q + y = xR(x).$$

This has genus $g = \frac{1}{2}(q-1) \deg R(x)$ by [8, VI.4.1]. We consider both the curve and the quadratic form over $K$. The number of points in $K$-projective space on $C_R$ is

$$\#C_R(K) = qN(Q_K^R) + 1 = q^k + \Lambda(Q_K^R)(q-1)\sqrt{q^k+w} + 1,$$

where $w = \dim \mathrm{rad}(Q_K^R)$. We will compare this to the Hasse-Weil bound

$$\#C_R(K) \leq q^k + 1 + 2g\sqrt{q^k} = q^k + 1 + (q-1)\ell\sqrt{q^k},$$

where $\ell = \deg R(x)$. Clearly equality will hold in the Hasse-Weil bound only if $k$ is even.

**Theorem 3.1.** Suppose $k = 2m$ and the top coefficient $\epsilon_{m-1} = 1$. Then the number of points on $C_R$ equals the Hasse-Weil bound iff one of the following holds

1. $t$ is odd, $R = x + A2$ and $4|k$,
2. $t$ is even, $R = x + A3$ and $6|k$. 
Proof. Note that \( \deg R(x) = \ell = m - 1 \). The number of points on \( C_R \) equals the Hasse-Weil bound iff

\[
\Lambda(Q^K_R)(q - 1)\sqrt{q^{k+w}} = (q - 1)q^{m-1}\sqrt{q^k} \\
\Lambda(Q^K_R)\sqrt{q^w} = q^{m-1} \\
w = 2(m - 1) = k - 2 \quad \text{and} \quad \Lambda(Q^K_R) = 1.
\]

This holds, by (1.7), iff either (1) or (2) hold. \( \square \)

The restriction that \( \epsilon_{m-1} = 1 \) is necessary.

Example. Let \( k = 12 \) so that \( \ell = 5 \). Set \( R = x + x^4 + x^{16} \). Then \( \epsilon_4 = 1 \) and \( \epsilon_5 = 0 \). In particular, the genus of \( C_R \) is \( g = 2^{4-1} = 8 \). Also \( \dim \text{rad}(Q^K_R) = 8 \) and \( \Lambda(Q^K_R) = 1 \). This may be checked as follows:

Let \( \delta \) satisfy \( \delta^6 = \delta + 1 \). Set

\[
a_1 = \delta^{28} \quad b_1 = \delta^{56} \quad a_2 = \delta^7 \quad b_2 = \delta^{35}.
\]

Then \( a_1b_1 + a_2b_2 = 1 \). Set

\[
\epsilon_i = a_1^{2^i}b_1 + a_1b_1^{2^i} + a_2^{2^i}b_2 + a_2b_2^{2^i}.
\]

Then we may compute that \( \epsilon_1 = 0, \epsilon_2 = 1, \epsilon_3 = 0, \epsilon_4 = 1, \epsilon_5 = 0, \epsilon_6 = 0 \) and \( \epsilon_{i+6} = \epsilon_i \) for \( i \geq 1 \). Thus

\[
R = \text{tr}(a_1x)\text{tr}(b_1x) + \text{tr}(a_2x)\text{tr}(b_2x) \\
Q^K_R \simeq x_1x_2 + x_3x_4,
\]

giving the stated dimension of the radical and the invariant \( \Lambda \).

Now

\[
N(Q^K_R) = \frac{1}{2}(2^{12} + \sqrt{2^{12}+8}) = \frac{1}{2}(2^{12} + 2^{10}) \\
#C_R(K) = 1 + 2^{12} + 2^{10}.
\]

The Hasse-Weil bound is \( 1 + 2^{12} + 2 \cdot 8\sqrt{2^{12}} = 1 + 2^{12} + 2^{10} \). Hence there are other Artin-Schreier curves meeting the Hasse-Weil bound besides those of (3.1).
4. Factoring linearized polynomials.
Here we will restrict to the case $q = 2$. For $R = \sum_{j=0}^{h} \epsilon_j x^{2^j}$ define

$$R^*(x) = \sum_{j=1}^{h} \epsilon_j (x^{2^h+j} + x^{2^h-j}).$$

Then by [4, Lemma 8]

$$\text{rad}Q^K_R = \{a \in K : R^*(a) = 0\}.$$ 

Notice that $R^*$ is a self-reciprocal, linearized polynomial and that any self-reciprocal, linearized polynomial of degree $2h$ arises in this way. If $S$ is a self-reciprocal, linearized polynomial we will say $T$ is the associated form if $T$ is linearized and $T^* = S$.

**Proposition 4.1.** Suppose $k$ is even and $2h = k - 2$. Let $S$ be a self-reciprocal, linearized polynomial of degree $2h$ with associated form $T$. The following are equivalent:

1. $S$ divides $x^{2^k} + x$.
2. All irreducible factors of $S$ have degree $d$, where $d$ divides $k$.
3. Either $6|k$ and $T = A_3$; or $4|k$ and $T = A_2$.

**Proof.** $(1) \leftrightarrow (2)$ is clear. $(1)$ implies every root of $S$ lies in $K = GF(2^k)$. Since $Q^K_T$ has a radical consisting of the roots of $S$ in $K$, we have $\dim \text{rad}Q^K_T = k - 2$ and so of codimension 2. This gives (3). Conversely, (3) gives $Q^K_T$ has codimension 2 radical and so every root of $S$ lies in $K$. $\Box$

**Proposition 4.2.** Let $k$ be odd and $2h = k - 1$. Let $S$ be a self-reciprocal, linearized polynomial of degree $2h$ with associated form $T$. The following are equivalent:

1. $S$ divides $(x^{2^k} + x)(x^{2^k} + x + 1)$.
2. Every irreducible factor of $S$ either has degree $d$ (where $d|k$), or has the form $p(x^2 + x + 1)$, where $p$ is irreducible of degree $d$ (where again $d|k$).
3. $3|k$ and $T = A_3$.

**Proof.** $(1) \rightarrow (2)$. Let $q(x)$ be an irreducible factor of $S$. Then $q$ divides $q_k$ or $q_k + 1$, where $q_k = x^{2^k} + x$. In the first case, we have $\deg q = d$, where $d|k$. So suppose we are in the second case.

Now the roots of $S$ not in $K$ look like $a + \beta$, where $a \in K$ is a root and $\beta^2 = \beta + 1$. Namely, say $S(\alpha) = 0$ and $\alpha \notin K$. Then $q_k(\alpha) = 1$. Now

$$\beta^{2^j} = \begin{cases} \beta, & \text{if } j \text{ is even} \\ \beta^2, & \text{if } j \text{ is odd.} \end{cases}$$

In particular, $q_k(\beta) = 1$. Since either both $h \pm i$ are even or both are odd, we have $\beta^{h+i} + \beta^{h-i} = 0$ and so $S(\beta) = 0$. $S$ and $q_k$ are linearized so that their roots are additive. Hence $S(a + \beta) = 0$ and $q_k(a + \beta) = 0$. Thus $a = \alpha + \beta$ is a root of $S$ in $K$. 

Pick a root of \( q(x) \), say \( a + \beta \), where \( a \in K \) is also a root of \( S \). Now \( a^2 + a \) is also a root of \( S \). Let \( p(x) \) be the irreducible polynomial of \( a^2 + a \). Set \( d = \deg p; \) note that \( d|k. \) Set \( q_0(x) = p(x^2 + x + 1) \). Now

\[
q_0(a + \beta) = p(a^2 + a + \beta^2 + \beta + 1) = p(a^2 + a) = 0.
\]

Thus \( q(x) \) divides \( q_0(x) \). We will be done if we show \( \deg q = 2d \), the same as \( \deg q_0 \).

Now \( \deg q = [F(a + \beta) : F] \). We have

\[
(a + \beta)^2 + (a + \beta) + 1 = a^2 + a.
\]

Hence

\[
F \subset F(a^2 + a) \subset F(a + \beta).
\]

Moreover, if \( a + \beta \in F(a^2 + a) \) then \( \beta \in F(a) \). But \( a \in K \) and \( [K : F] \) is odd, so this is impossible. Hence

\[
[F(a + \beta) : F] \geq 2[F(a^2 + a) : F] = 2 \deg p = 2d.
\]

Thus \( q(x) = q_0(x) = p(x^2 + x + 1) \).

(2) \( \rightarrow \) (1). Let \( \pi_1 \) be the product of irreducible factors of \( S \) that are of degree \( d \), with \( d|k \). Then \( \pi_1|q_k \). Let \( \pi_2 \) be the product of the irreducible factors of \( S \) of type \( p(x^2 + x + 1) \), with \( p \) irreducible of degree \( d, d|k \). Let \( \pi_3 \) be the product of the \( p \)'s. Then \( \pi_2(x) = \pi_3(x^2 + x + 1) \) and \( \pi_3|q_k \). Hence \( \pi_2 \) divides

\[
q_k(x^2 + x + 1) = x^{2k+1} + x^{2k} + x^2 + x = q_k^2 + q_k = q_k(q_k + 1).
\]

Moreover, no root of \( \pi_2 \) is in \( K \) (as each irreducible factor has even degree). Thus \( \pi_2 \) divides \( q_k + 1 \). And so \( S = \pi_1 \pi_2 \) divides \( q_k(q_k + 1) \).

(1) \( \rightarrow \) (3). Let \( A \) denote the roots of \( S \) that are also roots of \( q_k \) and let \( B \) be the roots of \( S \) that are also roots of \( q_k + 1 \). As before, \( S(\beta) = 0 \) and \( \beta \notin K \). The map \( A \rightarrow B \) by \( a \leftrightarrow a + \beta \) is bijective. Hence \( |A| = 2^{k-2} \). Now \( \text{rad}Q^K_T = A \) and so has codimension 2. Apply the main theorem (2.4) to get (3).

(3) \( \rightarrow \) (1). We have that the codimension of \( \text{rad}Q^K_T \) is 2 so that \( 2^{k-2} \) roots of \( S \) lie in \( K \). The other roots of \( S \) are \( a + \beta \), for \( a \in K \) a root of \( S \). Now each root \( a \in K \) is a root of \( q_k \). And for each \( a + \beta \) we have

\[
(a + \beta)^{2k} + (a + \beta) + 1 = (a^{2k} + a) + (\beta^{2k} + \beta + 1) = 0,
\]

as \( k \) is odd. So \( S \) divides \( q_k(q_k + 1) \). \( \Box \)

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