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STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS
ON MANIFOLDS

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Outline

• Theory of stochastic functional differential equations (SFDE’s) in flat space: Itô and Nisio ([IN], Kushner ([Ku]), Mohammed ([Mo₂], [Mo₃]) and Mohammed-Scheutzow ([MoS₁], [MoS₂]).

• Objective: to constrain the solution to live on a smooth submanifold of Euclidean space.

• Main difficulty: Tangent space along a solution path is random (cf. unlike flat case).
• Difficulty resolved by pulling back the calculus on the tangent space at the starting point of the initial semi-martingale using stochastic parallel transport. Get SFDE on a linear space of semimartingales with values in the tangent space at a given point on the manifold.

• Solve SFDE on flat space by Picard’s iteration method. (cf. Driver [Dr]). But two levels of randomness: 
  (1) stochastic parallel transport over initial semimartingale path;
  (2) driving Brownian motion.
Law of solution at a given time may not be absolutely continuous with respect to law of initial semimartingale.

- Example of SDDE on the manifold with a type of Markov property in space of semimartingales.

- Regularity of solution of SDDE in initial semimartingale: stochastic Chen-Souriau calculus (Léandre [Le2], [Le3]). Requires Fréchet topology on semimartingales.
The Existence Theorem

Notation:

\( M \) smooth compact Riemannian manifold, dimension \( d \).

Delay \( \delta > 0, \ T > 0. \)

\((\Omega, \mathcal{F}_t, t \geq -\delta, P)\) filtered probability space-usual conditions.

\( W : [-\delta, \infty) \times \Omega \to \mathbb{R}^p \) Brownian motion on \\
\( (\Omega, \mathcal{F}_t, t \geq -\delta, P), \ W(-\delta) = 0. \)

\( (p = 1 \text{ for simplicity.}) \)
any smooth finite-dimensional Riemannian manifold; \( x \in N \).

\[ S([-\delta, T], N; -\delta, x) := \text{space of all } N\text{-valued} \]

\((\mathcal{F}_t)_{t \geq -\delta}\)-adapted continuous semimartingales

\[ \gamma : [-\delta, T] \times \Omega \rightarrow N \]

with \( \gamma(-\delta) = x \).
The Itô Map:

Fix \( x \in M \).

\( T(M) := \) tangent bundle over \( M \).

Define the Itô map by

\[
S([-\delta, T], M; -\delta, x) \ni \gamma \rightarrow \tilde{\gamma} \in S([-\delta, T], T_x(M); -\delta, 0)
\]

\[
d\tilde{\gamma}(t) = \tau_{t, -\delta}^{-1}(\gamma) \circ d\gamma(t)
\]

\[
\tilde{\gamma}(-\delta) = 0
\]

(Stratonovich).

\( \tau_{t, -\delta}(\gamma) := \) (stochastic) parallel transport from \( x = \gamma(-\delta) \) to \( \gamma(t) \) along semimartingale \( \gamma \).(\cite{E.E}, \cite{Em})

Itô map is a bijection.
\[ \tilde{S}_2^T := \text{Hilbert space of all semimartingales } \tilde{\gamma} \in S([-\delta, T], T_x(M); -\delta, 0) \text{ such that} \]
\[
\tilde{\gamma}(t) = \int_{-\delta}^{t} A(s) \, dW(s) + \int_{-\delta}^{t} B(s) \, ds, \quad 0 \leq t \leq T
\]
\tag{2}
and

\[ \|\tilde{\gamma}\|_2^2 := E\left[\int_{-\delta}^{T} |A(s)|^2 \, ds\right] + E\left[\int_{-\delta}^{T} |B(s)|^2 \, ds\right] < \infty \quad (3) \]

\(A(s), B(s) \in T_x(M)\) adapted previsible processes-\textit{characteristics} of \(\tilde{\gamma}\) (or \(\gamma\)).

\(\| \cdot \|_2\) gives slightly different topology than traditional semi-martingale topologies ([D.M]).

\(S_2^T := \text{image of } \tilde{S}_2^T \text{ under the Itô map with induced topology.}\)
Let $\gamma \in \mathcal{S}_T^2$, $t \in [-\delta, T]$. Set

$$\gamma^t(s) := \gamma(s \wedge t), \quad s \in [-\delta, T].$$

Then $(\tilde{\gamma}^t) = (\tilde{\gamma})^t$.

Evaluation map

$$e : [0, T] \times \mathcal{S}_T^2 \to L^0(\Omega, M)$$

$$e(t, \gamma) := \gamma(t)$$

Vector bundle $L^0(\Omega, T(M))$ over $L^0(\Omega, M)$ with fiber over $Z \in L^0(\Omega, M)$ given by

$$L^0(\Omega, T(M))_Z := \{ Y : Y(\omega) \in T_{Z(\omega)}M \text{ a.a. } \omega \in \Omega \}$$

$$e^*L^0(\Omega, T(M)) := \text{ pull-back bundle of } L^0(\Omega, T(M)) \text{ over } [0, T] \times \mathcal{S}_T^2 \text{ by } e.$$
A SFDE on $M$ is a map

$$F : [0, T] \times S^T \rightarrow L^0(\Omega, T(M))$$

such that $F(t, \gamma^t) \in T_{\gamma(t)}(M)$ a.s. for all $\gamma \in S^T_2$, $0 \leq t \leq T$. I.e. $F$ is a section of $e^*L^0(\Omega, T(M))$.

Consider SFDE

$$dx(t) = F(t, x^t) \circ dW(t), \quad t \geq 0 \right\}
\quad x^0 = \gamma^0 \quad \left\} \quad (4)$$

- Pullback SFDE (4) over $T_x(M)$.

Then:

$$d\tilde{x}(t) = \tau_{t,-\delta}(x^t)^{-1} F(t, x^t) \circ dW(t) \right\}
\quad = \tilde{F}(t, \tilde{x}^t) \circ dW(t), \quad t \geq 0 \quad \left\} \quad (5)$$

$$\tilde{x}^0 = \tilde{\gamma}^0$$
(t, \tilde{\gamma}) \mapsto \tilde{F}(t, \tilde{\gamma}) := \tau_{t,-\delta}^{-1}(\gamma)F(t, \gamma) can be viewed as a functional

\[ [0, T] \times \tilde{S}_2^T \to L^0(\Omega, T_x(M)) \]
on the flat space \( \tilde{S}_2^T \),

- Impose “boundedness” and “Lipschitz condition” on \( F \) in terms of \( \tilde{F} \) to get existence and uniqueness:
Hypothesis H.1 (Delay Condition):

\[ \tilde{F}(t, \tilde{\gamma}^t) = \tilde{F}(t, \tilde{\gamma}^{t-\delta}) \] (6)

The Stratonovich equation (5) now becomes also the Itô equation:

\[ d\tilde{x}(t) = \tilde{F}(t, \tilde{x}^{(t-\delta)}) \, dW(t) \]

\[ \tilde{x}^0 = \tilde{\gamma}^0 \] (7)
Hypothesis H.2:

(i) “Boundedness”. There exists a deterministic constant $c_1$ such that

$$|\tilde{F}(t, \tilde{\gamma})| < c_1 < \infty, \quad \text{a.s.}$$

for all $(t, \tilde{\gamma}) \in [0, T] \times \tilde{S}_2^T$.

(ii) “Local Lipschitz property”. Suppose $\tilde{\gamma}, \tilde{\gamma}' \in S_2^T$ have characteristics $(A(\cdot), B(\cdot))$ and $(A'(\cdot), B'(\cdot))$ respectively which are a.s. bounded by a deterministic constant $R$. Then

$$E[|\tilde{F}(t, \tilde{\gamma}^t) - \tilde{F}(t, (\tilde{\gamma}')^t)|^2] \leq K(R)\|\tilde{\gamma}^t - (\tilde{\gamma}')^t\|_2^2 \quad (8)$$
Example:

\[ X := \text{a smooth vector field on } M. \]

SDDE:

\[ dx(t) = \tau_{t,-\delta}(x)X(x(t-\delta)), \quad t > 0 \quad (9) \]

with

\[ F(t, \gamma) := \tau_{t,-\delta}(\gamma)X(\gamma(t-\delta)); \]

and

\[ \tilde{F}(t, \tilde{\gamma}^t) = \tau_{t-\delta,-\delta}^{-1}(\gamma^t)X(\gamma^t(t-\delta)). \]

\( \tilde{F} \) satisfies (H.1) and (H.2)(i) because parallel transport is a rotation and \( M \) is compact.
For (H.2)(ii) embed $M$ (isometrically) into $R^{d'}$ and extend the Riemannian structure over $R^{d'}$: the Riemannian metric has bounded derivatives of all orders and is uniformly non-degenerate. Extend the Levi-Civita connection over $M$ to a connection which preserves the metric over $R^{d'}$ on the trivial tangent bundle of $R^{d'}$ with Christoffel symbols having bounded derivatives of all order. The pair $(\gamma(t), \tau_t, -\delta)$ corresponds to a process $\hat{x}(t) \in R^{d'} \times R^{d'} \times d'$ which solves the Stratonovitch SDE:

\[
\begin{align*}
    d\hat{x}(t) &= \hat{Z}(\hat{x}(t)) \circ A(t) \, dW(t) + \hat{Z}(\hat{x}(t)) B(t) \, dt \\
    \hat{x}(-\delta) &= (x, Id_{T_x(M)})
\end{align*}
\]

(10)
on $R^{d'} \times R^{d' \times d'}$

$\hat{Z}$ is Lipschitz with derivatives of all orders bounded (uniformly in $A(.)$ and $B(.)$).

(10) in Itô form:

$$d\hat{x}(t) = \hat{Z}(\hat{x}(t))A(t)\,dW(t) + \hat{Y}(\hat{x}(t))A(t)^2 \,dt + \hat{Z}(\hat{x}(t))B(t) \,dt$$

(11)

In (11), $A(t) \in T_x(M)$, but we consider the one-dimensional case $d = 1$ for simplicity.

$\hat{Y}$ satisfies same hypotheses as the vector field $\hat{Z}$.

$\hat{x}(A, B)$ denotes dependence of $\hat{x}$ on $A$ and $B$. 

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Lemma 1.

Suppose

\[ |A(t)| + |B(t)| + |A'(t)| + |B'(t)| \leq R, \]

a.s. for all \( t \in [-\delta, T] \) and some deterministic \( R > 0 \).

Then there exists a constant \( K(R) > 0 \) such that:

\[
E[\sup_{-\delta \leq s \leq t} |\hat{x}(A, B)(s) - \hat{x}(A', B')(s)|^2] \\
\leq K(R) E[\int_{-\delta}^{t} (|A(s) - A'(s)|^2 + |B(s) - B'(s)|^2) ds] 
\]

(12)

Proof.

Follows from (11) by Burkholder’s inequality and Gronwall’s lemma. \( \square \)

Put \( t = 0 \) in Lemma to show that SDDE (9) satisfies (H.2)(ii).
Theorem 1.

Assume hypotheses (H.1) and (H.2).

Suppose that \( \gamma^0 \in S^0_2 \) has characteristics \( (A(t), B(t)), t \in [-\delta, 0], \) a.s. bounded by a deterministic constant \( C > 0 \).

Then the SFDE (4) has a unique global solution \( x \) such that \( x|[-\delta, T] \in S^T_2 \) for every \( T > 0 \).

Proof.

Sufficient to prove theorem for the SFDE (7) in flat space.

Define \( \tilde{x}^n \) inductively:

\[
\begin{align*}
    d\tilde{x}^{n+1}(t) &= \tilde{F}(t, \tilde{x}^{n,t-\delta}) \, dW(t), \quad t \geq 0 \\
    \tilde{x}^{n+1,0} &= \tilde{\gamma}^0
\end{align*}
\]

(13)

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By (H.2)(i),(ii),

\[ \|\tilde{x}^{n+1,t} - \tilde{x}^{n,t}\|_2^2 \leq C \int_0^t E[|\tilde{F}(s,\tilde{x}^{n,s-\delta}) - \tilde{F}(s,\tilde{x}^{n-1,s-\delta})|^2]ds \]
\[ \leq C \int_0^t \|\tilde{x}^{n,s} - \tilde{x}^{n-1,s}\|_2^2 ds \]  

(14)

By induction:

\[ \|\tilde{x}^{n+1,t} - \tilde{x}^{n,t}\|_2^2 \leq \frac{C^n t^n}{n!} \]  

(15)

This gives existence.

For uniqueness, take two solutions \( \tilde{x}^1, \tilde{x}^2 \) of (7). By (H.2)(i), their characteristics are a.s. bounded. Then

\[
\begin{align*}
\frac{d\tilde{x}^1(t)}{dt} &= \tilde{F}(t, \tilde{x}^1(t-\delta)) dW(t) \\
\frac{d\tilde{x}^2(t)}{dt} &= \tilde{F}(t, \tilde{x}^2(t-\delta)) dW(t) \\
\tilde{x}^{1,0} &= \tilde{x}^{2,0} = \tilde{\gamma}^0
\end{align*}
\]  

(16)
imply

$$\|\tilde{x}^{1,t} - \tilde{x}^{2,t}\|_2^2 \leq C \int_0^t \|\tilde{x}^{1,s} - \tilde{x}^{2,s}\|_2^2 ds$$  \hspace{1cm} (17)

Hence  $$\|\tilde{x}^{1,t} - \tilde{x}^{2,t}\|_2^2 = 0.$$  \hspace{1cm}  \Box

Continuous dependence on initial process:

Theorem 2.

Assume hypotheses (H.1) and (H.2). Let $B_T \subset S_T^2$ be the family of all $\gamma \in S_T^2$ with characteristics $(A, B)$ a.s. uniformly bounded on $[-\delta, 0]$ by a deterministic constant. Denote by $x(\gamma^0)$ the unique solution of SFDE (4) with initial semimartingale $\gamma^0 \in B^0$. Then the mapping

$$B^0 \ni \gamma^0 \mapsto x(\gamma^0) \in B^T$$

is continuous.
Proof.

Let $\tilde{\gamma}^0, (\tilde{\gamma}')^0$ have characteristics $(A, B), (A', B')$ uniformly bounded on $[-\delta, 0]$ by a deterministic constant. Let $\tilde{x}(A, B)$ and $\tilde{x}(A', B')$ be corresponding solutions of (5).

By Burkholder’s inequality and (H.2)(ii):

$$\|\tilde{x}^t(A, B) - \tilde{x}^t(A', B')\|_2^2 \leq \|\tilde{\gamma}^0 - (\tilde{\gamma}')^0\|_2^2 + K \int_0^t \|\tilde{x}^s(A, B) - \tilde{x}^s(A', B')\|_2^2 ds$$  \hspace{1cm} (18)

By Gronwall’s lemma:

$$\|\tilde{x}(A, B) - \tilde{x}(A', B')\|_2^2 \leq C\|\tilde{\gamma}^0 - (\tilde{\gamma}')^0\|_2^2$$  \hspace{1cm} (19)

□
Example-Markov Behavior.

Consider the SDDE:

\[ dx(t) = \tau_{t,t-\delta}(x) X(x(t-\delta)) dW(t) \]
\[ x^0 = \gamma^0, \]

(20)

with \( \gamma^0(-\delta) = x \in M \).

Replace \( x \) by a random variable \( Z \in L^0(\Omega, M) \) independent of \( W(t), t \geq -\delta \).

Fix \( t_0 > 0 \). The process \( x(t), t \geq t_0 \) solves the SDDE:

\[ dx'(t) = \tau_{t,t-\delta}(x') X(x'(t-\delta)) dW(t), t \geq t_0 \]
\[ x'(s) = x(s), s \in [t_0 - \delta, t_0] \]

(21)
$x(t_0 - \delta)$ is independent of $dW(t), t \geq t_0 - \delta$, and parallel transport in (20) depends only on the path between $t - \delta$ and $t$.

Uniqueness implies

$$x'(t) = x(t), \quad t \geq t_0.$$ 

For any semi-martingale $\gamma(t), t \geq -\delta$ in $M$, let $\gamma_t := \gamma|_{t - \delta, t}$.

$x(\cdot)(\gamma^0)(W) :=$ solution of (20) with initial condition $\gamma^0$.

Then

$$x(t)(\gamma^0)(W) = x(t - t')(x_{t'}^{(\gamma^0)})(W(t' + \cdot)), \quad t \geq t' \quad (22)$$

$W(t' + \cdot) :=$ Brownian shift

$$s \mapsto W(t' + s) - W(t').$$
Differentiability in Chen-Souriau Sense:

Consider family of SDDE’s:

\[
\begin{align*}
\frac{dx(t)}{dt}(u) &= \tau_{t, t-\delta}(x^t(u))X(x(t - \delta)(u)) \circ dW(t), \quad t \geq 0 \\
\frac{x^0(\delta)}{u} &= \gamma^0(u)
\end{align*}
\]

parametrized by \( u \in U \), open subset of \( \mathbb{R}^n \).

Embed \( M \) into \( \mathbb{R}^d' \).

Seek differentiability of \( x(t)(u) \) in \( u \). Can use Kolmogorov’s lemma, Sobolev’s imbedding theorem because \( u \) is finite-dimensional.

Flat version of (23) given by SDDE (9) with an added parameter \( u \).
For a parametrized semimartingale $\gamma(u)$ on $M$, the couple

$$(\gamma(u), \tau_{t,-\delta}(\gamma(u))) = \hat{x}_t$$

satisfies an Itô SDE depending on the parameter $u$:

$$d\hat{x}(t) = \hat{Z}(\hat{x}(t))A(u)(t)dW(t) + \hat{Y}(\hat{x}(t))A(u)(t)^2 dt + \hat{Z}(\hat{x}(t))B(u)(t) dt$$

(24)

$\hat{Z}$ and $\hat{Y}$ have bounded derivatives of all orders.

Introduce family of norms:

$$\|\tilde{\gamma}\|_p := E[\int_{-\delta}^T |A(s)|^p ds + \int_{-\delta}^T |B(s)|^p ds].$$

(25)

on the space $\tilde{S}_T^\infty$ of all semimartingales

$$\tilde{\gamma} \in S([-\delta, T], T_x(M); -\delta, 0)$$
where \( \tilde{\gamma}(t) = \int_{-\delta}^{t} A(s) \, dW(s) + \int_{-\delta}^{t} B(s) \, ds, \ 0 \leq t \leq T \) and \( \|\tilde{\gamma}\|_p \) is finite for every \( p \geq 1 \).

Suppose \( A(u)(\cdot) \) and \( B(u)(\cdot) \) are bounded by a deterministic constant \( C \) independent of \( u \), and

\[
u \mapsto (A(u)(\cdot), B(u)(\cdot))
\]

is Fréchet smooth in the the Fréchet space \( \bar{S}_T^\infty \) defined by the family of norms \( \| \cdot \|_p \).
Theorem 3.

Consider the parametrized SDDE’s:

\[
\begin{aligned}
dx(t)(u) &= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta)(u)) \circ dW(t), \quad t \geq 0, \\
x^0(u) &= \gamma^0(u)
\end{aligned}
\]

(26)

where \(X\) is smooth and \(\gamma^0(u)\) is smooth in \(u\) as above. Then \(x(t)(u)\) has a version which is a.s. smooth in \(u\).

Theorem also holds if noise has a smooth parameter \(u\):

\[
\begin{aligned}
dx(t)(u) &= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta))(\circ A(u)(t) dW(t) + B(u)(t) dt) \\
&= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta))(\circ A(u)(t) dW(t) + B(u)(t) dt)
\end{aligned}
\]

(27)

with initial conditions \(x^0(u) = \gamma^0(u)\).
Smooth functional in Chen-Souriau sense:

**Definition 1**

A *stochastic diffeology* is a family of *stochastic plots* $\phi(u)(t)$ for $u \in U$, any open subset of Euclidean space $\mathbb{R}^n$, where

(i) 

$$
\phi(u)(t) = \begin{cases}
\int_{-\delta}^{t} A(u)(s) \, dW(s) + \int_{\delta}^{t} B(u)(s) \, ds, & t < 0 \\
\int_{0}^{t} A(u)(s) \, dW(s) + \int_{0}^{t} B(u)(s) \, ds, & t \geq 0
\end{cases}
$$

(ii) $A(u)(\cdot)$ and $B(u)(\cdot)$ are a.s. bounded in $u$ by a deterministic constant $C$ and are Fréchet smooth in the norms $\|\cdot\|_p$. 
**Definition 2:**

A functional

\[ G : \mathcal{S}([-\delta, 0], T_x(M); -\delta, 0) \times C([0, T], \mathbb{R}) \to M \]

is *smooth in the Chen-Souriau sense* if it satisfies the following:

(i) To each stochastic plot \( \phi(u)(\cdot)(\omega) \), associate a functional \( G_\phi(u)(\omega) \) which has a smooth version in \( u \) for all \( \omega \) in a set \( \Omega_\phi \) of probability 1.

(ii) Let \( j : U_1 \to U_2 \) be a smooth deterministic map from an open subset \( U_1 \) of \( R^{n_1} \) into an open subset \( U_2 \) of \( R^{n_2} \). Let \( \phi^2(u_2)(\cdot)(\omega) \) be a stochastic plot over \( U_2 \).
Let \( \phi^1(u_1)(\cdot)(\omega) \) be the composite plot \( \phi^2(j \circ u_1)(\cdot)(\omega) \). Then

\[
G_{\phi^1}(u_1)(\omega) = G_{\phi^2}(j \circ u_1)(\omega)
\]

for all \( \omega \in \Omega_{\phi^1} \cap \Omega_{\phi^2} \).

(iii) Let \( \phi^1(u)(\cdot)(\omega) \), \( \phi^2(u)(\cdot)(\omega) \) be stochastic plots over \( U \). Suppose there exists a random measurable map \( \Psi \) defined on a subset of strictly positive probability and which maps \( \Omega_{\phi^1} \) into \( \Omega_{\phi^2} \) and is such that \( \phi^1(u)(\cdot)(\omega) = \phi^2(u)(\cdot)(\Psi \omega) \) for a.a. \( \omega \). Then

\[
F_{\phi^1}(u)(\omega) = F_{\phi^2}(u)(\Psi \omega)
\]

for a.a. \( \omega \).
The solution $x(\gamma^0)(t)(W)$ of the SDDE has a version which is a smooth Chen-Souriau functional in $(\gamma^0, W)$. 
Proof of Theorem 3-Outline.

\( \alpha := \) multi-index.

\( D^\alpha := \) partial derivatives of order \( \alpha \).

- For a parametrized semimartingale \( \gamma(u) \) on \( M \), the couple

\[
(\gamma(u), \tau_{-\delta}^{-1}(\gamma(u))) := \hat{x}(t)(u)
\]

satisfies an Itô SDE depending on the parameter \( u \):

\[
d\hat{x}(t)(u) = \hat{Z}(\hat{x}(t)(u))A(u)(t)\,dW(t) + \hat{Y}(\hat{x}(t)(u))A(u)(t)^2\,dt + \hat{Z}(\hat{x}(t)(u))B(u)(t)\,dt
\]

Since the inverse of the parallel transport is bounded, then \( \hat{Z} \) and \( \hat{Y} \) have bounded derivatives of all orders. If
\( \gamma(u) \in S^T_\infty \) has a.s. bounded characteristics \((A(u), B(u))\) which are smooth in \( u \) into the Fréchet space \( S^T_\infty \), then the pair \( \hat{x}(t)(u) := (\gamma(u), \tau_{t,-\delta}(\gamma(u))) \) has characteristics Fréchet smooth in \( u \). Follows by differentiating above SDE and applying Burkholder’s inequality and Gronwall’s lemma.

- Approximate the SDDE

\[
\begin{align*}
    dx(t)(u) &= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta)(u)) \circ dW(t), \ t \geq 0, \\
    x^0(u) &= \gamma^0(u)
\end{align*}
\]

by the sequence of SDDE’s:

\[
\begin{align*}
    d\tilde{x}^n(t)(u) &= g(\tilde{x}^n((t-\delta)_n)(u))dW(t) \\
    \tilde{x}^{n,0}(u) &= \tilde{\gamma}^0(u)
\end{align*}
\]

(26)
$(t - \delta)_n$ is the unique $k2^{-n}$ such that $t - \delta \in [k2^{-n}, (k + 1)2^{-n})$,
\[ \hat{x}^n(t) := (x^n(t), \tau_t^{n, -1}) , \]
given $g(y, z) := zX(y)$, where $z$ represents parallel transport (orthogonal matrix), $y \in M$.

Then $g$ is bounded and has bounded derivatives of all orders.
\[ \tilde{\gamma}(t)^0(u) := \int_{t-\delta}^{t} A^0_s(u)dw_s + \int_{t-\delta}^{t} B^0_sds \text{ for } t < 0 \]
where $A^0(u)(\cdot)$ and $B^0(u)(\cdot)$ are bounded independently of $u$ and differentiable in $u$ in all the $L^p$ semi-martingale norms $\| \cdot \|_p$.

Hence $\tilde{\gamma}(t)^0(u)$ has $u$-derivatives of all orders in all $L^p$ semi-martingale norms.
Follows from Kolmogorov’s lemma and Burkholder’s inequality.

- $\tilde{x}^n(t)(u)$ is a.s. differentiable in $u$ and

$$dD^\alpha \tilde{x}^n(t)(u)$$

$$= Dg(\hat{x}^n((t - \delta)_n)(u))D^\alpha \hat{x}^n((t - \delta)_n)(u) \, dW(t) + l.o.$$ 

where $l.o.$ are terms containing lower-order derivatives of $\tilde{x}^n(t)(u)$.

- Get uniform estimate:

$$\sup_{u \in U} \|D^\alpha \tilde{x}^n(\cdot)(u)\|_p \leq C(p, \alpha)$$

- Use SDDE for $\tilde{x}^n$ to get

$$\sup_{u \in U} \|D^\alpha \tilde{x}^n(\cdot)(u) - D^\alpha \tilde{x}^m(\cdot)(u)\|_p \to 0$$

as $n, m \to \infty$, for all $p$. 

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\[ D^\alpha \hat{x}^n(\cdot)(u) \text{ and } D^\alpha \tilde{x}^n(\cdot)(u) \] are Cauchy sequences in all \( L^p \) semi-martingale norms. By Sobolev’s imbedding theorem, \( \hat{x}^n(\cdot)(u) \) and \( \tilde{x}^n(\cdot)(u) \) converge to required smooth version of the solution of the SDDE.
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