THE STABLE MANIFOLD THEOREM FOR STOCHASTIC DIFFERENTIAL EQUATIONS

Oslo, Norway: August 11, 2000

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SDE's: Stable Manifolds

- Formulate a *Local Stable Manifold Theorem* for SDE's driven by Brownian motion (or general noise with stationary ergodic increments): Stratonovich or Itô type.

- Start with the existence of a stochastic flow for SDE.

- Concept of a hyperbolic stationary trajectory. The stationary trajectory is a solution of the forward/backward anticipating SDE for all time (Stratonovich case).
• Existence of a stationary random family of asymptotically invariant stable and unstable manifolds within a stationary neighborhood of the hyperbolic stationary solution.

• Stable and unstable manifolds dynamically characterized using forward and backward solutions of anticipating versions of the (Stratonovich) SDE.

• Proof based on Ruelle-Oseledec (nonlinear) multiplicative ergodic theory and anticipating stochastic calculus.
Formulation of the Theorem

Stratonovich SDE on $\mathbb{R}^d$

\[ dx(t) = h(x(t)) \, dt + \sum_{i=1}^{m} g_i(x(t)) \circ dW_i(t), \quad (I) \]

driven by $m$-dimensional Brownian motion $W := (W_1, \ldots, W_m)$.

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P) :=$ canonical filtered Wiener space.

$\Omega :=$ space of all continuous paths $\omega : \mathbb{R} \rightarrow \mathbb{R}^m$, $\omega(0) = 0$, in Euclidean space $\mathbb{R}^m$, with compact open topology;

$\mathcal{F} :=$ Borel $\sigma$-field of $\Omega$;

$\mathcal{F}_t :=$ sub-$\sigma$-field of $\mathcal{F}$ generated by the evaluations $\omega \rightarrow \omega(u)$, $u \leq t$, $t \in \mathbb{R}$.

$P :=$ Wiener measure on $\Omega$. 

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$h : \mathbb{R}^d \rightarrow \mathbb{R}^d, 1 \leq i \leq m, C^k_b$ vector fields on $\mathbb{R}^d$; viz. $h$ has all derivatives $D^j h, 1 \leq j \leq k$, continuous and globally bounded, $D^k h$ Hölder continuous with exponent $\delta \in (0, 1)$.

$g_i, 1 \leq i \leq m$, globally bounded and $C^{k+1}_b$.

$\theta : \mathbb{R} \times \Omega \rightarrow \Omega$ is the (ergodic) Brownian shift

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbb{R}, \omega \in \Omega.$$ 

Let $\phi : \mathbb{R} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ be the stochastic flow generated by (I) $(\phi(t, \cdot, \omega) = [\phi(-t, \cdot, \theta(t, \omega))]^{-1}, t < 0)$. Then $\phi$ is a perfect cocycle:

$$\phi(t_1 + t_2, \cdot, \omega) = \phi(t_2, \cdot, \theta(t_1, \omega)) \circ \phi(t_1, \cdot, \omega),$$
for all $t_1, t_2 \in \mathbb{R}$ and all $\omega \in \Omega$ ([I-W], [A-S], [A]).

Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of $\mathbb{R}^d$. $(\phi, \theta)$ is a "random vector-bundle morphism" over the "base" probability space $\Omega$. 
The Cocycle
Definition

The SDE (1) has a \textit{stationary trajectory} if there exists an $\mathcal{F}$-measurable random variable $Y : \Omega \to \mathbb{R}^d$ such that

$$\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

(1)

for all $t \in \mathbb{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $\phi(t, Y) = Y(\theta(t))$. 
Examples of Stationary Solutions

1. Fixed points:

\[ d\phi(t) = h(\phi(t))\, dt + \sum_{i=1}^{m} g_i(\phi(t)) \circ dW_i(t) \]

\[ h(x_0) = g_i(x_0) = 0, \quad 1 \leq i \leq m \]

Take \( Y(\omega) = x_0 \) for all \( \omega \in \Omega \).

2. Linear affine case \( d = 1 \):

\[ d\phi(t) = \lambda \phi(t)\, dt + dW(t) \]

\( \lambda > 0 \) fixed, \( W(t) \in \mathbb{R} \). Take

\[ \phi(t, x, \omega) = e^{\lambda t} \left[ x + \int_{0}^{t} e^{-\lambda u} dW(u) \right], \]

\[ Y(\omega) := -\int_{0}^{\infty} e^{-\lambda u} dW(u), \]

\[ \theta(t, \omega)(s) = \omega(t + s) - \omega(t). \]
Check that $\phi(t,Y(\omega),\omega) = Y(\theta(t,\omega))$, using integration by parts and variation of parameters.

3. Affine linear SDE in $d = 2$:

$$d\phi(t) = A\phi(t)\, dt + GdW(t)$$

with $A$ a fixed hyperbolic $2 \times 2$-diagonal matrix; $G$ a constant $2 \times 2$-matrix, and $W$ 2-dimensional Brownian motion.

4. Non-linear transforms of (3) under a global diffeomorphism.

5. Invariant measure for SDE: Enlarge probability space ([M-S.3]).
Let $\phi(t, Y)$ be a stationary solution of (I). Cocycle property of $\phi$ implies that the linearization

$$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$$

along the stationary solution is also a $d \times d$-matrix-valued cocycle. Using Kolmogorov's theorem, the random variables

$$\sup_{x \in \mathbb{R}^d} \frac{\|D_2\phi(t, x)\|}{(1 + |x|^\gamma)}, \gamma > 0,$$

have moments of all orders. If $E \log^+ |Y| < \infty$, then $E \log^+ \|D_2\phi(1, Y)\| < \infty$. Apply Oseledec's Theorem to get a non-random finite Lyapunov spectrum:

$$\lim_{n \to \infty} \frac{1}{n} \log |D_2\phi(n, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbb{R}^d).$$
Spectrum takes finitely many fixed values \( \{\lambda_i\}_{i=1}^p \) with non-random multiplicities \( q_i \), \( 1 \leq i \leq p \), and \( \sum_{i=1}^p q_i = d \) ([Ru.1], Theorem I.6).

**Definition**

Stationary trajectory \( \phi(t,Y) \) of (I) is hyperbolic if \( E \log^+ |Y(\cdot)| < \infty \), and if the linearized cocycle \( (D_2\phi(n,Y(\omega),\omega), \theta(n,\omega)) \) has a non-vanishing Lyapunov spectrum

\[
\{\lambda_p < \cdots < \lambda_{i_0+1} < \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_2 < \lambda_1\}
\]

i.e. \( \lambda_i \neq 0 \) for all \( 1 \leq i \leq p \).

Define \( \lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\} \) if at least one \( \lambda_i < 0 \). If all \( \lambda_i > 0 \), set \( \lambda_{i_0} = -\infty \). (This implies that \( \lambda_{i_0-1} \) is the smallest
positive Lyapunov exponent of the linearized flow, if at least one $\lambda_i > 0$; in case all $\lambda_i$ are negative, set $\lambda_{i_0} = \infty$.)

Let $\rho \in \mathbb{R}^+$, $x \in \mathbb{R}^d$.

$B(x, \rho)$: open ball in $\mathbb{R}^d$, center $x$ and radius $\rho$;

$\bar{B}(x, \rho)$: corresponding closed ball;

$\mathcal{K}(\mathbb{R}^d)$: the class of all non-empty compact subsets of $\mathbb{R}^d$ with Hausdorff metric $d^*$:

\[
d^*(A_1, A_2) := \sup \{d(x, A_1) : x \in A_2\} \lor \sup \{d(y, A_2) : y \in A_1\}
\]

where $A_1, A_2 \in \mathcal{K}(\mathbb{R}^d)$;

\[
d(x, A_i) := \inf \{|x - y| : y \in A_i\}, x \in \mathbb{R}^d, i = 1, 2;
\]

$\mathcal{B}(\mathcal{K}(\mathbb{R}^d))$ := Borel $\sigma$-algebra on $\mathcal{K}(\mathbb{R}^d)$ with respect to the metric $d^*$. 
Theorem 1 (The Stable Manifold Theorem) (M. + Scheutzow, AOP ’99)

Assume that the coefficients of SDE (1) satisfy the given hypotheses. Suppose $\phi(t, Y)$ is a hyperbolic stationary trajectory of (1) with $E \log^+ |Y| < \infty$.

Fix $\varepsilon_1 \in (0, -\lambda_{i_0})$ and $\varepsilon_2 \in (0, \lambda_{i_0-1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$,

(ii) $\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1), \beta_i \geq \rho_i > 0, i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k, \varepsilon}$ ($\varepsilon \in (0, \delta)$) submanifolds $\tilde{S}(\omega), \tilde{U}(\omega)$ of $\overline{B}(Y(\omega), \rho_1(\omega))$ and $\overline{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:
(a) \( \tilde{S}(\omega) \) is the set of all \( x \in \bar{B}(Y(\omega), \rho_1(\omega)) \) such that

\[
|\phi(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}
\]

for all integers \( n \geq 0 \). Furthermore,

\[
\limsup_{t \to \infty} \frac{1}{t} \log |\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0} \quad (2)
\]

for all \( x \in \tilde{S}(\omega) \). Each stable subspace \( S(\omega) \) of the linearized flow \( D_2 \phi \) is tangent at \( Y(\omega) \) to the submanifold \( \tilde{S}(\omega), \) viz. \( T_{Y(\omega)} \tilde{S}(\omega) = S(\omega) \). In particular, \( \dim \tilde{S}(\omega) = \dim S(\omega) \) and is non-random.

(b) \[
\limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup_{x_1 \neq x_2} \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right] \leq \lambda_{i_0}.
\]

(c) (Cocycle-invariance of the stable manifolds):

There exists \( \tau_1(\omega) \geq 0 \) such that

\[
\phi(t, \cdot, \omega)(\tilde{S}(\omega)) \subseteq \tilde{S}(\theta(t, \omega)), \quad t \geq \tau_1(\omega). \quad (3)
\]
Also

\[ D_2\phi(t, Y(\omega), \omega)(S(\omega)) = S(\theta(t, \omega)), \quad t \geq 0. \] (4)

(d) \( \tilde{U}(\omega) \) is the set of all \( x \in B(Y(\omega), \rho_2(\omega)) \) with the property that

\[ |\phi(-n, x, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{(-\lambda_{i_0-1} + e_2)n} \] (5)

for all integers \( n \geq 0 \). Also

\[ \limsup_{t \to \infty} \frac{1}{t} \log |\phi(-t, x, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}. \] (6)

for all \( x \in \tilde{U}(\omega) \). Furthermore, the unstable subspace \( U(\omega) \) of \( D_2\phi \) is the tangent space to \( \tilde{U}(\omega) \) at \( Y(\omega) \), viz. \( T_{Y(\omega)}\tilde{U}(\omega) = U(\omega) \). In particular, \( \dim \tilde{U}(\omega) = \dim U(\omega) \) and is non-random.
(e) \[
\limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup_{x_1 \neq x_2, x_1, x_2 \in \tilde{U}(\omega)} \left\{ \frac{\phi(-t, x_1, \omega) - \phi(-t, x_2, \omega)}{|x_1 - x_2|} \right\} \right] \leq -\lambda_{i_0-1}.
\]

(f) (Cocycle-invariance of the unstable manifolds):

There exists \( \tau_2(\omega) \geq 0 \) such that

\[
\phi(-t, \cdot, \omega)(\tilde{U}(\omega)) \subseteq \tilde{U}(\theta(-t, \omega)), \quad t \geq \tau_2(\omega). \quad (7)
\]

Also

\[
D_2\phi(-t, Y(\omega), \omega)(U(\omega)) = U(\theta(-t, \omega)), \quad t \geq 0. \quad (8)
\]

(g) The submanifolds \( \tilde{U}(\omega) \) and \( \tilde{S}(\omega) \) are transversal, viz.

\[
\mathbb{R}^d = T_{Y(\omega)}\tilde{U}(\omega) \oplus T_{Y(\omega)}\tilde{S}(\omega). \quad (9)
\]

(h) The mappings

\[
\Omega \to \mathcal{K}(\mathbb{R}^d), \quad \Omega \to \mathcal{K}(\mathbb{R}^d),
\]

\[
\omega \mapsto \tilde{S}(\omega) \quad \omega \mapsto \tilde{U}(\omega)
\]
are \((\mathcal{F}, \mathcal{B}(\mathcal{K}(\mathbb{R}^d)))\)-measurable.

Assume, further, that \(h, g_i, 1 \leq i \leq m\), are \(C^\infty_b\). Then the local stable and unstable manifolds \(\tilde{S}(\omega), \tilde{U}(\omega)\) are \(C^\infty\).
$t > \tau_1(\omega)$

A picture is worth a 1000 words!
$Y(\theta(-t, \omega))$ 

$t > \tau_2(\omega)$
Sketch of Proof

Linearization and Substitution

Assume regularity conditions on the coefficients $h, g_i$. By the Substitution Rule, $\phi(t, Y(\omega), \omega)$ is a stationary solution of the anticipating Stratonovich SDE

$$d\phi(t, Y) = h(\phi(t, Y)) \, dt + \sum_{i=1}^{m} g_i(\phi(t, Y)) \circ dW_i(t), \quad t > 0 \right\}
\phi(0, Y) = Y. \quad (II)$$

([N-P]).

Linearize the SDE (I) along the stationary trajectory. By substitution, match the solution of the linearized equation
with the linearized cocycle $D_2\phi(t, Y(\omega), \omega)$. Hence $D_2\phi(t, Y(\omega), \omega), t \geq 0$, solves the SDE:

\[
\begin{align*}
    dD_2\phi(t, Y) &= Dh(\phi(t, Y))D_2\phi(t, Y) \, dt \\
                   &\quad + \sum_{i=1}^{m} Dg_i(\phi(t, Y))D_2\phi(t, Y) \circ dW_i(t), \quad t > 0 \\
    D_2\phi(0, Y) &= I.
\end{align*}
\]

(III)

$D_2, D$ denotes spatial (Fréchet) derivatives.

Similarly, the backward trajectories $\phi(t, Y), D_2\phi(t, Y), t < 0,$ solve the corresponding backward Stratonovich SDE’s:

\[
\begin{align*}
    d\phi(t, Y) &= -h(\phi(t, Y)) \, dt - \sum_{i=1}^{m} g_i(\phi(t, Y)) \circ dW_i(t), \quad t < 0 \\
    \phi(0, Y) &= Y.
\end{align*}
\]

(II−)
\[
dD_2\phi(t, Y) = -Dh(\phi(t, Y))D_2\phi(t, Y)\, dt
- \sum_{i=1}^{m} Dg_i(\phi(t, Y))D_2\phi(t, Y) \circ dW_i(t), \quad t < 0
\]
\[
D_2\phi(0, Y) = I. \quad (III^-)
\]

Above SDE's (II)-(III) give dynamic characterizations of the stable and unstable manifolds.

The following lemma is used to construct the shift-invariant sure event appearing in the statement of the local stable manifold theorem. Gives "perfect versions" of the ergodic theorem and Kingman's subadditive ergodic theorem.
Lemma 1

(i) Let \( h : \Omega \to \mathbb{R}^+ \) be \( \mathcal{F} \)-measurable and such that
\[
\int_{\Omega} \sup_{0 \leq u \leq 1} h(\theta(u, \omega)) \, dP(\omega) < \infty.
\]
Then there is a sure event \( \Omega_1 \in \mathcal{F} \) such that \( \theta(t, \cdot)(\Omega_1) = \Omega_1 \) for all \( t \in \mathbb{R} \), and
\[
\lim_{t \to \infty} \frac{1}{t} h(\theta(t, \omega)) = 0
\]
for all \( \omega \in \Omega_1 \).

(ii) Suppose \( f : \mathbb{R}^+ \times \Omega \to \mathbb{R} \cup \{-\infty\} \) is a measurable process on \((\Omega, \mathcal{F}, P)\) satisfying the following conditions
\[
(a) \ E \sup_{0 \leq u \leq 1} f^+(u) < \infty, \ E \sup_{0 \leq u \leq 1} f^+(1-u, \theta(u)) < \infty
\]
\[
(b) \ f(t_1 + t_2, \omega) \leq f(t_1, \omega) + f(t_2, \theta(t_1, \omega)) \text{ for all } t_1, t_2 \geq 0
\]
and all \( \omega \in \Omega \).
Then there is sure event $\Omega_2 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_2) = \Omega_2$ for all $t \in \mathbb{R}$, and a fixed number $f^* \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{t \to \infty} \frac{1}{t} f(t, \omega) = f^*$$

for all $\omega \in \Omega_2$.

**Proof**

[Mo.1], Lemma 7. $\square$
Theorem 2 ([O], 1968)

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(\theta : \mathbb{R}^+ \times \Omega \to \Omega\) a measurable family of ergodic \(P\)-preserving transformations. Let \(T : \mathbb{R}^+ \times \Omega \to L(\mathbb{R}^d)\) be measurable, such that \((T, \theta)\) is an \(L(\mathbb{R}^d)\)-valued cocycle. Suppose that

\[
E \sup_{0 \leq t \leq 1} \log^+ \|T(t, \cdot)\| < \infty, \quad E \sup_{0 \leq t \leq 1} \log^+ \|T(1-t, \theta(t, \cdot))\| < \infty.
\]

Then there is a set \(\Omega_0 \in \mathcal{F}\) of full \(P\)-measure such that \(\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0\) for all \(t \in \mathbb{R}^+\), and for each \(\omega \in \Omega_0\), the limit

\[
\lim_{t \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)
\]

exists in the uniform operator norm. Each \(\Lambda(\omega)\) has a discrete non-random spectrum

\[
e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_p}
\]
where the $\lambda_i$'s are distinct. Each $e^{\lambda_i}$ has an eigen-space $F_i(\omega)$ and a fixed non-random multiplicity $m_i := \text{dim}F_i(\omega)$.

Define

$$E_1(\omega) := \mathbb{R}^d, \quad E_i(\omega) := \left[ \bigoplus_{j=1}^{i-1} F_j(\omega) \right]^\perp, \quad 1 < i \leq p.$$ 

Then

$$E_p(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = \mathbb{R}^d$$

$$\lim_{t \to \infty} \frac{1}{t} \log \|T(t, \omega)x\| = \lambda_i(\omega), \quad \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega),$$

and

$$T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))$$

for all $t \geq 0, \ 1 \leq i \leq p$.

**Proof.**

Based on the discrete version of Oseledec’s multiplicative ergodic theorem.
and Lemma 1. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]), “perfect” infinite-dimensional version and application to SFDE’s. □
Spectral Theorem

\[ T(t, \omega) \]

\[ E_1 = \mathbb{R}^d \]

\[ E_2(\omega) \]

\[ E_3(\omega) \]

\[ \Omega \]

\[ \omega \]

\[ \theta(t, \omega) \]
Apply Theorem 2 with

\[ T(t, \omega) := D_2 \phi(t, Y(\omega), \omega) \]

Then linearized cocycle has random invariant stable and unstable subspaces \( \{ S(\omega), U(\omega) : \omega \in \Omega \} \):

\[ D_2 \phi(t, Y(\omega), \omega)(S(\omega)) = S(\theta(t, \omega)), \]
\[ D_2 \phi(-t, Y(\omega), \omega)(U(\omega)) = U(\theta(-t, \omega)), \quad t \geq 0. \]

[Mo.1].
\[ U(w) + s(w) \]

\[ u(e(t, w)) \]

\[ s(e(t, w)) \]

\[ \Omega \]

\[ \omega \]

\[ \theta(t, \cdot) \]

\[ D_2 \phi(t, Y(\omega), \omega) \]
Estimates on the non-linear cocycle

Theorem 3 (M. + Scheutzow [M-S.2])

There exists a jointly measurable modification of the trajectory random field of (I) (with initial condition $x$ at $t = s$), denoted by $\{\phi_{s,t}(x) : -\infty < s, t < \infty, x \in \mathbb{R}^d\}$, having the following properties:

The cocycle $\phi : \mathbb{R} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ is given by

$$\phi(t, x, \omega) := \phi_{0,t}(x, \omega), \quad x \in \mathbb{R}^d, \omega \in \Omega, t \in \mathbb{R}.$$  

Then for all $\omega \in \Omega$, $\epsilon \in (0, \delta)$, $\gamma, \rho, T > 0, 1 \leq |\alpha| \leq k$, $\phi(t, \cdot, \omega)$ is $C^{k,\epsilon}$, $0 < \epsilon < \delta$, and the quantities

$$\sup_{0 \leq s, t \leq T, x \in \mathbb{R}^d} \frac{|\phi_{s,t}(x, \omega)|}{[1 + |x|(|\log^+ |x||)|\gamma]}, \quad \sup_{0 \leq s, t \leq T, x \in \mathbb{R}^d} \frac{\|D^\alpha_x \phi_{s,t}(x, \omega)\|}{(1 + |x|\gamma)},$$

$$\sup_{x \in \mathbb{R}^d} \sup_{0 \leq s, t \leq T, \rho, 0 < |x' - x| \leq \rho} \frac{\|D^\alpha_x \phi_{s,t}(x, \omega) - D^\alpha_x \phi_{s,t}(x', \omega)\|}{|x - x'|\epsilon (1 + |x|)\gamma},$$

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are finite. The random variables defined by the above expressions have $p$-th moments for all $p \geq 1$. 

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Ruelle’s Non-linear Ergodic Theorem

Theorem 4 ([Ru.1], 1979)

Let \( \Omega \ni \omega \mapsto F_\omega \in C^{k,\epsilon}(\mathbb{R}^d, 0; \mathbb{R}^d, 0) \) be measurable such that \( E \log^+ \| F \| \overline{B}(0, 1) \|_{k,\epsilon} < \infty \). Set \( F^n(\omega) := F_\theta(n-1, \omega)^o \cdots o F_\theta(1, \omega) o F_\omega \). Suppose \( \lambda < 0 \) is not in the spectrum of the cocycle \( (DF_\omega^n(0), \theta(n, \omega)) \). Then there is a sure event \( \Omega_0 \in F \) such that \( \theta(1, \cdot)(\Omega_0) \subseteq \Omega_0 \), and measurable functions \( 0 < \alpha(\omega) < \beta(\omega) < 1, \gamma(\omega) > 1 \) with the following properties:

(a) If \( \omega \in \Omega_0 \), the set

\[
V_\omega^\lambda := \{ x \in \overline{B}(0, \alpha(\omega)) : |F_\omega^n(x)| \leq \beta(\omega)e^{n\lambda} \text{ for all } n \geq 0 \}
\]

is a \( C^{k,\epsilon} \) submanifold of \( \overline{B}(0, \alpha(\omega)) \).

(b) If \( x_1, x_2 \in V_\omega^\lambda \), then

\[
|F_\omega^n(x_1) - F_\omega^n(x_2)| \leq \gamma(\omega)|x_1 - x_2|e^{n\lambda}
\]
for all integers $n \geq 0$. If $\lambda' < \lambda$ and $[\lambda', \lambda]$ is disjoint from the spectrum of $(DF^n_\omega(0), \theta(n, \omega))$, then there exists a measurable $\gamma'(\omega) > 1$ such that

$$|F^n_\omega(x_1) - F^n_\omega(x_2)| \leq \gamma'(\omega)|x_1 - x_2|e^{n\lambda'}$$

for all $x_1, x_2 \in V_\omega^\lambda$ and all integers $n \geq 0$.

**Proof**

[Ru.1], Theorem 5.1, p. 292.
Construction of the Stable/Unstable Manifolds

- Use auxiliary cocycle \((Z, \theta)\):

\[
Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \quad (16)
\]

for \(t \in \mathbb{R}, x \in \mathbb{R}^d, \omega \in \Omega\). Set \(\tau := \theta(1, \cdot) : \Omega \to \Omega\). Define maps \(F_\omega, F_\omega^n : \mathbb{R}^d \to \mathbb{R}^d\):

\[
F_\omega(x) := Z(1, x, \omega) \quad x \in \mathbb{R}^d
\]

\[
F_\omega^\tau := F_{\tau^(-1)}(\omega) \circ \cdots \circ F_{\tau(\omega)} \circ F_\omega
\]

for all \(\omega \in \Omega\). Then cocycle property for \(Z\) gives \(F_\omega^n = Z(n, \cdot, \omega)\) for each \(n \geq 1\).

\(F_\omega\) is \(C^{k, \epsilon} (\epsilon \in (0, \delta))\) and \((DF_\omega)(0) = D_2\phi(1, Y(\omega), \omega)\).

- Integrability of the map

\[
\omega \mapsto \log^+ \|D_2\phi(1, Y(\omega), \omega)\|_{L(\mathbb{R}^d)}
\]
(Lemma 2) implies discrete cocycle 
\((DF^m_\omega)(0), \theta(n, \omega), n \geq 0\) has same non-random Lyapunov spectrum as that of linearized continuous cocycle

\[(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0),\]

viz. \(\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}\), where each \(\lambda_i\) has fixed multiplicity \(q_i, 1 \leq i \leq m\) (Lemma 2).

- If \(\lambda_i > 0\) for all \(1 \leq i \leq m\), then take \(\tilde{S}(\omega) := \{Y(\omega)\}\) for all \(\omega \in \Omega\). Theorem is trivial in this case. Hence assume there is at least one \(\lambda_i < 0\).

- Use discrete non-linear ergodic theorem of Ruelle (Theorem 4) and its proof to obtain a sure event \(\Omega_1^* \in \mathcal{F}\)
such that $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbb{R}$, $\mathcal{F}$-measurable positive random variables $\rho_1, \beta_1 : \Omega_1^* \to (0, \infty)$, $\rho_1 < \beta_1$, and a random family of $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifolds of $\bar{B}(0, \rho_1(\omega))$ denoted by $\tilde{S}_d(\omega)$, $\omega \in \Omega_1^*$, and satisfying the following properties for each $\omega \in \Omega_1^*$: $\tilde{S}_d(\omega)$ is the set of all $x \in \bar{B}(0, \rho_1(\omega))$ such that

$$|Z(n, x, \omega)| \leq \beta_1(\omega)e^{(\lambda_i_0 + \epsilon_1)n}, \quad n \in \mathbb{Z}^+ \quad (21)$$

$\tilde{S}_d(\omega)$ is tangent at 0 to the stable subspace $S(\omega)$ of the linearized flow $D_2\phi$, viz. $T_0\tilde{S}_d(\omega) = S(\omega)$. Therefore dim $\tilde{S}_d(\omega)$ is non-random by ergodicity of $\theta$. Also

$$\limsup_{n \to \infty} \frac{1}{n} \log \left[ \sup_{x_1, x_2 \in \tilde{S}_d(\omega), x_1 \neq x_2} \frac{|Z(n, x_1, \omega) - Z(n, x_2, \omega)|}{|x_1 - x_2|} \right] \leq \lambda_i_0. \quad (22)$$
The $\theta(t, \cdot)$-invariant sure event $\Omega_1^* \in \mathcal{F}$ is constructed using the ideas in Ruelle's proof (of Theorem 5.1 in [Ru.1], p. 293), combined with the estimate (10) of Lemma 2 and the subadditive ergodic theorem (Lemma 1 (ii)).

- For each $\omega \in \Omega_1^*$, let $\tilde{S}(\omega)$ be as defined in part (a) of the theorem. Then by definition of $\tilde{S}_d(\omega)$ and $z$:

$$\tilde{S}(\omega) = \tilde{S}_d(\omega) + Y(\omega). \quad (23)$$

Since $\tilde{S}_d(\omega)$ is a $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifold of $B(0, \rho_1(\omega))$, then $\tilde{S}(\omega)$ is a $C^{k, \epsilon}$ ($\epsilon \in (0, \delta)$) submanifold of $B(Y(\omega), \rho_1(\omega))$. Furthermore, $T_{Y(\omega)}\tilde{S}(\omega) = T_0\tilde{S}_d(\omega) = S(\omega)$. 

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Hence \( \dim \tilde{S}(\omega) = \dim S(\omega) = \sum_{i=i_0}^{m} q_i \), and is non-random.

- (22) implies that
\[
\limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)| \leq \lambda_{i_0} \quad (24)
\]
for all \( \omega \) in \( \Omega_1^* \) and all \( x \in \tilde{S}_d(\omega) \). Lemma 4 implies there is a sure event \( \Omega_2^* \subseteq \Omega_1^* \) such that \( \theta(t, \cdot)(\Omega_2^*) = \Omega_2^* \) for all \( t \in \mathbb{R} \), and
\[
\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| \leq \lambda_{i_0} \quad (25)
\]
for all \( \omega \in \Omega_2^* \) and all \( x \in \tilde{S}_d(\omega) \). Therefore (2) holds.

- To prove (b), let \( \omega \in \Omega_1^* \). By (22), there is a positive integer \( N_0 := N_0(\omega) \) (independent of \( x \in \tilde{S}_d(\omega) \)) such that
$Z(n, x, \omega) \in \bar{B}(0, 1)$ for all $n \geq N_0$. Let $\Omega^*_4 := \Omega^*_2 \cap \Omega_3$, where $\Omega_3$ is the shift-invariant sure event defined in the proof of Lemma 4. Then $\Omega^*_4$ is a sure event and $\theta(t, \cdot)(\Omega^*_4) = \Omega^*_4$ for all $t \in \mathbb{R}$. By cocycle property, Mean-Value theorem and the ergodic theorem (Lemma 1(i)), we get (b).

- To prove the invariance property (4), apply the Oseledec theorem to $(D_2 \phi(t, Y(\omega), \omega), \theta(t, \omega))$. Get a sure $\theta(t, \cdot)$-invariant event, also denoted by $\Omega^*_1$, such that

$D_2 \phi(t, Y(\omega), \omega)(S(\omega)) \subseteq S(\theta(t, \omega))$ for all $t \geq 0$ and all $\omega \in \Omega^*_1$. Equality holds because $D_2 \phi(t, Y(\omega), \omega)$ is injective and $\dim S(\omega) = \dim S(\theta(t, \omega))$ for all $t \geq 0$ and all $\omega \in \Omega^*_1$. 

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- To prove the asymptotic invariance property (3), use ideas from Ruelle’s Theorems 5.1 and 4.1 in [Ru.1], to pick random variables $\rho_1, \beta_1$ and a sure event (also denoted by) $\Omega^*_1$ such that $\theta(t, \cdot)(\Omega^*_1) = \Omega^*_1$ for all $t \in \mathbb{R}$, and for any $\epsilon \in (0, \epsilon_1)$ and every $\omega \in \Omega^*_1$, there exists a positive $K_1^\epsilon(\omega)$ for which the inequalities

$$
\rho_1(\theta(t, \omega)) \geq K_1^\epsilon(\omega) \rho_1(\omega) e^{(\lambda_{i_0} + \epsilon) t},
$$

$$
\beta_1(\theta(t, \omega)) \geq K_1^\epsilon(\omega) \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon) t}
$$

hold for all $t \geq 0$. Use (b) to obtain a sure event $\Omega^*_5 \subseteq \Omega^*_4$ such that $\theta(t, \cdot)(\Omega^*_5) = \Omega^*_5$ for all $t \in \mathbb{R}$, and for any $0 < \epsilon < \epsilon_1$
and \( \omega \in \Omega^*_4 \), there exists \( \beta^e(\omega) > 0 \) (independent of \( x \)) with

\[
|\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \beta^e(\omega) e^{(\lambda_{i_0} + \epsilon)t} \tag{27}
\]

for all \( x \in \mathcal{S}(\omega), t \geq 0 \). Fix \( t \geq 0, \omega \in \Omega^*_5 \) and \( x \in \mathcal{S}(\omega) \). Let \( n \) be a non-negative integer. Then the cocycle property and (27) imply that

\[
|\phi(n, \phi(t, x, \omega), \theta(t, \omega)) - Y(\theta(n, \theta(t, \omega)))| = |\phi(n + t, x, \omega) - Y(\theta(n + t, \omega))| \\
\leq \beta^e(\omega) e^{(\lambda_{i_0} + \epsilon)(n+t)} \\
\leq \beta^e(\omega) e^{(\lambda_{i_0} + \epsilon)t} e^{(\lambda_{i_0} + \epsilon_1)n}.
\tag{28}
\]

If \( \omega \in \Omega^*_5 \), then it follows from (26), (27), (28) and the definition of \( \mathcal{S}(\theta(t, \omega)) \) that
there exists $\tau_1(\omega) > 0$ such that $\phi(t, x, \omega) \in \tilde{S}(\theta(t, \omega))$ for all $t \geq \tau_1(\omega)$. This proves asymptotic invariance.

• Prove (d), the existence of the local unstable manifolds $\tilde{U}(\omega)$, by running both the flow $\phi$ and the shift $\theta$ backward in time getting the cocycle $(\tilde{Z}(t, \cdot, \omega), \tilde{\theta}(t, \omega), t \geq 0)$:

$$\tilde{\phi}(t, x, \omega) := \phi(-t, x, \omega), \quad \tilde{Z}(t, x, \omega) := Z(-t, x, \omega),$$

$$\tilde{\theta}(t, \omega) := \theta(-t, \omega)$$

for all $t \geq 0, \omega \in \Omega$. The linearized flow $(D_2\tilde{\phi}(t, Y(\omega), \omega), \tilde{\theta}(t, \omega), t \geq 0)$ is an $L(\mathbb{R}^d)$-valued perfect cocycle with a non-random finite Lyapunov spectrum $\{-\lambda_1 < -\lambda_2 < \cdots < -\lambda_i < -\lambda_{i+1} < \cdots < -\lambda_m\}$ where $\{\lambda_m < \cdots < \lambda_1 < \lambda_2 < \cdots \}$. 
\[ \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1 \] is the Lyapunov spectrum of the forward linearized flow \((D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)\). Apply first part of the proof to get stable manifolds for the backward flow \(\tilde{\phi}\) satisfying assertions (a), (b), (c). This gives unstable manifolds for the original flow \(\phi\), and (d), (e), (f) automatically hold.

- Measurability of the stable manifolds follows from the representations:

\[
\tilde{S}(\omega) = Y(\omega) + \tilde{S}_d(\omega) \tag{29}
\]

\[
\tilde{S}_d(\omega) = \lim_{n \to \infty} \tilde{B}(0, \rho_1(\omega)) \cap \bigcap_{i=1}^{n} f_i(\cdot, \omega)^{-1}(\tilde{B}(0, 1)) \tag{30}
\]

\[
f_i(x, \omega) := \beta_1(\omega)^{-1} e^{-(\lambda_{i_0} + \epsilon_1)i} Z(i, x, \omega), \quad x \in \mathbb{R}^d, \quad \omega \in \Omega^*_1,
\]
for all integers $i \geq 0$. (Above limit is taken in the metric $d^*$ on $\mathcal{K}(\mathbb{R}^d)$. ) Use joint continuity of translation and measurability of $\gamma$, $f_i$, $\rho_1$, finite intersections and the continuity of the maps

$$\mathbb{R}^+ \ni r \mapsto \overline{B}(0, r) \in \mathcal{K}(\mathbb{R}^d).$$

$$\text{Hom}(\mathbb{R}^d) \ni f \mapsto f^{-1}(\overline{B}(0, 1)) \in \mathcal{K}(\mathbb{R}^d).$$

- For $h, g_i$ in $C^\infty_b$, can adapt above argument to give a sure event in $\mathcal{F}$, also denoted by $\Omega^*$ such that $\tilde{s}(\omega), \tilde{u}(\omega)$ are $C^\infty$ for all $\omega \in \Omega^*$.
Some Technical Lemmas

\[ \| \cdot \|_{k, \epsilon} := C^{k, \epsilon}-\text{norm on } C^{k, \epsilon} \text{ mappings } B(0, \rho) \to \mathbb{R}^d. \]

Lemma 2

Assume that \( \log^+ |Y(\cdot)| \) is integrable. Then the cocycle \( \phi \) satisfies

\[ \int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \| \phi(t_2, Y(\theta(t_1, \omega)) + (\cdot), \theta(t_1, \omega)) \|_{k, \epsilon} dP(\omega) < \infty \]

(10)

for any fixed \( 0 < T, \rho < \infty \) and any \( \epsilon \in (0, \delta) \). Furthermore, the linearized flow \( (D_2 \phi(t, Y(\omega), \omega), \theta(t, \omega)), t \geq 0, \) is an \( L(\mathbb{R}^d) \)-valued perfect cocycle and

\[ \int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \| D_2 \phi(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega)) \|_{L(\mathbb{R}^d)} dP(\omega) < \infty \]

(11)
for any fixed $0 < T < \infty$. The forward cocycle 
$(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t > 0)$ has a non-random finite Lyapunov spectrum \{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}. Each Lyapunov exponent $\lambda_i$ has a non-random multiplicity $q_i$, $1 \leq i \leq m$, and $\sum_{i=1}^{m} q_i = d$. The backward linearized cocycle $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t < 0)$, admits a “backward” non-random finite Lyapunov spectrum:

$$\lim_{t \to -\infty} \frac{1}{t} \log |D_2\phi(t, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbb{R}^d),$$

taking values in \{-\lambda_i\}_{i=1}^{m}$ with non-random multiplicities $q_i$, $1 \leq i \leq m$, and $\sum_{i=1}^{m} q_i = d$. 

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The Auxiliary Cocycle

To apply Ruelle's discrete non-linear ergodic theorem ([Ru.1], Theorem 5.1, p. 292), introduce the following auxiliary cocycle $Z : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$. This a "centering" of the flow $\phi$ about the stationary solution:

$$Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega))$$

(16)

for $t \in \mathbb{R}, x \in \mathbb{R}^d, \omega \in \Omega$.

Lemma 3

$(Z, \theta)$ is a perfect cocycle on $\mathbb{R}^d$ and $Z(t, 0, \omega) = 0$

for all $t \in \mathbb{R}$, and all $\omega \in \Omega$.  

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The proof of the local stable-manifold theorem (Theorem 1) uses a discretization argument that requires the following lemma.

**Lemma 4**

Suppose that \( \log^+ |Y(\cdot)| \) is integrable. Then there is a sure event \( \Omega_3 \in \mathcal{F} \) with the following properties:

(i) \( \theta(t, \cdot)(\Omega_3) = \Omega_3 \) for all \( t \in \mathbb{R} \),

(ii) For every \( \omega \in \Omega_3 \) and any \( x \in \mathbb{R}^d \), the statement

\[
\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| < 0
\]  

implies

\[
\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| = \limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)|.
\]  

(18)
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DYNAMICS
OF
STOCHASTIC SYSTEMS
WITH MEMORY

Oslo, Norway : August 12, 2000

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Deterministic ODE’s: Stable Manifolds

ODE on $\mathbb{R}^d$:

$$dx(t) = h(x(t)) \, dt \quad (ODE)$$

driven by a vector field $h : \mathbb{R}^d \to \mathbb{R}^d$, $C^k$; viz. all derivatives $D^j h, 1 \leq j \leq k$, continuous and globally bounded.

Assume hyperbolic equilibrium at 0: $h(0) = 0$; $Dh(0) \in L(\mathbb{R}^d)$ has all eigenvalues off imaginary axis.

Then (ODE) has a $C^k$ flow $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ s.t.

(i) $\phi(\cdot, x) = \text{unique solution of (ODE) through } x \in \mathbb{R}^d$.

(ii) $\phi(t, 0) = 0, t \in \mathbb{R}$.

(iii) Group property:

$$\phi(t_1 + t_2, \cdot) = \phi(t_2, \cdot) \circ \phi(t_1, \cdot), \quad t_1, t_2 \in \mathbb{R}$$

(iv) Local flow-invariant stable/unstable $C^k$ manifolds in a neighborhood of 0.

Properties (i)-(iv) are “generic” among all vector fields.
The Flow
Local Stable/Unstable Manifolds

\[ \phi(t, \cdot) \]

\[ \mathbb{R} \quad \mathbb{R}^d \]

\[ 0 \quad t \]

\[ \tilde{s} \quad \tilde{u} \]
What happens

if vector field

is noisy??
Stable Manifolds

Outline

- Smooth cocycles in Hilbert space. Stationary trajectories.
- Linearization of a cocycle along a stationary trajectory.
- Ergodic theory of cocycles in Hilbert space.
- Hyperbolicity of stationary trajectories. Lyapunov exponents.
- Cocycles generated by stochastic systems with memory. Via random diffeomorphism groups.
- Local Stable Manifold Theorem for stochastic differential equations with memory (SFDE's): Existence of smooth stable and unstable manifolds in a neighborhood of a hyperbolic stationary trajectory.
- Proof: Ruelle-Oseledec multiplicative ergodic theory+perfection techniques.
The Cocycle

\((\Omega, \mathcal{F}, P) := \text{complete probability space.}\)

\(\theta : \mathbb{R}^+ \times \Omega \to \Omega\) a \(P\)-preserving (ergodic) semigroup on \((\Omega, \mathcal{F}, P)\).

\(E := \text{real (separable) Hilbert space, norm } \| \cdot \|, \text{Borel } \sigma\)-algebra.

**Definition.**

\(k = \text{non-negative integer, } \epsilon \in (0, 1].\) A \(C^{k,\epsilon}\) perfect cocycle \((X, \theta)\) on \(E\) is a measurable random field \(X : \mathbb{R}^+ \times E \times \Omega \to E\) such that:

(i) For each \(\omega \in \Omega\), the map \(\mathbb{R}^+ \times E \ni (t, x) \mapsto X(t, x, \omega) \in E\) is continuous; for fixed \((t, \omega) \in \mathbb{R}^+ \times \Omega\), the map \(E \ni x \mapsto X(t, x, \omega) \in E\) is \(C^{k,\epsilon}\) (\(D^k X(t, x, \omega)\) is \(C^\epsilon\) in \(x\)).

(ii) \(X(t_1 + t_2, \cdot, \omega) = X(t_2, \cdot, \theta(t_1, \omega)) \circ X(t_1, \cdot, \omega)\) for all \(t_1, t_2 \in \mathbb{R}^+\), all \(\omega \in \Omega\).

(iii) \(X(0, x, \omega) = x\) for all \(x \in E, \omega \in \Omega\).
Cocycle Property

Vertical solid lines represent random fibers: copies of $E$. $(X, \theta)$ is a “vector-bundle morphism”.

![Diagram of cocycle property with arrows and variables](image)

$X(t_1, \omega)$, $X(t_2, \theta(t_1, \omega))$, $X(t_1, x, \omega)$, $X(t_1 + t_2, x, \omega)$
Definition

A random variable $Y : \Omega \rightarrow E$ is a *stationary point* for the cocycle $(X, \theta)$ if

$$X(t, Y(\omega), \omega) = Y(\theta(t, \omega))$$

(1)

for all $t \in \mathbb{R}$ and every $\omega \in \Omega$. Denote stationary trajectory (1) by $X(t, Y) = Y(\theta(t))$. 
Linearization. Hyperbolicity.

Linearize a $C^{k,c}$ cocycle $(X, \theta)$ along a stationary random point $Y$: Get an $L(E)$-valued cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega))$. (Follows from cocycle property of $X$ and chain rule.)

**Theorem.** (Oseledec-Ruelle)

Let $T : \mathbb{R}^+ \times \Omega \to L(E)$ be strongly measurable, such that $(T, \theta)$ is an $L(E)$-valued cocycle, with each $T(t, \omega)$ compact. Suppose that

$$E \sup_{0 \leq t \leq 1} \log^+ \|T(t, \cdot)\|_{L(E)} < \infty, \quad E \sup_{0 \leq t \leq 1} \log^+ \|T(1-t, \theta(t, \cdot))\|_{L(E)} < \infty.$$ 

Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0$ for all $t \in \mathbb{R}^+$, and for each $\omega \in \Omega_0$,

$$\lim_{t \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)$$

exists in the uniform operator norm. $\Lambda(\omega)$ is self-adjoint with a non-random spectrum

$$e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots$$

where the $\lambda_i$'s are distinct. Each $e^{\lambda_i}$ has a fixed finite non-random multiplicity $m_i$ and eigen-space $F_i(\omega)$, with $m_i := \dim F_i(\omega)$. Define

$$E_1(\omega) := E, \quad E_i(\omega) := \left[\bigoplus_{j=1}^{i-1} F_j(\omega)\right]^{\perp}, \quad i > 1, \quad E_\infty := \ker \Lambda(\omega).$$
Then

\[ E_\infty \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = E, \]

\[
\lim_{t \to \infty} \frac{1}{t} \log \|T(t,\omega)x\| = \begin{cases} 
\lambda_i & \text{if } x \in E_i(\omega) \setminus E_{i+1}(\omega), \\
-\infty & \text{if } x \in E_\infty(\omega),
\end{cases}
\]

and

\[ T(t,\omega)(E_i(\omega)) \subseteq E_i(\theta(t,\omega)) \]

for all \( t \geq 0, \ i \geq 1 \).

**Proof.**

Based on discrete version of Oseledec’s multiplicative ergodic theorem and the perfect ergodic theorem. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]). \( \square \)

**Lyapunov Spectrum:**

\( \{\lambda_1, \lambda_2, \lambda_3, \cdots \} := \text{Lyapunov spectrum of } (T, \theta). \)
Definition

A stationary point $Y(\omega)$ of $(X, \theta)$ is hyperbolic if the linearized cocycle $(DX(t,Y(\omega),\omega),\theta(t,\omega))$ has a non-vanishing Lyapunov spectrum \{\dots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}, viz. $\lambda_i \neq 0$ for all $i \geq 1$. 
Let $i_0 > 1$ be s.t. $\lambda_{i_0} < 0 < \lambda_{i_0-1}$.

Assume $X(t, \cdot, \omega)$ locally compact and

$$E \log^+ \sup_{0 \leq t_1, t_2 \leq r} \|D_2X(t_2, Y(\theta(t_1)), \theta(t_1))\|_{L(E)} < \infty.$$  

By Oseledec-Ruelle Theorem, there is a sequence of closed finite-codimensional (Oseledec) spaces

$$\cdots E_{i-1}(\omega) \subset E_i(\omega) \subset \cdots \subset E_2(\omega) \subset E_1(\omega) = E,$$

$$E_i(\omega) = \{ x \in E : \lim_{t \to \infty} \frac{1}{t} \log \|DX(t, Y(\omega), \omega)(x)\| \leq \lambda_i \} , \quad i \geq 1,$$

for all $\omega \in \Omega^*$, a sure event in $\mathcal{F}$ satisfying $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$.

Let $\{U(\omega), S(\omega) : \omega \in \Omega^*\}$ be the unstable and stable subspaces associated with the linearized cocycle $(DX, \theta)$ ([Mo.1], Theorem 4, Corollary 2; [M-S.1], Theorem 5.3). Then get a measurable invariant splitting

$$E = U(\omega) \oplus S(\omega), \quad \omega \in \Omega^*,$$

$$DX(t, Y(\omega), \omega)(U(\omega)) = U(\theta(t, \omega)), \quad DX(t, Y(\omega), \omega)(S(\omega)) \subseteq S(\theta(t, \omega)),$$

for all $t \geq 0$, with exponential dichotomies

$$\|DX(t, Y(\omega), \omega)(x)\| \geq \|x\|e^{\delta_1 t} \quad \text{for all} \quad t \geq \tau_1^*, \quad x \in U(\omega),$$
\[ \|DX(t, Y(\omega), \omega)(x)\| \leq \|x\|e^{-\delta t} \quad \text{for all} \quad t \geq \tau^*_2, x \in S(\omega), \]

with \( \tau^*_i = \tau^*_i(x, \omega) > 0, i = 1, 2 \), random times and \( \delta_i > 0, i = 1, 2 \), fixed.
Nonlinear Stochastic Systems with Memory

“Regular” Itô SFDE with finite memory:

\[
\begin{aligned}
dx(t) &= H(x(t), x_t) dt + \sum_{i=1}^{m} G_i(x(t)) dW_i(t), \\
(x(0), x_0) &= (v, \eta) \in M_2 := \mathbb{R}^d \times L^2([-r, 0], \mathbb{R}^d)
\end{aligned}
\]

Solution segment \( x_t(s) := x(t + s), \ t \geq 0, \ s \in [-r, 0] \).

\( m \)-dimensional Brownian motion \( W := (W_1, \ldots, W_m) \), \( W(0) = 0 \).

Ergodic Brownian shift \( \theta \) on Wiener space \( (\Omega, \mathcal{F}, P) \).

\( \bar{\mathcal{F}} := P \)-completion of \( \mathcal{F} \).

State space \( M_2 \), Hilbert with usual norm \( \| \cdot \| \).

Can allow for “smooth memory” in diffusion coefficient.

\( H : M_2 \to \mathbb{R}^d, C^{k,\delta} \), globally bounded.

\( G : \mathbb{R}^d \to L(\mathbb{R}^m, \mathbb{R}^d), C^{k+1,\delta}_b; \ G := (G_1, \ldots, G_m) \).

\( B((v, \eta), \rho) \) open ball, radius \( \rho \), center \( (v, \eta) \in M_2 \);

\( \bar{B}((v, \eta), \rho) \) closed ball.

Then (I) has a stochastic semiflow \( X : \mathbb{R}^+ \times M_2 \times \Omega \to M_2 \)
with \( X(t, (v, \eta), \cdot) = (x(t), x_t) \). \( X \) is \( C^{k,\epsilon} \) for any \( \epsilon \in (0, \delta) \), takes
bounded sets into relatively compact sets in $M_2$. $(X, \theta)$ is a perfect cocycle on $M_2$ ([M-S.4]).

**Theorem.** ([M-S], 1999) (Local Stable and Unstable Manifolds)

Assume smoothness hypotheses on $H$ and $G$. Let $Y : \Omega \to M_2$ be a hyperbolic stationary point of the SFDE (I) such that $E(\|Y(\cdot)\|^{\epsilon_0}) < \infty$ for some $\epsilon_0 > 0$.

Suppose the linearized cocycle $(DX(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)$ of (I) has a Lyapunov spectrum $\{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$. Define $\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all finite $\lambda_i$ are positive, set $\lambda_{i_0} = -\infty$. (This implies that $\lambda_{i_0 - 1}$ is the smallest positive Lyapunov exponent of the linearized semiflow, if at least one $\lambda_i > 0$; in case all $\lambda_i$ are negative, set $\lambda_{i_0 - 1} = \infty$.)

Fix $\epsilon_1 \in (0, -\lambda_{i_0})$ and $\epsilon_2 \in (0, \lambda_{i_0 - 1})$. Then there exist

(i) a sure event $\Omega^* \in \mathcal{F}$ with $\theta(t, \cdot)(\Omega^*) = \Omega^*$ for all $t \in \mathbb{R}$,

(ii) $\mathcal{F}$-measurable random variables $\rho_i, \beta_i : \Omega^* \to (0, 1)$, $\beta_i > \rho_i > 0$, $i = 1, 2$, such that for each $\omega \in \Omega^*$, the following is true:

There are $C^{k,\epsilon} (\epsilon \in (0, \delta))$ submanifolds $\tilde{S}(\omega), \tilde{U}(\omega)$ of $\bar{B}(Y(\omega), \rho_1(\omega))$ and $\bar{B}(Y(\omega), \rho_2(\omega))$ (resp.) with the following properties:

(a) $\tilde{S}(\omega)$ is the set of all $(v, \eta) \in \bar{B}(Y(\omega), \rho_1(\omega))$ such that

$$\|X(nr, (v, \eta), \omega) - Y(\theta(nr, \omega))\| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)nr}$$
for all integers $n \geq 0$. Furthermore,

$$\limsup_{t \to \infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega) - Y(\theta(t, \omega))\| \leq \lambda_{i_0}$$

for all $(v, \eta) \in \mathcal{S}(\omega)$. Each stable subspace $S(\omega)$ of the linearized semiflow $DX$ is tangent at $Y(\omega)$ to the submanifold $\mathcal{S}(\omega)$, viz. $T_{Y(\omega)}\mathcal{S}(\omega) = S(\omega)$. In particular, $\text{codim} \mathcal{S}(\omega) = \text{codim} S(\omega)$, is fixed and finite.

(b) $\limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup \left\{ \frac{\|X(t, (v_1, \eta_1), \omega) - X(t, (v_2, \eta_2), \omega)\|}{\|(v_1, \eta_1) - (v_2, \eta_2)\|} : (v_1, \eta_1) \neq (v_2, \eta_2), (v_1, \eta_1), (v_2, \eta_2) \in \mathcal{S}(\omega) \right\} \right] \leq \lambda_{i_0}$.

(c) (Cocycle-invariance of the stable manifolds):

There exists $\tau_1(\omega) \geq 0$ such that

$$X(t, \cdot, \omega)(\mathcal{S}(\omega)) \subseteq \mathcal{S}(\theta(t, \omega))$$

for all $t \geq \tau_1(\omega)$. Also

$$DX(t, Y(\omega), \omega)(S(\omega)) \subseteq S(\theta(t, \omega)), \quad t \geq 0.$$  

(d) $\mathcal{U}(\omega)$ is the set of all $(v, \eta) \in \mathcal{B}(Y(\omega), \rho_2(\omega))$ with the property that there is a unique “history” process $y(\cdot, \omega) : \{-nr : n \geq$
0 \to M_2$ such that $y(0, \omega) = (v, \eta)$ and for each integer $n \geq 1$, one has $X(r, y(-nr, \omega), \theta(-nr, \omega)) = y(-(n - 1)r, \omega)$ and

$$\|y(-nr, \omega) - Y(\theta(-nr, \omega))\|_{M_2} \leq \beta_2(\omega)e^{-(\lambda_{i_0} - \epsilon_2)nr}.$$ 

Furthermore, for each $(v, \eta) \in \tilde{U}(\omega)$, there is a unique continuous-time “history” process also denoted by $y(\cdot, \omega) : (-\infty, 0] \to M_2$ such that $y(0, \omega) = (v, \eta)$, $X(t, y(s, \omega), \theta(s, \omega)) = y(t + s, \omega)$ for all $s \leq 0, 0 \leq t \leq -s$, and

$$\limsup_{t \to \infty} \frac{1}{t} \log \|y(-t, \omega) - Y(\theta(-t, \omega))\| \leq -\lambda_{i_0}.$$

Each unstable subspace $U(\omega)$ of the linearized semiflow $DX$ is tangent at $Y(\omega)$ to $\tilde{U}(\omega)$, viz. $T_{Y(\omega)}\tilde{U}(\omega) = U(\omega)$. In particular, $\dim \tilde{U}(\omega)$ is finite and non-random.

(e) Let $y(\cdot, (v_i, \eta_i), \omega), i = 1, 2$, be the history processes associated with $(v_i, \eta_i) = y(0, (v_i, \eta_i), \omega) \in \tilde{U}(\omega), i = 1, 2$. Then

$$\limsup_{t \to \infty} \frac{1}{t} \log \left[ \sup \left\{ \frac{\|y(-t, (v_1, \eta_1), \omega) - y(-t, (v_2, \eta_2), \omega)\|}{\|(v_1, \eta_1) - (v_2, \eta_2)\|} : (v_1, \eta_1) \neq (v_2, \eta_2), (v_i, \eta_i) \in \tilde{U}(\omega), i = 1, 2 \right\} \right] \leq -\lambda_{i_0}.$$

(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ such that

$$\tilde{U}(\omega) \subseteq X(t, \cdot, \theta(-t, \omega))(\tilde{U}(\theta(-t, \omega))).$$
for all $t \geq \tau_2(\omega)$. Also
\[ DX(t, \cdot, \theta(-t, \omega))(U(\theta(-t, \omega))) = U(\omega), \quad t \geq 0; \]
and the restriction
\[ DX(t, \cdot, \theta(-t, \omega))|U(\theta(-t, \omega)) : U(\theta(-t, \omega)) \to U(\omega), \quad t \geq 0, \]
is a linear homeomorphism onto.

(g) The submanifolds $\tilde{U}(\omega)$ and $\tilde{S}(\omega)$ are transversal, viz.
\[ M_2 = T_{\gamma(\omega)}\tilde{U}(\omega) \oplus T_{\gamma(\omega)}\tilde{S}(\omega). \]

Assume, in addition, that $H, G$ are $C^\infty_b$. Then the local stable and unstable manifolds $\tilde{S}(\omega), \tilde{U}(\omega)$ are $C^\infty$.

Figure summarizes essential features of Stable Manifold Theorem:
Stable Manifold Theorem

\[ X(t, \cdot, \omega) \]

\[ S(\omega) \]

\[ U(\omega) \]

\[ \theta(t, \cdot) \]

\[ t > \tau_1(\omega) \]

A picture is worth a 1000 words!
Example

Affine linear sfde:
\[
\begin{align*}
 dx(t) &= H(x(t), x_t) \, dt + G \, dW(t), \quad t > 0 \\
x(0) &= v \in \mathbb{R}^d, \quad x_0 = \eta \in L^2([-r, 0], \mathbb{R}^d)
\end{align*}
\]

\[
H : M_2 \to \mathbb{R}^d \text{ continuous linear map, } G \text{ a fixed } (d \times p)\text{-matrix, and } W \text{ } p\text{-dimensional Brownian motion. Assume that the }(d \times d)\text{-matrix-valued FDE}
\[
dy(t) = H \circ (y(t), y_t) \, dt
\]

has a semiflow

\[
T_t : L(\mathbb{R}^d) \times L^2([-r, 0], L(\mathbb{R}^d)) \to L(\mathbb{R}^d) \times L^2([-r, 0], L(\mathbb{R}^d)), t \geq 0,
\]

which is uniformly asymptotically stable. Set

\[
Y := \int_{-\infty}^{0} T_{-u}(I, 0)G \, dW(u)
\]

where \( I \) is the identity \((d \times d)\)-matrix. Integration by parts and

\[
W(t, \theta(t_1, \omega)) = W(t + t_1, \omega) - W(t_1, \omega), \quad t, t_1 \in \mathbb{R},
\]

imply that \( Y \) has a measurable version satisfying (1). \( Y \) is Gaussian and thus has finite moments of all orders. See
([Mo], Pitman Books, 1984, Theorem 4.2, Corollary 4.2.1, pp. 208-217.) More generally, when $H$ is hyperbolic, one can show that a stationary point of $(I'')$ exists ([Mo]).

For general white-noise case with invariant measure, get stationary point in $M_2$ by enlarging probability space. Conversely, let $Y: \Omega \to M_2$ be a stationary point independent of the Brownian motion $W(t), t \geq 0$. Then $\rho := P \circ Y^{-1}$ (distribution of $Y$) is an invariant measure for the one-point motion (by independence of $Y$ and $W$).
Outline of Proof

• By definition, a stationary random point \( Y(\omega) \in M_2 \) is invariant under the semiflow \( X \); viz \( X(t, Y) = Y(\theta(t, \cdot)) \) for all times \( t \).

• Linearize the semiflow \( X \) along the stationary point \( Y(\omega) \) in \( M_2 \). By stationarity of \( Y \) and the cocycle property of \( X \), this gives a linear perfect cocycle \( (DX(t, Y), \theta(t, \cdot)) \) in \( L(M_2) \), where \( D = \) spatial (Fréchet) derivatives.

• Ergodicity of \( \theta \) allows for the notion of hyperbolicity of a stationary solution of (I) via Oseledec-Ruelle theorem: Use local compactness of the semiflow for times greater than the delay \( r \) ([M-S.4]), and apply multiplicative ergodic theorem to get a discrete non-random Lyapunov spectrum \( \{\lambda_i : i \geq 1\} \) for the linearized cocycle. \( Y \) is hyperbolic if \( \lambda_i \neq 0 \) for every \( i \).

• Assume that \( \|Y\|^{\epsilon_0} \) is integrable (for small \( \epsilon_0 \)). Variational method of construction of the semiflow shows that the linearized cocycle satisfies hypotheses of “perfect versions” of ergodic theorem and Kingman’s sub-additive ergodic theorem. These refined versions give
invariance of the Oseledec spaces under the continuous-time linearized cocycle. Thus the stable/unstable subspaces will serve as tangent spaces to the local stable/unstable manifolds of the non-linear semiflow $X$.

- Establish continuous-time integrability estimates on the spatial derivatives of the non-linear cocycle $X$ in a neighborhood of the stationary point $Y$. Estimates follow from the variational construction of the stochastic semiflow coupled with known global spatial estimates for finite-dimensional stochastic flows.

- Introduce the auxiliary perfect cocycle

$$Z(t, \cdot, \omega) := X(t, \cdot + Y(\omega), \omega) - Y(\theta(t, \omega)), \quad t \in \mathbb{R}^+, \omega \in \Omega.$$ 

Refine arguments in ([Ru.2], Theorems 5.1 and 6.1) to construct local stable/unstable manifolds for the discrete cocycle $(Z(nr, \cdot, \omega), \theta(nr, \omega))$ near 0 and hence (by translation) for $X(nr, \cdot, \omega)$ near $Y(\omega)$ for all $\omega$ sampled from a $\theta(t, \cdot)$-invariant sure event in $\Omega$. This is possible because of the continuous-time integrability estimates, the perfect ergodic theorem and the perfect subadditive ergodic theorem. By interpolating between delay periods of length $r$ and further refining
the arguments in [Ru.2], show that the above manifolds also serve as local stable/unstable manifolds for the continuous-time semiflow \( X \) near \( Y \).

- Final key step: Establish the asymptotic invariance of the local stable manifolds under the stochastic semiflow \( X \). Use arguments underlying the proofs of Theorems 4.1 and 5.1 in [Ru.2] and some difficult estimates using the continuous-time integrability properties, and the perfect subadditive ergodic theorem. Asymptotic invariance of the local unstable manifolds follows by employing the concept of a stochastic history process for \( X \) coupled with similar arguments to the above. Existence of history process compensates for the lack of invertibility of the semiflow.
REFERENCES


STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH CONSTRAINTS

Oslo, Norway: August 14, 2000

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Outline

- Theory of stochastic functional differential equations (SFDE's) in flat space: Itô and Nisio ([IN], Kushner ([Ku]), Mohammed ([Mo$_2$], [Mo$_3$]) and Mohammed-Scheutzow ([MoS$_1$], [MoS$_2$]).

- **Objective:** to constrain the solution to live on a smooth submanifold of Euclidean space.

- **Main difficulty:** Tangent space along a solution path is random (cf. unlike flat case).
• Difficulty resolved by pulling back the calculus on the tangent space at the starting point of the initial semimartingale using stochastic parallel transport. Get SFDE on a linear space of semimartingales with values in the tangent space at a given point on the manifold.

• Solve SFDE on flat space by Picard’s iteration method. (cf. Driver [Dr]). But two levels of randomness:
  (1) stochastic parallel transport over initial semimartingale path;
  (2) driving Brownian motion.
Law of solution at a given time may not be absolutely continuous with respect to law of initial semimartingale.

- Example of SDDE on the manifold with a type of Markov property in space of semimartingales.

- Regularity of solution of SDDE in initial semimartingale: stochastic Chen-Souriau calculus (Léandre [Le₂], [Le₃]). Requires Fréchet topology on semimartingales.
The Existence Theorem

Notation:

$M$ smooth compact Riemannian manifold, dimension $d$.

Delay $\delta > 0$, $T > 0$.

$(\Omega, \mathcal{F}_t, t \geq -\delta, P)$ filtered probability space—usual conditions.

$W : [-\delta, \infty) \times \Omega \to \mathbb{R}^p$ Brownian motion on $(\Omega, \mathcal{F}_t, t \geq -\delta, P)$, $W(-\delta) = 0$.

($p = 1$ for simplicity.)
\(N\) any smooth finite-dimensional Riemannian manifold; \(x \in N\).

\[S([-\delta, T], N; -\delta, x) := \text{space of all } N\text{-valued } (\mathcal{F}_t)_{t \geq -\delta}\text{-adapted continuous semimartingales}
\]

\[\gamma : [-\delta, T] \times \Omega \to N\]

with \(\gamma(-\delta) = x\).
The Itô Map:

Fix $x \in M$.

$T(M) :=$ tangent bundle over $M$.

Define the Itô map by

$S([-\delta, T], M; -\delta, x) \ni \gamma \rightarrow \tilde{\gamma} \in S([-\delta, T], T_x(M); -\delta, 0)\$

$$d\tilde{\gamma}(t) = \tau_{t,-\delta}(\gamma) \circ d\gamma(t)$$

$$\tilde{\gamma}(-\delta) = 0$$

(Stratonovich).

$\tau_{t,-\delta}(\gamma) :=$ (stochastic) parallel transport from $x = \gamma(-\delta)$ to $\gamma(t)$ along semi-martingale $\gamma$.([E.E], [Em])

Itô map is a bijection.
\( \tilde{S}_2^T := \) Hilbert space of all semimartingales \( \tilde{\gamma} \in \mathcal{S}([-\delta,T]; T_x(M); -\delta,0) \) such that

\[
\tilde{\gamma}(t) = \int_{-\delta}^{t} A(s) \, dW(s) + \int_{-\delta}^{t} B(s) \, ds, \quad -\delta \leq t \leq T
\]  

and

\[
\|\tilde{\gamma}\|_2^2 := E[\int_{-\delta}^{T} |A(s)|^2 \, ds] + E[\int_{-\delta}^{T} |B(s)|^2 \, ds] < \infty
\]

A(s), B(s) \in T_x(M) adapted previsible processes-characteristics of \( \tilde{\gamma} \) (or \( \gamma \)). \( \| \cdot \|_2 \) gives slightly different topology than traditional semi-martingale topologies ([D.M]).

\( S_2^T := \) inverse image of \( \tilde{S}_2^T \) under the Itô map with induced topology.
Let $\gamma \in S^T_2$, $t \in [-\delta, T]$. Set

$$\gamma^t(s) := \gamma(s \wedge t), \quad s \in [-\delta, T].$$

Then $\circledast(\gamma^t) = (\circledast\gamma)^t$.

Evaluation map

$$e : [0, T] \times S^T_2 \to L^0(\Omega, M)$$

$$e(t, \gamma) := \gamma(t)$$

Vector bundle $L^0(\Omega, T(M))$ over $L^0(\Omega, M)$ with fiber over $Z \in L^0(\Omega, M)$ given by

$$L^0(\Omega, T(M))_Z := \{Y : Y(\omega) \in T_{Z(\omega)} M \text{ a.a. } \omega \in \Omega\}$$

$$e^*L^0(\Omega, T(M)) := \text{pull-back bundle of} \ L^0(\Omega, T(M)) \text{ over } [0, T] \times S^T_2 \text{ by } e.$$
A SFDE on $M$ is a map

$$F : [0, T] \times S_2^T \to L^0(\Omega, T(M))$$

such that $F(t, \gamma^t) \in T_{\gamma(t)}(M)$ a.s. for all $\gamma \in S_2^T$, $0 \leq t \leq T$. I.e. $F$ is a section of $e^*L^0(\Omega, T(M))$.

Consider SFDE

$$dx(t) = F(t, x^t) \circ dW(t), \quad t \geq 0 \quad \{(4)\}$$

$$x^0 = \gamma^0$$

- Pullback SFDE (4) over $T_x(M)$. Then:

$$d\tilde{x}(t) = \tau^{-1}_{t,-\delta(x^t)}(x^t) F(t, x^t) \circ dW(t)$$

$$= \tilde{F}(t, \tilde{x}^t) \circ dW(t), \quad t \geq 0 \quad \{(5)\}$$

$$\tilde{x}^0 = \tilde{\gamma}^0$$
\((t, \tilde{\gamma}) \mapsto \tilde{F}(t, \tilde{\gamma}) := \tau_{t,-\delta}^{-1}(\gamma)F(t, \gamma)\) can be viewed as a functional

\[
[0, T] \times \tilde{S}_2^T \to L^0(\Omega, T_x(M))
\]
on the flat space \(\tilde{S}_2^T\),

- Use Stratonovich correction \(\Delta \tilde{F}(t, \tilde{\gamma}^t)\) and impose “boundedness” and “Lipschitz condition” on \(F\) in terms of \(\tilde{F}\) to get existence and uniqueness:
Hypothesis (H):

(i) “Boundedeness”. There exists a deterministic constant $C_1$ such that

$$|\tilde{F}(t, \tilde{\gamma}^t)| + |\Delta \tilde{F}(t, \tilde{\gamma}^t)| < C_1 < \infty, \text{ a.s.} \quad (6)$$

for all $(t, \tilde{\gamma}) \in [0, T] \times \tilde{S}_T$.

(ii) “Local Lipschitz property”. Suppose $\tilde{\gamma}, \tilde{\gamma}' \in S^T_2$ have characteristics $(A(.), B(.))$ and $(A'(.), B'(.))$ respectively which are a.s. bounded by a deterministic constant $R$. Then

$$E[|\tilde{F}(t, \tilde{\gamma}^t) - \tilde{F}(t, (\tilde{\gamma}')^t)|^2 + |\Delta \tilde{F}(t, \tilde{\gamma}^t) - \Delta \tilde{F}(t, (\tilde{\gamma}')^t)|^2] \leq K(R)\|\tilde{\gamma}^t - (\tilde{\gamma}')^t\|_2^2 \quad (7)$$
Examples:

1. $x :=$ a smooth section of $\mathcal{L}_k$-frame bundle $L(\mathbb{R}^k, T(M)) \to M$.

SDDE:

$$dx(t) = \tau_{t, t-\delta}(x)X(x(t-\delta)), \quad t > 0 \quad (8)$$

with

$$F(t, \gamma) := \tau_{t, t-\delta}(\gamma)X(\gamma(t-\delta));$$

and

$$\tilde{F}(t, \tilde{\gamma}) = \tau_{t-\delta, -\delta}^{-1}(\gamma)X(\gamma(t-\delta)). \quad (8')$$

$\tilde{F}$ satisfies (H)(i) because parallel transport is a rotation and $M$ is compact.
2. $X_1, X_2 := \text{smooth sections of } k\text{-frame bundle } L(\mathbb{R}^k, T(M)) \to M$.

SFDE:

$$dx_{c,t} = \left\{ \int_{t-\delta}^{t} \tau_{t,s}(x_{c,\cdot})X_1(x_{c,s})ds + X_2(x_{c,t}) \right\} \circ dw_t,$$

(9)

for $0 < t < T$.

For (H)(ii) embed $M$ (isometrically) into $\mathbb{R}^{d'}$ and extend the Riemannian structure over $\mathbb{R}^{d'}$: the Riemannian metric has bounded derivatives of all orders and is uniformly non-degenerate. Extend the Levi-Civita connection over $M$ to a connection which preserves the metric over $\mathbb{R}^{d'}$ on the trivial tangent bundle of $\mathbb{R}^{d'}$ with
Christoffel symbols having bounded derivatives of all order. The pair \((\gamma(t),\tau_t,-\delta)\) corresponds to a process \(\hat{x}(t) \in R^{d' \times R^{d' \times d'}}\) which solves the Stratonovitch SDE:

\[
d\hat{x}(t) = \hat{Z}(\hat{x}(t)) \circ A(t) \, dW(t) + \hat{Z}(\hat{x}(t)) B(t) \, dt
\]

\(\hat{x}(-\delta) = (x, Id_{Tx(M)})\)

\(\hat{Z}\) is smooth (and hence has derivatives of all orders bounded over the range of existence of \(\hat{x}\)).

(10) in Itô form:

\[
d\hat{x}(t) = \hat{Z}(\hat{x}(t)) A(t) \, dW(t) + \hat{Y}(\hat{x}(t)) A(t)^2 \, dt
\]

\(\hat{Z}(\hat{x}(t)) B(t) \, dt\)

\(\hat{Z}\) is smooth (and hence has derivatives of all orders bounded over the range of existence of \(\hat{x}\)).
In (11), \( A(t) \in T_x(M) \), but we consider the one-dimensional case \( d = 1 \) for simplicity.

\( \hat{y} \) satisfies same hypotheses as the vector field \( \hat{z} \).

\( \hat{x}(A, B) \) denotes dependence of \( \hat{x} \) on \( A \) and \( B \).

**Lemma 1.**

Suppose

\[
|A(t)| + |B(t)| + |A'(t)| + |B'(t)| \leq R,
\]

a.s. for all \( t \in [-\delta, T] \) and some deterministic \( R > 0 \).
Then there exists a constant $K(R) > 0$ such that:

$$E\left[ \sup_{-\delta \leq s \leq t} |\hat{x}(A, B)(s) - \hat{x}(A', B')(s)|^2 \right]$$

$$\leq K(R)E\left[ \int_{-\delta}^{t} (|A(s) - A'(s)|^2 + |B(s) - B'(s)|^2) \, ds \right]$$

(12)

Proof.

Follows from (11) by Burkholder’s inequality and Gronwall’s lemma.

Put $t = 0$ in Lemma to show that SFDE’s (8) and (9) satisfy (H)(ii).

Theorem 1.

Assume hypotheses (H).

Suppose that $\gamma^0 \in S_2^0$ has characteristics $(A(t), B(t))$, $t \in [-\delta, 0]$, a.s. bounded by a deterministic constant $C > 0$. 

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Then the SFDE (4) has a unique global solution $x$ such that $x|[-\delta, T] \in S^T_2$ for every $T > 0$.

**Proof.**

Sufficient to prove theorem for the SFDE (5) in flat space.

Define $\tilde{x}^n$ inductively:

$$
\begin{aligned}
&dx^{n+1}(t) = \tilde{F}(t, \tilde{x}^n, t) dW(t) + \Delta \tilde{F}(t, \tilde{x}^n, t) dt, \quad t \geq 0 \\
&\tilde{x}^{n+1,0} = \tilde{\gamma}^0
\end{aligned}
$$

By (H)(i),(ii),

$$
||\tilde{x}^{n+1, t} - \tilde{x}^n, t||_2^2 \leq C \int_0^t ||\tilde{x}^{n, s} - \tilde{x}^{n-1, s}||_2^2 ds
$$

By induction:

$$
||\tilde{x}^{n+1, t} - \tilde{x}^n, t||_2^2 \leq \frac{C^{n+1}t^n}{n!}
$$
This gives existence.

For uniqueness, take two solutions $\tilde{x}^1, \tilde{x}^2$ of (5). By (H)(i), their characteristics are a.s. bounded. Then

$$
\begin{align*}
    d\tilde{x}^1(t) &= \tilde{F}(t, \tilde{x}^{1,t}) \, dW(t) + \Delta \tilde{F}(t, \tilde{x}^{1,t}) \, dt \\
    d\tilde{x}^2(t) &= \tilde{F}(t, \tilde{x}^{2,t}) \, dW(t) + \Delta \tilde{F}(t, \tilde{x}^{2,t}) \, dt \\
\end{align*}
$$

(16)

$$\tilde{x}^{1,0} = \tilde{x}^{2,0} = \tilde{\gamma}^0$$

implies

$$
\|\tilde{x}^{1,t} - \tilde{x}^{2,t}\|_2^2 \leq C \int_0^t \|\tilde{x}^{1,s} - \tilde{x}^{2,s}\|_2^2 \, ds
$$

(17)

Hence $\|\tilde{x}^{1,t} - \tilde{x}^{2,t}\|_2^2 = 0$. □
Under the Delay Condition:

\[ \tilde{F}(t, \tilde{\gamma}^t) = \tilde{F}(t, \tilde{\gamma}^{t-\delta}) \]  

the Stratonovich equation (5) now becomes also the Itô equation:

\[
\begin{align*}
\dot{x}(t) &= \tilde{F}(t, \tilde{x}^{(t-\delta)}) \, dW(t) \\
\tilde{x}^0 &= \tilde{\gamma}^0
\end{align*}
\]

Existence and uniqueness hold by forward steps of length \( \delta \).
Continuous dependence on initial process:

**Theorem 2.**

Assume hypotheses (H). Let $\mathcal{B}^T \subset S_2^T$ be the family of all $\gamma \in S_2^T$ with characteristics $(A, B)$ a.s. uniformly bounded on $[-\delta, 0]$ by a deterministic constant. Denote by $x(\gamma^0)$ the unique solution of SFDE (4) with initial semimartingale $\gamma^0 \in \mathcal{B}^0$. Then the mapping

$$\mathcal{B}^0 \ni \gamma^0 \mapsto x(\gamma^0) \in \mathcal{B}^T$$

is continuous.
Proof.

Let \( \gamma^0, (\gamma')^0 \) have characteristics \((A, B), (A', B')\) uniformly bounded on \([-\delta, 0]\) by a deterministic constant. Let \( \tilde{x}(A, B) \) and \( \tilde{x}(A', B') \) be corresponding solutions of (5). By Burkholder’s inequality and (H)(ii):

\[
\|\tilde{x}^t(A, B) - \tilde{x}^t(A', B')\|_2^2 \\
\leq \|\gamma^0 - (\gamma')^0\|_2^2 + K \int_0^t \|\tilde{x}^s(A, B) - \tilde{x}^s(A', B')\|_2^2 ds
\]

(18)

By Gronwall’s lemma:

\[
\|\tilde{x}(A, B) - \tilde{x}(A', B')\|_2^2 \leq C\|\gamma^0 - (\gamma')^0\|_2^2
\]

(19)

\[\square\]
Example-Markov Behavior.

Consider the SDDE:

\[
\begin{align*}
    dx(t) &= \tau_{t;t-\delta}(x)X(x(t-\delta)) \, dW(t) \\
    x^0 &= \gamma^0,
\end{align*}
\]  

(20)

with \( \gamma^0(-\delta) = x \in M \).

Replace \( x \) by a random variable \( Z \in L^0(\Omega, M) \) independent of \( W(t), t \geq -\delta \).

Fix \( t_0 > 0 \). The process \( x(t), t \geq t_0 \) solves the SDDE:

\[
\begin{align*}
    dx'(t) &= \tau_{t,t-\delta}(x')X(x'(t-\delta)) \, dW(t), t \geq t_0 \\
    x'(s) &= x(s), \ s \in [t_0 - \delta, t_0]
\end{align*}
\]  

(21)
$x(t_0 - \delta)$ is independent of $dW(t), t \geq t_0 - \delta$, and parallel transport in (20) depends only on the path between $t - \delta$ and $t$.

Uniqueness implies

$$x'(t) = x(t), \ t \geq t_0.$$

For any semi-martingale $\gamma(t), t \geq -\delta$ in $M$, let $\gamma_t := \gamma|[t - \delta, t]$.

$x(\cdot)(\gamma^0)(W) :=$ solution of (20) with initial condition $\gamma^0$.

Then

$$x(t)(\gamma^0)(W) = x(t - t')(x_{t'}(\gamma^0))(W(t' + \cdot)), t \geq t' \quad (22)$$

$W(t' + \cdot) :=$ Brownian shift

$$s \mapsto W(t' + s) - W(t').$$
Differentiability in Chen-Souriau Sense:

Consider family of SDDE’s:

\[
\begin{align*}
\frac{dx(t)(u)}{dt} &= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta)(u)) \circ dW(t), \ t \geq 0 \\
x^0(u) &= \gamma^0(u)
\end{align*}
\]  

(23)

parametrized by \( u \in U \), open subset of \( \mathbb{R}^n \).

Embed \( M \) into \( \mathbb{R}^{d'} \).

Seek differentiability of \( x(t)(u) \) in \( u \). Can use Kolmogorov’s lemma, Sobolev’s imbedding theorem because \( u \) is finite-dimensional.

Flat version of (23) given by SDDE \( (8') \) with an added parameter \( u \).
For a parametrized semimartingale $\gamma(u)$ on $M$, the couple

$$(\gamma(u), \tau_t, -\delta(\gamma(u))) = \hat{x}_t$$

satisfies an Itô SDE depending on the parameter $u$:

$$d\hat{x}(t) = \hat{Z}(\hat{x}(t)) A(u)(t) dW(t) + \hat{Y}(\hat{x}(t)) A(u)(t)^2 dt$$

$$+ \hat{Z}(\hat{x}(t)) B(u)(t) dt$$

(24)

$\hat{Z}$ and $\hat{Y}$ are smooth.

Introduce family of norms:

$$||\tilde{\gamma}||_p^p := E[\int_{-\delta}^T |A(s)|^p ds + \int_{-\delta}^T |B(s)|^p ds]$$

(25)

on the space $\tilde{S}_T^\infty$ of all semimartingales

$$\tilde{\gamma} \in \mathcal{S}([-\delta, T], T_x(M); -\delta, 0)$$

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where \( \tilde{\gamma}(t) = \int_{-\delta}^{t} A(s) \, dW(s) + \int_{-\delta}^{t} B(s) \, ds, \ 0 \leq t \leq T \)
and \( \|\tilde{\gamma}\|_p \) is finite for every \( p \geq 1 \).

Suppose \( A(u)(\cdot) \) and \( B(u)(\cdot) \) are bounded by a deterministic constant \( C \) independent of \( u \), and

\[
 u \mapsto (A(u)(\cdot), B(u)(\cdot))
\]

is Fréchet smooth in the Fréchet space \( \mathcal{S}_T^{\infty} \) defined by the family of norms \( \| \cdot \|_p. \)
Theorem 3.

Consider the parametrized SDDE's:

\[
\begin{align*}
\frac{dx(t)(u)}{dt} &= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta)(u)) \circ dW(t), \quad t \geq 0, \\
x^0(u) &= \gamma^0(u)
\end{align*}
\]

where \( X \) is smooth and \( \gamma^0(u) \) is smooth in \( u \) as above. Then \( x(t)(u) \) has a version which is a.s. smooth in \( u \).

Theorem also holds if noise has a smooth parameter \( u \):

\[
\frac{dx(t)(u)}{dt} = \tau_{t,t-\delta}(x^t(u))X(x(t-\delta))(\circ A(u)(t) \, dW(t) + B(u)(t) \, dt)
\]

with initial conditions \( x^0(u) = \gamma^0(u) \).
Proof of Theorem 3-Outline.

\[ \alpha := (\alpha_1, \cdots, \alpha_k) \text{ multi-index.} \]

\[ D^\alpha := \text{partial derivatives of order} \]

\[ |\alpha| := \sum_{i=1}^{k} \alpha_i. \]

- For a parametrized semimartingale \( \gamma(u) \) on \( M \), the couple

\[ (\gamma(u), \tau_{t,-\delta}(\gamma(u))) := \hat{x}(t)(u) \]

satisfies an Itô SDE depending on the parameter \( u \):

\[ d\hat{x}(t)(u) = \hat{Z}(\hat{x}(t)(u))A(u)(t)\,dW(t) \]

\[ + \hat{Y}(\hat{x}(t)(u))A(u)(t)^2\,dt + \hat{Z}(\hat{x}(t)(u))B(u)(t)\,dt \]

Since the inverse of the parallel transport is bounded, then \( \hat{Z} \) and \( \hat{Y} \) have
bounded derivatives of all orders. If \( \gamma(u) \in \mathcal{S}_\infty^T \) has a.s. bounded characteristics \((A(u), B(u))\) which are smooth in \( u \) into the Fréchet space \( \mathcal{S}_\infty^T \), then the pair \( \hat{x}(t)(u) := (\gamma(u), \tau_{t,-\delta}^{-1}(\gamma(u))) \) has characteristics Fréchet smooth in \( u \). Follows by differentiating above SDE and applying Burkholder's inequality and Gronwall's lemma.

- Write the SDDE

\[
\begin{align*}
\frac{dx(t)(u)}{dt} &= \tau_{t,t-\delta}(x^t(u)) X(x(t-\delta)(u)) \circ dW(t), \quad t \geq 0, \\
\quad x^0(u) &= \gamma^0(u)
\end{align*}
\]

in the form:

\[
\begin{align*}
\frac{d\tilde{x}(t)(u)}{dt} &= g(\tilde{x}(t-\delta)(u)) dW(t) \\
\quad \tilde{x}^0(u) &= \tilde{\gamma}^0(u)
\end{align*}
\]
where \( \hat{x}(t) := (x(t), \tau_{t, -\delta}^{-1}(x)) \),
\( g(y, z) := zX(y) \), and \( z \) represents parallel transport (orthogonal matrix), \( y \in M \).
Then \( g \) is bounded and has bounded derivatives of all orders.
\( \tilde{\gamma}(t)^0(u) := \int_{-\delta}^{t} A_s^0(u) dw_s + \int_{-\delta}^{t} B_s^0 ds \) for \( t < 0 \)
where \( A^0(u)(\cdot) \) and \( B^0(u)(\cdot) \) are bounded independently of \( u \) and differentiable in \( u \) in all the \( L^p \) semi-martingale norms \( \| \cdot \|_p \).
Hence \( \tilde{\gamma}(t)^0(u) \) has \( u \)-derivatives of all orders in all \( L^p \) semi-martingale norms.
Follows from Kolmogorov’s lemma and Burkholder’s inequality.
• For $t \in [0, \delta]$, $\tilde{x}(t)(u)$ is a.s. differentiable in $u$ and

$$dD^\alpha \tilde{x}(t)(u)$$

$$= Dg(\tilde{x}(t - \delta)(u))D^\alpha \tilde{x}(t - \delta)(u) \, dW(t) + l.o.$$  

where $l.o.$ are terms containing lower-order derivatives of $\tilde{x}(t)(u)$.

• Get estimate:

$$\sup_{u \in U} \|D^\alpha \tilde{x}(\cdot)(u)\|_p \leq C(p, \alpha)$$

• Use forward steps of length $\delta$ to prove that $\tilde{x}(t)(u)$ has a smooth version in $u$ for all $t \in [0, T]$.  

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REFERENCES


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