The Stable Manifold Theorem for SDE's (Probability Seminar, University of California, Irvine)

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THE STABLE MANIFOLD THEOREM

FOR SDE’S


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Outline

• Formulate a *Local Stable Manifold Theorem* for stochastic differential equations (SDE’s) (Stratonovich or Itô SDE’s-driven by Brownian motion or spatial Kunita-type semimartingales with stationary ergodic increments.)

• Start with the existence of a stochastic flow for SDE.

• Concept of a hyperbolic stationary trajectory. The stationary trajectory is a solution of the forward /backward anticipating SDE for all time (Stratonovich case).

• Existence of a stationary random family of asymptotically invariant stable and unstable manifolds within a stationary neighborhood of the hyperbolic stationary solution.

• The stable and unstable manifolds are dynamically characterized using forward and backward solutions of anticipating versions of the (Stratonovich) SDE.

• Proof based on Ruelle-Oseledec (non-linear) multiplicative ergodic theory and anticipating stochastic calculus.
Formulation of The Theorem

Stratonovich SDE

\[ dx(t) = h(x(t)) \, dt + \sum_{i=1}^{m} g_i(x(t)) \circ dW_i(t), \]  

(I)

on \( \mathbb{R}^d \) driven by \( m \)-dimensional Brownian motion \( W := (W_1, \cdots, W_m) \).

\( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P) := \) canonical filtered Wiener space.

\( \Omega := \) space of all continuous paths \( \omega : \mathbb{R} \to \mathbb{R}^m, \omega(0) = 0 \), in Euclidean space \( \mathbb{R}^m \), with compact open topology;

\( \mathcal{F} := \) Borel \( \sigma \)-field of \( \Omega \);

\( \mathcal{F}_t := \) sub-\( \sigma \)-field of \( \mathcal{F} \) generated by the evaluations \( \omega \to \omega(u), \ u \leq t, \ t \in \mathbb{R} \).

\( P := \) Wiener measure on \( \Omega \).

\( h, g_i : \mathbb{R}^d \to \mathbb{R}^d, 1 \leq i \leq m, \) vector fields on \( \mathbb{R}^d \). For some \( k \geq 1, \delta \in (0,1), h \) is \( C^{k,\delta}_b \), viz. \( h \) has all derivatives \( D^j h, 1 \leq j \leq k \), continuous and globally bounded, \( D^k h \) Hölder continuous with exponent \( \delta \).

\( g_i, 1 \leq i \leq m, \) globally bounded and in \( C^{k+1,\delta}_b \).

\( \theta : \mathbb{R} \times \Omega \to \Omega \) is the (ergodic) Brownian shift

\[ \theta(t, \omega)(s) := \omega(t+s) - \omega(t), \quad t, s \in \mathbb{R}, \omega \in \Omega. \]
Let $\phi : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ be the stochastic flow generated by (I) $(\phi(t, \cdot, \omega) = [\phi(-t, \cdot, \theta(t, \omega))]^{-1}, t < 0)$. Then $\phi$ is a perfect cocycle:

$$
\phi(t + s, \cdot, \omega) = \phi(t, \cdot, \theta(s, \omega)) \circ \phi(s, \cdot, \omega),
$$

for all $s, t \in \mathbb{R}$ and all $\omega \in \Omega$ ([I-W], [A-S], [A]).

Figure illustrates the cocycle property. Vertical solid lines represent random fibers consisting of copies of $\mathbb{R}^d$. $(\phi, \theta)$ is a “random vector-bundle morphism” over the “base” probability space $\Omega$. 
The Cocycle

$\Omega \xrightarrow{\phi(t_1, \cdot, \omega)} R^d \xrightarrow{\phi(t_2, \cdot, \theta(t_1, \omega))} R^d \xrightarrow{\phi(t_1 + t_2, \cdot, \omega)} R^d$

$\Omega \xleftarrow{\theta(t_1, \cdot)} \xrightarrow{\theta(t_2, \cdot)} \xrightarrow{\theta(t_1 + t_2, \cdot)}$

$t = 0 \quad \omega \quad \theta(t_1, \omega) \quad \theta(t_1 + t_2, \omega)

\begin{align*}
\phi(t_1, x, \omega) \\
\phi(t_1, x, \omega)
\end{align*}
Definition

The SDE (I) has a stationary trajectory if there exists an \( \mathcal{F} \)-measurable random variable \( Y : \Omega \rightarrow \mathbb{R}^d \) such that

\[
\phi(t, Y(\omega), \omega) = Y(\theta(t, \omega))
\]

for all \( t \in \mathbb{R} \) and every \( \omega \in \Omega \). Denote stationary trajectory (1) by \( \phi(t, Y) = Y((\theta(t)) \).

If (1) holds on a sure event \( \Omega_t \) that may depend on \( t \), then there are “perfect” versions of the stationary random variable \( Y \) and of the flow \( \phi \) such that (1) and the cocycle property hold for all \( \omega \in \Omega \) ([Sc]).

Let \( \phi(t, Y) \) be a stationary solution of (I). Cocycle property of \( \phi \) implies that the linearization

\[
(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))
\]

along the stationary solution is also a \( d \times d \)-matrix-valued cocycle. Using Kolmogorov’s theorem, the random variables

\[
\sup_{x \in \mathbb{R}^d} \frac{|D_2\phi(t, x)|}{(1 + |x| \gamma)}, \gamma > 0,
\]
have moments of all orders. If $E \log^+ |Y| < \infty$, then $E \log^+ |D_2\phi(1,Y)| < \infty$. Apply Oseledec’s Theorem to get a non-random finite Lyapunov spectrum:

$$\lim_{n \to \infty} \frac{1}{n} \log |D_2\phi(n,Y(\omega),\omega)(v(\omega))|, \quad v \in L^0(\Omega,\mathbb{R}^d).$$

Spectrum takes finitely many values $\{\lambda_i\}_{i=1}^p$ with non-random multiplicities $q_i$, $1 \leq i \leq p$, and $\sum_{i=1}^p q_i = d$ ([Ru.1], Theorem I.6).

**Definition**

Stationary trajectory $\phi(t,Y)$ of (I) is hyperbolic if $E \log^+ |Y(\cdot)| < \infty$, and if the linearized cocycle $(D_2\phi(n,Y(\omega),\omega),\theta(n,\omega))$ has a non-vanishing Lyapunov spectrum

$$\{\lambda_p < \cdots < \lambda_{i_0+1} < \lambda_{i_0} < 0 < \lambda_{i_0-1} < \cdots < \lambda_2 < \lambda_1\}$$

i.e. $\lambda_i \neq 0$ for all $1 \leq i \leq p$.

Define $\lambda_{i_0} := \max\{\lambda_i : \lambda_i < 0\}$ if at least one $\lambda_i < 0$. If all $\lambda_i > 0$, set $\lambda_{i_0} = -\infty$. (This implies that $\lambda_{i_0-1}$ is the smallest positive Lyapunov exponent of the linearized flow, if at least one $\lambda_i > 0$; in case all $\lambda_i$ are negative, set $\lambda_{i_0-1} = \infty$.)
Let $\rho \in \mathbb{R}^+, x \in \mathbb{R}^d$.

$B(x, \rho) := \text{open ball in } \mathbb{R}^d, \text{ center } x \text{ and radius } \rho$;

$\bar{B}(x, \rho) := \text{corresponding closed ball};$

$\mathcal{C}(\mathbb{R}^d) := \text{the class of all non-empty compact subsets of } \mathbb{R}^d$ with Hausdorff metric $d^*$:

$d^*(A_1, A_2) := \sup\{d(x, A_1) : x \in A_2\} \vee \sup\{d(y, A_2) : y \in A_1\} \text{ where } A_1, A_2 \in \mathcal{C}(\mathbb{R}^d)$;

$d(x, A_i) := \inf\{|x - y| : y \in A_i\}, \ x \in \mathbb{R}^d, \ i = 1, 2$;

$\mathcal{B}(\mathcal{C}(\mathbb{R}^d)) := \text{Borel } \sigma\text{-algebra on } \mathcal{C}(\mathbb{R}^d)$ with respect to the metric $d^*$. 
**Theorem 1** (The Stable Manifold Theorem) (M. + Scheutzow, 1997)

Assume that the coefficients of SDE (I) satisfy the given hypotheses. Suppose \( \phi(t,Y) \) is a hyperbolic stationary trajectory of (I) with \( E \log^+ |Y| < \infty \).

Fix \( \epsilon_1 \in (0, -\lambda_{i_0}) \) and \( \epsilon_2 \in (0, \lambda_{i_0 - 1}) \). Then there exist

(i) a sure event \( \Omega^* \in \mathcal{F} \) with \( \theta(t, \cdot)(\Omega^*) = \Omega^* \) for all \( t \in \mathbb{R} \),

(ii) \( \mathcal{F} \)-measurable random variables \( \rho_i, \beta_i : \Omega^* \to [0, \infty), \beta_i > \rho_i > 0, i = 1, 2 \), such that for each \( \omega \in \Omega^* \), the following is true:

There are \( C^{k,\epsilon} (\epsilon \in (0, \delta)) \) submanifolds \( \tilde{S}(\omega), \tilde{U}(\omega) \) of \( \bar{B}(Y(\omega), \rho_1(\omega)) \) and \( \bar{B}(Y(\omega), \rho_2(\omega)) \) (resp.) with the following properties:

(a) \( \tilde{S}(\omega) \) is the set of all \( x \in \bar{B}(Y(\omega), \rho_1(\omega)) \) such that

\[
|\phi(n, x, \omega) - Y(\theta(n, \omega))| \leq \beta_1(\omega) e^{(\lambda_{i_0} + \epsilon_1)n}
\]

for all integers \( n \geq 0 \). Furthermore,

\[
\limsup_{t \to \infty} \frac{1}{t} \log |\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \lambda_{i_0} \quad (2)
\]

for all \( x \in \tilde{S}(\omega) \). Each stable subspace \( S(\omega) \) of the linearized flow \( D_2\phi \) is tangent at \( Y(\omega) \) to the submanifold \( \tilde{S}(\omega) \), viz. \( T_{Y(\omega)}\tilde{S}(\omega) = S(\omega) \). In particular, \( \dim \tilde{S}(\omega) = \dim S(\omega) \) and is non-random.

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(b) \[\limsup_{t \to \infty} \frac{1}{t} \log \left( \sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{S}^i(\omega)}} \left\{ \frac{|\phi(t, x_1, \omega) - \phi(t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right) \leq \lambda_{i_0}.\]

(c) (Cocycle-invariance of the stable manifolds):

There exists \(\tau_1(\omega) \geq 0\) such that

\[\phi(t, \cdot, \omega)(\tilde{S}(\omega)) \subseteq \tilde{S}(\theta(t, \omega)), \quad t \geq \tau_1(\omega). \tag{3}\]

Also

\[D_2\phi(t, Y(\omega), \omega)(S(\omega)) = S(\theta(t, \omega)), \quad t \geq 0. \tag{4}\]

(d) \(\tilde{U}(\omega)\) is the set of all \(x \in \tilde{B}(Y(\omega), \rho_2(\omega))\) with the property that

\[|\phi(-n, x, \omega) - Y(\theta(-n, \omega))| \leq \beta_2(\omega) e^{(-\lambda_{i_0-1} + \epsilon_2)n} \tag{5}\]

for all integers \(n \geq 0\). Also

\[\limsup_{t \to \infty} \frac{1}{t} \log |\phi(-t, x, \omega) - Y(\theta(-t, \omega))| \leq -\lambda_{i_0-1}. \tag{6}\]

for all \(x \in \tilde{U}(\omega)\). Furthermore, the unstable subspace \(U(\omega)\) of \(D_2\phi\) is the tangent space to \(\tilde{U}(\omega)\) at \(Y(\omega)\), viz. \(T_{Y(\omega)}\tilde{U}(\omega) = U(\omega)\). In particular, \(\dim \tilde{U}(\omega) = \dim U(\omega)\) and is non-random.

(e) \[\limsup_{t \to \infty} \frac{1}{t} \log \left( \sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \tilde{U}(\omega)}} \left\{ \frac{|\phi(-t, x_1, \omega) - \phi(-t, x_2, \omega)|}{|x_1 - x_2|} \right\} \right) \leq -\lambda_{i_0-1}.\]
(f) (Cocycle-invariance of the unstable manifolds):

There exists $\tau_2(\omega) \geq 0$ such that

$$\phi(-t, \cdot, \omega)(\tilde{U}(\omega)) \subseteq \tilde{U}(\theta(-t, \omega)), \quad t \geq \tau_2(\omega). \quad (7)$$

Also

$$D_2\phi(-t, Y(\omega), \omega)(U(\omega)) = U(\theta(-t, \omega)), \quad t \geq 0. \quad (8)$$

(g) The submanifolds $\tilde{U}(\omega)$ and $\tilde{S}(\omega)$ are transversal, viz.

$$\mathbb{R}^d = T_{Y(\omega)}\tilde{U}(\omega) \oplus T_{Y(\omega)}\tilde{S}(\omega). \quad (9)$$

(h) The mappings

$$\Omega \to C(\mathbb{R}^d), \quad \Omega \to C(\mathbb{R}^d),$$

$$\omega \mapsto \tilde{S}(\omega) \quad \omega \mapsto \tilde{U}(\omega)$$

are $(\mathcal{F}, \mathcal{B}(C(\mathbb{R}^d)))$-measurable.

Assume, further, that $h, g_i, 1 \leq i \leq m$, are $C^\infty_b$. Then the local stable and unstable manifolds $\tilde{S}(\omega), \tilde{U}(\omega)$ are $C^\infty$. 
$t > \tau_1(\omega)$

A picture is worth a 1000 words!
\[
\begin{align*}
\theta(t, \omega) &
\quad \Omega \\
\tilde{S}(\theta(t, \omega)) &
\quad \mathbb{R}^d \\
\tilde{U}(\theta(t, \omega)) &
\quad \mathbb{R}^d \\
\phi(t, \omega) &
\quad \phi(-t, \cdot, \omega)
\end{align*}
\]

\[t > \tau_2(\omega)\]
Sketch of Proof

Linearization and Substitution

Assume regularity conditions on the coefficients $h, g_i$. By the Substitution Rule, $\phi(t, Y(\omega), \omega)$ is a stationary solution of the anticipating Stratonovich SDE

$$
\begin{align*}
\phi(0, Y) &= Y.
\end{align*}
$$

Linearize the SDE (I) along the stationary trajectory. By substitution, match the solution of the linearized equation with the linearized cocycle $D_2\phi(t, Y(\omega), \omega)$. Hence $D_2\phi(t, Y(\omega), \omega), t \geq 0$, solves the SDE:

$$
\begin{align*}
\phi(0, Y) &= Y. \\

D_2\phi(0, Y) &= I.
\end{align*}
$$

$I, D$ denotes spatial (Fréchet) derivatives.

Similarly, the backward trajectories

$$
\phi(t, Y), D_2\phi(t, Y), t < 0,
$$
solve the corresponding backward Stratonovich SDE’s:

\[
\begin{align*}
\frac{d\phi(t, Y)}{dt} &= -h(\phi(t, Y)) dt - \sum_{i=1}^{m} g_i(\phi(t, Y)) \circ \hat{d}W_i(t), \quad t < 0 \\
\phi(0, Y) &= Y. \\
\frac{dD_2\phi(t, Y)}{dt} &= -Dh(\phi(t, Y)) D_2\phi(t, Y) dt \\
&\quad - \sum_{i=1}^{m} Dg_i(\phi(t, Y)) D_2\phi(t, Y) \circ \hat{d}W_i(t), \quad t < 0 \\
D_2\phi(0, Y) &= I.
\end{align*}
\]

Above SDE’s (II)-(III) give dynamic characterizations of the stable and unstable manifolds.

The following lemma is used to construct the shift-invariant sure event appearing in the statement of the local stable manifold theorem. Gives “perfect versions” of the ergodic theorem and Kingman’s subadditive ergodic theorem.

**Lemma 1**

(i) Let \( h : \Omega \to \mathbb{R}^+ \) be \( \mathcal{F} \)-measurable and such that

\[
\int_{\Omega} \sup_{0 \leq u \leq 1} h(\theta(u, \omega)) \, dP(\omega) < \infty.
\]
Then there is a sure event $\Omega_1 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_1) = \Omega_1$ for all $t \in \mathbb{R}$, and

$$
\lim_{t \to \infty} \frac{1}{t} h(\theta(t, \omega)) = 0
$$

for all $\omega \in \Omega_1$.

(ii) Suppose $f : \mathbb{R}^+ \times \Omega \to \mathbb{R} \cup \{-\infty\}$ is a measurable process on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the following conditions

(a) $E \sup_{0 \leq u \leq 1} f^+(u) < \infty$, $E \sup_{0 \leq u \leq 1} f^+(1 - u, \theta(u)) < \infty$

(b) $f(t_1 + t_2, \omega) \leq f(t_1, \omega) + f(t_2, \theta(t_1, \omega))$ for all $t_1, t_2 \geq 0$ and all $\omega \in \Omega$.

Then there is sure event $\Omega_2 \in \mathcal{F}$ such that $\theta(t, \cdot)(\Omega_2) = \Omega_2$ for all $t \in \mathbb{R}$, and a fixed number $f^* \in \mathbb{R} \cup \{-\infty\}$ such that

$$
\lim_{t \to \infty} \frac{1}{t} f(t, \omega) = f^*
$$

for all $\omega \in \Omega_2$.

**Proof**

[Mo.1], Lemma 7. $\square$
Theorem 2 ([O], 1968)

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(\theta : \mathbb{R}^+ \times \Omega \to \Omega\) a measurable family of ergodic \(P\)-preserving transformations. Let \(T : \mathbb{R}^+ \times \Omega \to L(\mathbb{R}^d)\) be measurable, such that \((T, \theta)\) is an \(L(\mathbb{R}^d)\)-valued cocycle. Suppose that

\[
E \sup_{0 \leq t \leq 1} \log^+ \|T(t, \cdot)\| < \infty, \quad E \sup_{0 \leq t \leq 1} \log^+ \|T(1 - t, \theta(t, \cdot))\| < \infty.
\]

Then there is a set \(\Omega_0 \in \mathcal{F}\) of full \(P\)-measure such that \(\theta(t, \cdot)(\Omega_0) \subseteq \Omega_0\) for all \(t \in \mathbb{R}^+\), and for each \(\omega \in \Omega_0\), the limit

\[
\lim_{n \to \infty} [T(t, \omega)^* \circ T(t, \omega)]^{1/(2t)} := \Lambda(\omega)
\]

exists in the uniform operator norm. Each \(\Lambda(\omega)\) has a discrete non-random spectrum \(e^{\lambda_1} > e^{\lambda_2} > e^{\lambda_3} > \cdots > e^{\lambda_p}\)

where the \(\lambda_i\)’s are distinct. Each \(e^{\lambda_i}\) has an eigen-space \(F_i(\omega)\) and a fixed non-random multiplicity \(m_i := \dim F_i(\omega)\). Define

\[
E_1(\omega) := \mathbb{R}^d, \quad E_i(\omega) := \left[\oplus_{j=1}^{i-1} F_j(\omega)\right]^\perp, \quad 1 < i \leq p.
\]

Then

\[
E_p(\omega) \subset \cdots \subset E_{i+1}(\omega) \subset E_i(\omega) \cdots \subset E_2(\omega) \subset E_1(\omega) = \mathbb{R}^d
\]
\[
\lim_{t \to \infty} \frac{1}{t} \log \|T(t, \omega)x\| = \lambda_i(\omega), \quad \text{if} \quad x \in E_i(\omega) \setminus E_{i+1}(\omega),
\]
and
\[
T(t, \omega)(E_i(\omega)) \subseteq E_i(\theta(t, \omega))
\]
for all \( t \geq 0, \ 1 \leq i \leq p. \)

**Proof.**

Based on the discrete version of Oseledec’s multiplicative ergodic theorem and Lemma 1. ([Ru.1], I.H.E.S Publications, 1979, pp. 303-304; cf. Furstenberg & Kesten (1960), [Mo.1]), “perfect” infinite-dimensional version and application to SFDE’s. \( \square \)
**Spectral Theorem**

\[ T(t, \omega) \]

\[ E_1 = \mathbb{R}^d \]

\[ E_2(\omega) \]

\[ E_3(\omega) \]

\[ \theta(t, \cdot) \]

\[ \Omega \]

\[ \omega \]

\[ \theta(t, \omega) \]
Apply Theorem 2 with \( T(t, \omega) := D_2\phi(t, Y(\omega), \omega) \). Then linearized cocycle has random invariant stable and unstable subspaces \( \{ S(\omega), U(\omega) : \omega \in \Omega \} : \)

\[
D_2\phi(t, Y(\omega), \omega)(S(\omega)) = S(\theta(t, \omega)),
\]
\[
D_2\phi(-t, Y(\omega), \omega)(U(\omega)) = U(\theta(-t, \omega)), \quad t \geq 0.
\]

[Mo.1].
Estimates on the non-linear cocyle

**Theorem 3** (M. + Scheutzow [M-S.2])

There exists a jointly measurable modification of the trajectory random field of (I), denoted by \( \{ \phi_{s,t}(x) : -\infty < s, t < \infty, \ x \in \mathbb{R}^d \} \), with the following properties:

Define \( \phi : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d \) by

\[
\phi(t, x, \omega) := \phi_{0,t}(x, \omega), \quad x \in \mathbb{R}^d, \omega \in \Omega, t \in \mathbb{R}.
\]

Then for all \( \omega \in \Omega, \epsilon \in (0, \delta), \gamma, \rho, T > 0, 1 \leq |\alpha| \leq k \), \( \phi(t, \cdot, \omega) \) is \( C^{k, \epsilon} \), \( 0 < \epsilon < \delta \), and the quantities

\[
sup_{0 \leq s, t \leq T, x \in \mathbb{R}^d} \frac{|\phi_{s,t}(x, \omega)|}{[1 + |x| (\log^+ |x|) \gamma]}, \quad \sup_{0 \leq s, t \leq T, x \in \mathbb{R}^d} \frac{|D_x^{\alpha} \phi_{s,t}(x, \omega)|}{(1 + |x| \gamma)},
\]

\[
\sup_{x \in \mathbb{R}^d} \sup_{0 \leq s, t \leq T, 0 < |x'-x| \leq \rho} \frac{|D_x^{\alpha} \phi_{s,t}(x, \omega) - D_x^{\alpha} \phi_{s,t}(x', \omega)|}{|x - x'| \epsilon (1 + |x|)^{\gamma}},
\]

are finite. The random variables defined by the above expressions have \( p \)-th moments for all \( p \geq 1 \).
\[ \| \cdot \|_{k, \epsilon} := C^{k, \epsilon}-\text{norm on } C^{k, \epsilon} \text{ mappings } \bar{B}(0, \rho) \to \mathbb{R}^d. \]

**Lemma 2**

Assume that \( \log^+ |Y(\cdot)| \) is integrable. Then the cocycle \( \phi \) satisfies

\[
\int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \| \phi(t_2, Y(\theta(t_1, \omega)) + (\cdot), \theta(t_1, \omega)) \|_{k, \epsilon} dP(\omega) < \infty \tag{10}
\]

for any fixed \( 0 < T, \rho < \infty \) and any \( \epsilon \in (0, \delta) \). Furthermore, the linearized flow \( (D_2 \phi(t, Y(\omega), \omega), \theta(t, \omega)), \ t \geq 0 \), is an \( L(\mathbb{R}^d) \)-valued perfect cocycle and

\[
\int_{\Omega} \log^+ \sup_{-T \leq t_1, t_2 \leq T} \| D_2 \phi(t_2, Y(\theta(t_1, \omega)), \theta(t_1, \omega)) \|_{L(\mathbb{R}^d)} dP(\omega) < \infty \tag{11}
\]

for any fixed \( 0 < T < \infty \). The forward cocycle \( (D_2 \phi(t, Y(\omega), \omega), \theta(t, \omega), \ t > 0) \) has a non-random finite Lyapunov spectrum \( \{ \lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1 \} \). Each Lyapunov exponent \( \lambda_i \) has a non-random multiplicity \( q_i, 1 \leq i \leq m \), and \( \sum_{i=1}^{m} q_i = d \). The backward linearized cocycle \( (D_2 \phi(t, Y(\omega), \omega), \theta(t, \omega), t < 0) \), admits a “backward” non-random finite Lyapunov spectrum:

\[
\lim_{t \to -\infty} \frac{1}{t} \log |D_2 \phi(t, Y(\omega), \omega)(v(\omega))|, \quad v \in L^0(\Omega, \mathbb{R}^d),
\]

taking values in \( \{-\lambda_i\}_{i=1}^{m} \) with non-random multiplicities \( q_i, 1 \leq i \leq m \), and \( \sum_{i=1}^{m} q_i = d \).
The Auxiliary Cocycle

To apply Ruelle’s discrete non-linear ergodic theorem ([Ru.1], Theorem 5.1, p. 292), introduce the following auxiliary cocycle \( Z : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d \). This a “centering” of the flow \( \phi \) about the stationary solution:

\[
Z(t, x, \omega) := \phi(t, x + Y(\omega), \omega) - Y(\theta(t, \omega)) \tag{16}
\]

for \( t \in \mathbb{R}, x \in \mathbb{R}^d, \omega \in \Omega \).

**Lemma 3**

\((Z, \theta)\) is a perfect cocycle on \( \mathbb{R}^d \) and \( Z(t, 0, \omega) = 0 \) for all \( t \in \mathbb{R}, \) and all \( \omega \in \Omega \).

The proof of the local stable-manifold theorem (Theorem 1) uses a discretization argument that requires the following lemma.

**Lemma 4**

Suppose that \( \log^+ |Y(\cdot)| \) is integrable. Then there is a sure event \( \Omega_3 \in \mathcal{F} \) with the following properties:

(i) \( \theta(t, \cdot)(\Omega_3) = \Omega_3 \) for all \( t \in \mathbb{R} \),

(ii) For every \( \omega \in \Omega_3 \) and any \( x \in \mathbb{R}^d \), the statement

\[
\limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)| < 0 \tag{17}
\]

implies

\[
\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| = \limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)|. \tag{18}
\]
Ruelle’s Non-linear Ergodic Theorem

**Theorem 4** ([Ru.1], 1979)

Let $\Omega \ni \mapsto F_\omega \in C^{k,\epsilon}(\mathbb{R}^d, 0; \mathbb{R}^d, 0)$ be measurable such that $E \log^+ \|F|B(0, 1)\|_{k,\epsilon} < \infty$. Set $F^n(\omega) := F_{\theta(n-1, \omega)} \circ \cdots \circ F_{\theta(1, \omega)} \circ F_\omega$. Suppose $\lambda < 0$ is not in the spectrum of the cocycle $(DF^n(0), \theta(n, \omega))$. Then there is a sure event $\Omega_0 \in \mathcal{F}$ such that $\theta(1, \cdot)(\Omega_0) \subseteq \Omega_0$, and measurable functions $0 < \alpha(\omega) < \beta(\omega) < 1, \gamma(\omega) > 1$ with the following properties:

(a) If $\omega \in \Omega_0$, the set

$$V_\omega^\lambda := \{x \in B(0, \alpha(\omega)) : |F^n_\omega(x)| \leq \beta(\omega)e^{n\lambda} \text{ for all } n \geq 0\}$$

is a $C^{k,\epsilon}$ submanifold of $B(0, \alpha(\omega))$.

(b) If $x_1, x_2 \in V_\omega^\lambda$, then

$$|F^n_\omega(x_1) - F^n_\omega(x_2)| \leq \gamma(\omega)|x_1 - x_2|e^{n\lambda}$$

for all integers $n \geq 0$. If $\lambda' < \lambda$ and $[\lambda', \lambda]$ is disjoint from the spectrum of $(DF^n_\omega(0), \theta(n, \omega))$, then there exists a measurable $\gamma'(\omega) > 1$ such that

$$|F^n_\omega(x_1) - F^n_\omega(x_2)| \leq \gamma'(\omega)|x_1 - x_2|e^{n\lambda'}$$

for all $x_1, x_2 \in V_\omega^\lambda$ and all integers $n \geq 0$.

**Proof**

[Ru.1], Theorem 5.1, p. 292.
Construction of the Stable/Unstable Manifolds

- Use auxiliary cocycle \((Z, \theta)\). Set \(\tau := \theta(1, \cdot) : \Omega \to \Omega\). Define maps \(F_\omega, F^n_\omega : \mathbb{R}^d \to \mathbb{R}^d\):

\[
F_\omega(x) := Z(1, x, \omega) \quad x \in \mathbb{R}^d
\]
\[
F^n_\omega := F_\tau^{n-1}(\omega) \circ \cdots \circ F_\tau(\omega) \circ F_\omega
\]
for all \(\omega \in \Omega\). Then cocycle property for \(Z\) gives \(F^n_\omega = Z(n, \cdot, \omega)\) for each \(n \geq 1\). \(F_\omega\) is \(C^{k,\epsilon}\) (\(\epsilon \in (0, \delta)\)) and \((DF_\omega)(0) = D_2\phi(1, Y(\omega), \omega)\).

- Integrability of the map \(\omega \mapsto \log^+ \|D_2\phi(1, Y(\omega), \omega)\|_{L(\mathbb{R}^d)}\) (Lemma 2) implies discrete cocycle \((DF^n_\omega)(0), \theta(n, \omega), n \geq 0\) has same non-random Lyapunov spectrum as that of linearized continuous cocycle \((D_2\phi(t, Y(\omega), \omega), \theta(t, \omega), t \geq 0)\), viz. \(\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}\), where each \(\lambda_i\) has fixed multiplicity \(q_i, 1 \leq i \leq m\) (Lemma 2).

- If \(\lambda_i > 0\) for all \(1 \leq i \leq m\), then take \(\tilde{S}(\omega) := \{Y(\omega)\}\) for all \(\omega \in \Omega\). Theorem is trivial in this case. Hence assume there is at least one \(\lambda_i < 0\).

- Use discrete non-linear ergodic theorem of Ruelle (Theorem 4) and its proof to obtain a sure event \(\Omega^*_1 \in \mathcal{F}\) such that \(\theta(t, \cdot)(\Omega^*_1) = \Omega^*_1\) for all \(t \in \mathbb{R}\), \(\mathcal{F}\)-measurable positive random variables \(\rho_1, \beta_1 : \Omega^*_1 \to (0, \infty), \rho_1 < \beta_1\), and a random family of \(C^{k,\epsilon}\) (\(\epsilon \in (0, \delta)\)) submanifolds of \(\bar{B}(0, \rho_1(\omega))\) denoted by \(\tilde{S}_d(\omega), \omega \in \Omega^*_1\), and satisfying the following properties for each \(\omega \in \Omega^*_1\): \(\tilde{S}_d(\omega)\) is the set of all \(x \in \bar{B}(0, \rho_1(\omega))\) such that

\[
|Z(n, x, \omega)| \leq \beta_1(\omega)e^{(\lambda_{i_0} + \epsilon_1)n}, \quad n \in \mathbb{Z}^+ \quad (21)
\]
\( \tilde{S}_d(\omega) \) is tangent at 0 to the stable subspace \( S(\omega) \) of the linearized flow \( D_2\phi \), viz. \( T_0\tilde{S}_d(\omega) = S(\omega) \). Therefore \( \dim \tilde{S}_d(\omega) \) is non-random by ergodicity of \( \theta \). Also

\[
\limsup_{n \to \infty} \frac{1}{n} \log \left( \sup_{x_1 \neq x_2, x_1, x_2 \in \tilde{S}_d(\omega)} \frac{|Z(n, x_1, \omega) - Z(n, x_2, \omega)|}{|x_1 - x_2|} \right) \leq \lambda_{i_0}.
\]  

(22)

The \( \theta(t, \cdot) \)-invariant sure event \( \Omega_1^* \in \mathcal{F} \) is constructed using the ideas in Ruelle’s proof (of Theorem 5.1 in [Ru.1], p. 293), combined with the estimate (10) of Lemma 2 and the subadditive ergodic theorem (Lemma 1 (ii)).

• For each \( \omega \in \Omega_1^* \), let \( \tilde{S}(\omega) \) be as defined in part (a) of the theorem. Then by definition of \( \tilde{S}_d(\omega) \) and \( Z \):

\[
\tilde{S}(\omega) = \tilde{S}_d(\omega) + Y(\omega).
\]

(23)

Since \( \tilde{S}_d(\omega) \) is a \( C^{k, \epsilon} \) submanifold of \( \bar{B}(0, \rho_1(\omega)) \), then \( \tilde{S}(\omega) \) is a \( C^{k, \epsilon} \) submanifold of \( \bar{B}(Y(\omega), \rho_1(\omega)) \). Furthermore, \( T_Y(\omega) \tilde{S}(\omega) = T_0\tilde{S}_d(\omega) = S(\omega) \). Hence \( \dim \tilde{S}(\omega) = \dim S(\omega) = \sum_{i= i_0}^{m} q_i \), and is non-random.

• (22) implies that

\[
\limsup_{n \to \infty} \frac{1}{n} \log |Z(n, x, \omega)| \leq \lambda_{i_0}
\]

(24)

for all \( \omega \) in \( \Omega_1^* \) and all \( x \in \tilde{S}_d(\omega) \). Lemma 4 implies there is a sure event \( \Omega_2^* \subseteq \Omega_1^* \) such that \( \theta(t, \cdot)(\Omega_2^*) = \Omega_2^* \) for all \( t \in \mathbb{R} \), and

\[
\limsup_{t \to \infty} \frac{1}{t} \log |Z(t, x, \omega)| \leq \lambda_{i_0}
\]

(25)
for all $\omega \in \Omega_2^*$ and all $x \in \tilde{S}_d(\omega)$. Therefore (2) holds.

- To prove (b), let $\omega \in \Omega_1^*$. By (22), there is a positive integer $N_0 := N_0(\omega)$ (independent of $x \in \tilde{S}_d(\omega)$) such that $Z(n, x, \omega) \in \bar{B}(0, 1)$ for all $n \geq N_0$. Let $\Omega_4^* := \Omega_2^* \cap \Omega_3$, where $\Omega_3$ is the shift-invariant sure event defined in the proof of Lemma 4. Then $\Omega_4^*$ is a sure event and $\theta(t, \cdot)(\Omega_4^*) = \Omega_4^*$ for all $t \in \mathbb{R}$. By cocycle property, Mean-Value theorem and the ergodic theorem (Lemma 1(i)), we get (b).

- To prove the invariance property (4), apply the Oseledec theorem to $(D_2\phi(t, Y(\omega), \omega), \theta(t, \omega))$. Get a sure $\theta(t, \cdot)$-invariant event, also denoted by $\Omega_1^*$, such that $D_2\phi(t, Y(\omega), \omega)(S(\omega)) \subseteq S(\theta(t, \omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$. Equality holds because $D_2\phi(t, Y(\omega), \omega)$ is injective and $\dim S(\omega) = \dim S(\theta(t, \omega))$ for all $t \geq 0$ and all $\omega \in \Omega_1^*$.

- To prove the asymptotic invariance property (3), use ideas from Ruelle’s Theorems 5.1 and 4.1 in [Ru.1], to pick random variables $\rho_1, \beta_1$ and a sure event (also denoted by) $\Omega_1^*$ such that $\theta(t, \cdot)(\Omega_1^*) = \Omega_1^*$ for all $t \in \mathbb{R}$, and for any $\epsilon \in (0, \epsilon_1)$ and every $\omega \in \Omega_1^*$, and the inequalities

$$
\rho_1(\theta(t, \omega)) \geq \rho_1(\omega)e^{(\lambda_{i_0} + \epsilon_1)t}, \\
\beta_1(\theta(t, \omega)) \geq \beta_1(\omega)e^{(\lambda_{i_0} + \epsilon_1)t}
$$

hold for all $t \geq 0$ and every $\omega \in \Omega_1^*$. Use (b) to obtain a sure event $\Omega_5^* \subseteq \Omega_4^*$ such that $\theta(t, \cdot)(\Omega_5^*) = \Omega_5^*$ for all
\(t \in \mathbb{R}\), and for any \(0 < \epsilon < \epsilon_1\) and \(\omega \in \Omega^*_4\), there exists \(\beta(\omega) > 0\) (independent of \(x\)) with

\[
|\phi(t, x, \omega) - Y(\theta(t, \omega))| \leq \beta(\omega)e^{(\lambda_0 + \epsilon)t}
\]  

(27)

for all \(x \in \tilde{S}(\omega), t \geq 0\). Fix \(t \geq 0, \omega \in \Omega^*_5\) and \(x \in \tilde{S}(\omega)\). Let \(n\) be a non-negative integer. Then the cocycle property and (27) imply that

\[
|\phi(n, \phi(t, x, \omega), \theta(t, \omega)) - Y(\theta(n, \theta(t, \omega)))| = |\phi(n + t, x, \omega) - Y(\theta(n + t, \omega))| \\
\leq \beta(\omega)e^{(\lambda_0 + \epsilon)(n + t)} \\
\leq \beta(\omega)e^{(\lambda_0 + \epsilon)t}e^{(\lambda_0 + \epsilon_1)n}.
\]  

(28)

If \(\omega \in \Omega^*_5\), then it follows from (26), (27), (28) and the definition of \(\tilde{S}(\theta(t, \omega))\) that there exists \(\tau_1(\omega) > 0\) such that \(\phi(t, x, \omega) \in \tilde{S}(\theta(t, \omega))\) for all \(t \geq \tau_1(\omega)\). This proves asymptotic invariance.

• Prove (d), the existence of the local unstable manifolds \(\tilde{U}(\omega)\), by running both the flow \(\phi\) and the shift \(\theta\) backward in time getting the cocycle \((\tilde{Z}(t, \cdot, \omega), \tilde{\theta}(t, \omega), t \geq 0)\):

\[
\tilde{\phi}(t, x, \omega) := \phi(-t, x, \omega), \quad \tilde{Z}(t, x, \omega) := Z(-t, x, \omega), \quad \tilde{\theta}(t, \omega) := \theta(-t, \omega)
\]

for all \(t \geq 0, \omega \in \Omega\). The linearized flow \((D_2\tilde{\phi}(t, Y(\omega), \omega), \tilde{\theta}(t, \omega), t \geq 0)\) is an \(L(\mathbb{R}^d)\)-valued perfect cocycle with a non-random finite Lyapunov spectrum \(-\lambda_1 < -\lambda_2 < \cdots < -\lambda_i < -\lambda_{i+1} < \cdots < -\lambda_m\) where \(\{\lambda_m < \cdots < \lambda_{i+1} < \lambda_i < \cdots < \lambda_2 < \lambda_1\}\) is the Lyapunov spectrum of the forward linearized flow.
Apply first part of the proof to get stable manifolds for the backward flow $\tilde{\phi}$ satisfying assertions (a), (b), (c). This gives unstable manifolds for the original flow $\phi$, and (d), (e), (f) automatically hold.

- Measurability of the stable manifolds follows from the representations:

$$\tilde{S}(\omega) = Y(\omega) + \tilde{S}_d(\omega)$$  \hspace{1cm} (29)

$$\tilde{S}_d(\omega) = \lim_{n \to \infty} \bar{B}(0, \rho_1(\omega)) \cap \bigcap_{i=1}^{n} f_i(\cdot, \omega)^{-1}(\bar{B}(0, 1))$$  \hspace{1cm} (30)

$$f_i(x, \omega) := \beta_1(\omega)^{-1} e^{-(\lambda_{i_0} + \epsilon_1)i} Z(i, x, \omega), \quad x \in \mathbb{R}^d, \omega \in \Omega^*_1,$$

for all integers $i \geq 0$. (Above limit is taken in the metric $d^*$ on $C(\mathbb{R}^d)$.) Use joint continuity of translation and measurability of $Y, f_i, \rho_1$, finite intersections and the continuity of the maps

$$\mathbb{R}^+ \ni r \mapsto \bar{B}(0, r) \in C(\mathbb{R}^d).$$

$$\text{Hom}(\mathbb{R}^d) \ni f \mapsto f^{-1}(\bar{B}(0, 1)) \in C(\mathbb{R}^d).$$

- For $h, g_i$ in $C_b^\infty$, can adapt above argument to give a sure event in $\mathcal{F}$, also denoted by $\Omega^*$ such that $\tilde{S}(\omega), \tilde{U}(\omega)$ are $C^\infty$ for all $\omega \in \Omega^*$.
Examples of Stationary Solutions

1. Fixed points:

\[ d\phi(t) = h(\phi(t)) \, dt + \sum_{i=1}^{m} g_i(\phi(t)) \circ dW_i(t) \]

\[ h(x_0) = g_i(x_0) = 0, \quad 1 \leq i \leq m \]

Take \( Y(\omega) = x_0 \) for all \( \omega \in \Omega \).

2. Linear affine case \( d = 1 \):

\[ d\phi(t) = \lambda \phi(t) \, dt + dW(t) \]

\( \lambda > 0 \) fixed, \( W(t) \in \mathbb{R} \). Take

\[ Y(\omega) := -\int_{0}^{\infty} e^{-\lambda u} \, dW(u), \]

\[ \theta(t, \omega)(s) = \omega(t + s) - \omega(t). \]

Check that \( \phi(t, Y(\omega), \omega) = Y(\theta(t, \omega)) \), using integration by parts and variation of parameters.

3. Affine linear SDE in \( d = 2 \):

\[ d\phi(t) = A\phi(t) \, dt + GdW(t) \]

with \( A \) a fixed hyperbolic \( 2 \times 2 \)-diagonal matrix; \( G \) a constant matrix.

4. Non-linear transforms of (3) under a global diffeomorphism.
References


[Nu] Nualart, D., Analysis on Wiener space and anticipating stochastic calculus (to appear in) *St. Flour Notes*.


