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Numerics of Stochastic Systems with Memory
(Applied Mathematics and Numerical Analysis Seminars, University of Manchester)

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NUMERICS
OF STOCHASTIC SYSTEMS
WITH MEMORY

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Outline

- Develop two numerical schemes for solving stochastic differential systems with memory.
- Strong Euler scheme for SDDE’s and SFDE’s with continuous memory. Order of convergence 0.5.
- Strong Milstein scheme for SDDE’s. Order of convergence 1.
- For Milstein scheme, use infinite dimensional Itô formula for “tame” functions acting on segment process of solution of SDDE. Presence of memory in SDDE requires use of Malliavin calculus + anticipating stochastic analysis of Nualart and Pardoux.
Types of SFDE’s

SDDE:

\[ X(t) = \begin{cases} 
\eta(0) + \int_0^t g(s, \Pi_1(X_s)) \, dW(s) + \\
\int_0^t h(s, \Pi_2(X_s)) \, ds, & t \geq 0 \\
\eta(t), & -r \leq t < 0.
\end{cases} \]

\[ \Pi_i(\eta) := (\eta(s_{i,1}), \ldots, \eta(s_{i,k_i})) \in \mathbb{R}^{mk}, \quad \eta \in C \]

\( i = 1, 2. \)

SFDE with mixed discrete and continuous memory:

\[ X(t) = \eta(0) + \int_0^t g(s, \Pi_1(X_s), Q_1(X_s)) \, dW(s) \]
\[ + \int_0^t h(s, \Pi_2(X_s), Q_2(X_s)) \, ds, \quad t \in [0, a], \]
\[ X_0 = \eta \in C = C(J; \mathbb{R}^m) \]
Π₁, Π₂ two projections of discrete type;

Q₁, Q₂ two projections of continuous type:

\[ Q_i(\eta) := (Q_{i,1}(\eta), \cdots, Q_{i,m_i}(\eta)), \quad i = 1, 2, \]
\[ Q_{ij}(\eta) := \int_{-r}^{0} \phi_{ij}(\eta(s))a_{ij}(s) \, ds, \quad j = 1, \cdots, m_i. \]

\[ a_{ij} \in C^{\frac{1}{2}}(J), \text{ and } \phi_{ij} : \mathbb{R}^m \to \mathbb{R}, i = 1, 2, j = 1, \cdots, m_i. \]

General SFDE:

\[
X(t) = \begin{cases} 
\eta(0) + \int_{0}^{t} G(s, X_s) \, dW(s) \\
+ \int_{0}^{t} H(s, X_s) \, ds, & t \geq 0 \\
\eta(t), & -r \leq t < 0,
\end{cases}
\]
Numerical Schemes

Suppose rate of change of physical system depends on \textit{present state} and some noisy input. Model by SODE.

Rate of change depends on \textit{present} and \textit{past} states of the system: Model by SDDE or SFDE.


SDDE's and SFDE's cannot be solved explicitly: Need effective numerical techniques.

Numerical methods for SODE's: well developed; Kloeden and Platen, Kloeden, Platen and Schurz, McShane, Chapters 5 and 6), Hu, Talay, Protter, etc.. Cauchy-Maruyama scheme for
SFDE’s with continuous memory: On Delfour-Mitter state space $\mathbb{R}^m \times L^2([-r, 0], \mathbb{R}^m)$ developed by Ahmed, Elsanousi and Mohammed (Ahmed, M.Sc. thesis, Khartoum 1983), Baker and Buckwar. See also [M], 1984, p. 227, and Hu-Mohammed.

Aims.

- **Strong Euler schemes** for SFDE’s. Allows for several discrete delays and for SFDE’s with mixed discrete and continuous memory. Estimates in supremum norm on $C([-r, 0], \mathbb{R}^m)$ (cf. [A]).

- **Strong Milstein scheme** for SDDE’s. Solution of SDDE is non-anticipating. But need methods from *anticipating* stochastic analysis and Malliavin calculus to derive Itô’s
formula for segment process. Itô’s formula needed for convergence of Milstein scheme.
Preliminaries

\( \mathbb{R}^m := m \)-dimensional Euclidean space.

Euclidean norm \( |x| := \sqrt{x_1^2 + \cdots + x_m^2}, \ x = (x_1, \cdots, x_m) \in \mathbb{R}^m. \)

\( T := [0, a], \ J := [-r, 0], \)

\( C := C(J; \mathbb{R}^m), \ m, r, a > 0; \) sup norm:

\[
\| \eta \|_C := \sup_{-r \leq s \leq 0} |\eta(s)|, \quad \eta \in C.
\]

Projection \( \Pi : C \to \mathbb{R}^{mk} \) associated with \( s_1, \cdots, s_k \in [-r, 0]: \)

\[
\Pi(\eta) := (\eta(s_1), \cdots, \eta(s_k)) \in \mathbb{R}^{mk}, \quad \eta \in C
\]
Definition.

\[ \Phi \in C(T \times C(J; \mathbb{R}^m); \mathbb{R}) \] is tame if there exist \( \phi \in C(T \times \mathbb{R}^{mk}, \mathbb{R}) \) and a projection \( \Pi \) such that

\[ \Phi(t, \eta) = \phi(t, \Pi(\eta)) \]

for all \( t \in T \) and \( \eta \in C \).

Segment process \( X_t, t \in [0, a] \):

\[ X_t(s) = X(t + s), \quad t \in [0, a], \quad s \in [-r, 0]. \]

for continuous m-dimensional process \( \{X(t)\}_{t \in [-r, a]} \).

\( \{X_t\} \) is a \( C \)-valued or \( L^2(J; \mathbb{R}^m) \)-valued process.

Distinguish between finite-dimensional current state \( x(t) \) and infinite-dimensional segment \( X_t, t \in [0, a] \).
**Lipschitz Condition:**

\[ |g(t, x) - g(t, y)| \leq L|x - y|, \quad x, y \in \mathbb{R}^{mk_1} \]

\[ |h(t, z) - h(t, w)| \leq L|z - w|, \quad z, w \in \mathbb{R}^{mk_2} \]

for all \( t \in T; \ L > 0 \) constant.

**Boundedness Condition:**

\[ \sup_{0 \leq t \leq a} \left[ |g(t, 0)| + |h(t, 0)| \right] < \infty. \]

\( \Pi_1, \Pi_2: \) Two projections based on \( s_{1,1}, \ldots, s_{1,k_1} \in [-r, 0] \) and \( s_{2,1}, \ldots, s_{2,k_2} \in [-r, 0] \), respectively.

\( \{W(t) := (W^1(t), \ldots, W^d(t)) : t \geq 0\}, \ d\)-dimensional standard Brownian motion on \((\Omega, \mathcal{F}, P)\).

\( (\mathcal{F}_t)_{t \geq 0} = \) Brownian filtration.
\(\eta \in C([-r, 0]; \mathbb{R}^m) = \) random initial path independent of \(\{W(t) : t \geq 0\}\).

Itô SDDE’s:

\[
X(t) = \begin{cases} 
\eta(0) + \int_0^t g(s, \Pi_1(X_s)) dW(s) + \\
\int_0^t h(s, \Pi_2(X_s)) ds, & t \geq 0 \\
\eta(t), & -r \leq t < 0.
\end{cases}
\]

Lipschitz + bounded conditions imply SDDE has unique strong solution such that for each \(q \geq 1\), there exists a constant \(C = C(q, L, a) > 0\) such that

\[
E\|X_t\|_C^{2q} \leq C(1 + E\|\eta\|_C^{2q})
\]

for all \(\eta \in C, t \in [0, a]\) ([M], 1984).
**Strong versus Weak:**

SFDE’s do not lead to diffusions on Euclidean space. *(Highly degenerate infinite-dimensional diffusions on C.)* Hence no natural link to deterministic PDE’s. Strong schemes give information on sample paths dynamics, a.s. financial option-pricing models with delays.
Strong Euler Scheme

Develop Euler scheme for SFDE’s (discrete and/or continuous memory).

For simplicity, assume:

\[ a = \text{positive integer}, \ T := [0, a], \ J := [-1, 0]. \]

Uniform partitions: \( \pi_n := \{t_i : -l \leq i \leq n\} \) of \([-1, a]\)
such that \( t_i + s_{j,i} \in \pi_n \) for \( 1 \leq i \leq k_j \) and \( j = 1, 2 \).

\( \delta_n := |\pi_n| \)

\( X^n := X^{\pi_n}, \ n \) positive integer.

SFDE:

\[
X(t) = \begin{cases} 
\eta(0) + \int_0^t G(s, X_s) \, dW(s) \\
+ \int_0^t H(s, X_s) \, ds, \ t \geq 0 \\
\eta(t), \quad -r \leq t < 0,
\end{cases}
\]
Euler scheme for SFDE’s has strong order of convergence 0.5 (as in SODE).

Euler scheme for SFDE:

\[
X^n(t) = \begin{cases} 
X^n(t_i) + G(t_i, X^n_{t_i})(W(t) - W(t_i)) \\
+ H(t_i, X^n_{t_i})(t - t_i), & t \in (t_i, t_{i+1}], \quad t_i \in (0, a] \\
\eta^n(t), & -1 \leq t \leq 0 
\end{cases}
\]

Approx. initial path \(\eta^n \in C(J, \mathbb{R}^m)\) is prescribed (e.g. a piece-wise linear approximation of \(\eta\) using partition points \(\{t_{-l}, \cdots, t_0\}\)).

Error function \(Z^n:\)

\[
\begin{cases} 
Z^n(t) := X^n(t) - X(t), & 0 \leq t \leq a, \\
Z^n_0 := \eta^n - \eta.
\end{cases}
\]
Theorem 1.

Assume that the coefficients $G : T \times C([-r, 0], \mathbb{R}^m) \to L(\mathbb{R}^d; \mathbb{R}^m)$ and $H : T \times C([-r, 0], \mathbb{R}^m) \to \mathbb{R}^m$ in SFDE satisfy the following Lipschitz and regularity conditions:

$$\|G(t, \eta) - G(t, \xi)\| + |H(t, \eta) - H(t, \xi)| \leq L\|\eta - \xi\|_C, \ t \in T,$$

$$\sup_{0 \leq t \leq a} \left[\|G(t, 0)\| + |H(t, 0)|\right] < \infty.$$

$$\begin{cases} 
\|G(s, \eta) - G(t, \eta)\| \leq L_1(1 + \|\eta\|_C)|s - t|^\gamma, \ s, t \in T, \\
|H(s, \eta) - H(t, \eta)| \leq L_1(1 + \|\eta\|_C)|s - t|^\gamma, \ s, t \in T,
\end{cases}$$

for all $\eta, \xi \in C([-r, 0], \mathbb{R}^m)$, where $L$ and $L_1$ are positive constants.

Fix any integer $q \geq 2$. Suppose that $\eta : [-r, 0] \to L^q(\Omega, \mathbb{R}^m)$ is independent of $W$ and Hölder continuous with exponent $\gamma \in (0, 1]$, i.e., there is a
positive constant $K$ such that

$$E|\eta(s) - \eta(t)|^q \leq K|s - t|^\gamma q$$

for all $s, t \in [-r, 0]$. Suppose also that there is a positive constant $C' := C'(q)$ such that

$$E||\eta^n - \eta||_C^q \leq C' \delta_n^q$$

Then there is a constant $C'' := C''(q, a) > 0$, depending on $a$ and $q$, such that

$$E \sup_{0 \leq s \leq a} ||Z^n_s||_C^q \leq C'' \delta_n^{\tilde{\gamma} q}$$

where $\tilde{\gamma} := \gamma \wedge (1/2)$.

**Proof.**

Based on moment estimates:

$$E||X_t||_C^{2q} \leq C(1 + E||\eta||_C^{2q}), \ for \ q \geq 1$$
for all $\eta \in C, t \in [0, a]$ ([M], 1984), and Burkholder’s inequality. □

Theorem 1 applies to SDDE’s under Lipschitz and boundedness conditions. Also to SFDE’s with mixed discrete and continuous memory:

\[
X(t) = \eta(0) + \int_0^t g(s, \Pi_1(X_s), Q_1(X_s)) \, dW(s)
+ \int_0^t h(s, \Pi_2(X_s), Q_2(X_s)) \, ds, \quad t \in [0, a],
\]

\[
X_0 = \eta \in C = C(J; \mathbb{R}^m)
\]

$\Pi_1, \Pi_2$ two projections of discrete type;
$Q_1, Q_2$ two projections of continuous type:

\[
Q_i(\eta) := (Q_{i,1}(\eta), \cdots, Q_{i,m_i}(\eta)), \quad i = 1, 2,
\]

\[
Q_{ij}(\eta) := \int_{-1}^0 \phi_{ij}(\eta(s)) a_{ij}(s) \, ds, \quad j = 1, \cdots, m_i.
\]
\( a_{ij} \in C^{\frac{1}{2}}(J) \), and \( \phi_{ij} : \mathbb{R}^m \to \mathbb{R} \), \( i = 1, 2, j = 1, \ldots, m_i \), satisfy Lipschitz and linear growth conditions.

Euler scheme for SFDE with mixed discrete and continuous memory:

\[
X^n(t) = X^n(t_i) + g(t_i, \Pi_1(X^n_{t_i}), Q^n_1(X^n_{t_i}))(W(t) - W(t_i)) \\
+ h(t_i, \Pi_2(X^n_{t_i}), Q^n_2(X^n_{t_i}))(t - t_i), \quad t \in (t_i, t_{i+1}],
\]

\[
X^n(t) = \eta^n(t), \quad -r \leq t \leq 0,
\]

where \( Q^n_i(\eta), i = 1, 2 \), are approximations of \( Q_i(\eta) \) using partial sums of Riemann integral. Strong order of convergence 0.5 under Lipschitz and regularity conditions as in Theorem 1.
**Example: Exact convergence rate.**

One-dimensional SDDE:

\[
\begin{align*}
  dX(t) &= b(t)X(t - 1)\,dW(t), \quad 0 \leq t \leq a \\
  X(t) &= \eta(t), \quad -1 \leq t \leq 0.
\end{align*}
\]

Use partitions \(\{\pi_n(h)\}\) of \([-1, a]\) generated by a continuous (strictly positive) function \(h : [0, a] \to (0, \infty)\). For each integer \(n\), choose partition points \(t_{k,n} \equiv t_k\) of the partition \(\pi_n(h)\) in \([0, a]\) are chosen such that

\[
t_0 = 0, \quad \int_{t_k}^{t_{k+1}} h(s)\,ds = \frac{1}{n}, \quad k = 0, 1, \ldots, n - 1.
\]

i.e. subdivide interval in such a way that the areas under \(h\) over each subinterval are all equal to \(1/n\). Then

\[
\lim_{n \to \infty, t_k \to t} n(t_{k+1} - t_k) = 1/h(t).
\]
e.g. $h(t) \equiv 1 \implies (t_{k+1} - t_k) = 1/n$, $k = 0, 1, \ldots, n - 1$.

Euler scheme gives

$$X_{\pi_n}(t) = \begin{cases} X_{\pi_n}(t_k) + b(t_k)X_{\pi_n}(t_k - 1)(W(t) - W(t_k)), & t_k \leq t < t_{k+1}, \\ \eta(t), & t \in J, \end{cases}$$

for $0 \leq k \leq n-1$. By Theorem 1, there is a positive constant $C$ (independent of $n$) such that

$$nE|X(t) - X_{\pi_n}(t)|^2 \leq C,$$

for all $n \geq 1, t \in [0, a]$. Theorem 2 (below) show that as $n \to \infty$, the left hand side of the above inequality has a limit satisfying a deterministic DDE.
Theorem 2.

Suppose $\eta \in C^\gamma(J, \mathbb{R}^m), 1/2 < \gamma \leq 1$. Let $a \geq 1$.

Suppose $b : [0, a] \to \mathbb{R}$ is a bounded continuous function such that

$$|b(t) - b(s)| \leq K|t - s|^{(1/2)+\alpha}$$

for all $s, t \in [0, a]$ and some $K, \alpha > 0$. Let $X$ be the solution of the SDDE and $X^{\pi_n}$ its Euler approximation. Then $Z(t) := \lim_{n \to \infty} n \mathbb{E}|X(t) - X^{\pi_n}(t)|^2$ exists for each $t \in [0, a]$. Furthermore, $Z(t)$ satisfies the following deterministic linear DDE

$$Z'(t) = b^2(t)Z(t - 1) + b^2(t)b^2(t - 1)EX^2(t - 2)/h(t), \quad 1 < t < a,$$

$$Z(t) = 0, \quad -1 \leq t \leq 1,$$

where $EX^2(t)$ is given by the integral equation

$$EX^2(t) = \begin{cases} 
\eta(0)^2 + \int_0^t b^2(s)EX^2(s - 1) \, ds, & t \in [0, a], \\
\eta(t)^2, & t \in [-1, 0].
\end{cases}$$
Milstein Scheme

Strong higher order scheme for SDDE:

\[ X(t) = \begin{cases} 
\eta(0) + \int_0^t g(s, \Pi_1(X_s)) \, dW(s) \\
+ \int_0^t h(s, \Pi_2(X_s)) \, ds, & t \geq 0 \\
\eta(t), & -1 \leq t < 0,
\end{cases} \]

( \ r = 1 \).

Requires infinite-dimensional Itô formula for “tame” functions of segments of semimartingales or (anticipating) processes. Proof based on Nualart-Pardoux anticipating calculus techniques.
\(a = \text{positive integer}, \ T := [0, a], \ J := [-1, 0].\)

Uniform partitions: \(\pi_n := \{t_i : -l \leq i \leq n\}\) of \([-1, a]\)
such that \(t_i + s_{j,i} \in \pi_n\) for \(1 \leq i \leq k_j, \ j = 1, 2.\)

\(\delta_n := |\pi_n|\)

\(X^n := X^\pi_n, \ n \text{ positive integer}.\)

Milstein approximations for SDDE:

\[
X^{i,n}(t) = X^{i,n}(t_k) + h^i(t_k, \Pi_2(X^n_{t_k}))(t - t_k) + g^{il}(t_k, \Pi_1(X^n_{t_k}))(W^l(t) - W^l(t_k)) + \sum_{i_1, j_1, l} \frac{\partial g^{il}}{\partial x_{i_1 j_1}}(t_k, \Pi_1(X^n_{t_k}))g^{i_1 j_1}(t_k + s_{1,j_1}, \Pi_1(X^n_{t_k+s_{1,j_1}})) \times
\]
\[
\times 1_{[0,T]}(t_k + s_{1,j_1}) \times I_{l,l_1}(t_k + s_{1,j_1}, t + s_{1,j_1}; s_{1,j_1}),
\]

for \(t_k < t \leq t_{k+1}, \ i = 1, 2, \ldots, m,\) where

\[
I_{l,l_1}(t_0+s_{i,j}, t+s_{i,j}; s_{i,j}) = \int_{t_0}^{t} \int_{t_0+s_{i,j}}^{t_1+s_{i,j}} \diamond dW^l(t_2) \diamond dW^{l_1}(t_1).
\]

23
\( X^i, h^i, g^{il} \) = coordinates of \( X, h \) and \( g \) with respect to standard bases in Euclidean space.

Milstein scheme has strong order of convergence 1.

**Theorem 3.**

Consider the Milstein scheme for the SDDE \((r = 1)\). Let \( 0 < \gamma \leq 1 \). Suppose that \( \eta : [-1, 0] \to L^2(\Omega, \mathbb{R}^m) \) is Hölder continuous with exponent \( \frac{\gamma}{2} \), i.e. there is a positive constant \( K \) such that

\[
E|\eta(s) - \eta(t)|^2 \leq K|s - t|^{2\gamma}
\]

for all \( s, t \in J \). Suppose that \( g \in C^{1,2}(T \times \mathbb{R}^{mk_1}, L(\mathbb{R}^d, \mathbb{R}^m)) \), \( h \in C^{1,2}(T \times \mathbb{R}^{mk_2}, \mathbb{R}^m) \) and have bounded first and second spatial derivatives. Let

\[
\begin{align*}
Z^n(t) &:= X^n(t) - X(t), \quad 0 \leq t \leq a, \\
Z^n_0 &:= \eta^n_0 - \eta.
\end{align*}
\]
Assume that

$$\sup_{-1 \leq s \leq 0} E(|Z^n(s)|^2) \leq C' \delta_n^{2\gamma}$$

for some positive constant $C'$, where $\delta_n := |\pi_n|$. Then there exists a constant $C > 0$ (depending on $a$ and independent of $\pi_n$) such that

$$\sup_{-1 \leq s \leq a} E|Z^n(s)|^2 \leq C\delta_n^{2\gamma}$$

for any $n \geq 1$. 
Proof.

Itô’s formula for “tame” functionals

\[ C(J, \mathbb{R}^m) \to \mathbb{R} \]

of the segment \( X_t \). Use formula + moment estimates on weak derivatives of \( X \) to get global error estimate for the Milstein approximations. □

Example:

Formally expect something like:

\[
\begin{align*}
    f(W(t - 1), W(t)) - f(W(-1), W(0)) &= \int_0^t \frac{\partial f}{\partial x_2}(W(s - 1), W(s))dW(s) \\
    &+ \int_0^t \frac{\partial f}{\partial x_1}(W(s - 1), W(s))dW(s - 1) \quad (\text{anticipating!}) \\
    &+ \frac{1}{2} \left( \int_0^t \frac{\partial^2 f}{\partial x_1^2}(W(s - 1), W(s))ds + \int_0^t \frac{\partial^2 f}{\partial x_2^2}(W(s - 1), W(s))ds \right) \\
    &+ \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x_1 \partial x_2}(W(s - 1), W(s))dW(s - 1) dW(s)(= 0!)
\end{align*}
\]
LHS is adapted but anticipating integral on RHS.

$(\Omega, \mathcal{F}, P) :=$ probability space.

$W(t) := (W^1(t), \cdots, W^d(t)), t \geq 0, := d$-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, P)$.

$D := (D_1, \cdots, D_d) :=$ Malliavin differentiation operator associated with $\{W(t): t \geq 0\}$.

Pathwise-continuous process:

$$X(t) := \begin{cases} 
\eta(0) + \int_0^t u(s) \, dW(s) + \int_0^t v(s) \, ds, & t > 0, \\
\eta(t), & -r \leq t \leq 0,
\end{cases}$$

Skorohod integral. $\eta \in C, BV$.

$u = (u^1, \cdots, u^m)^T, u^i \in \mathbb{L}^{2,4}_{d,loc}$;

$v = (v^1, \cdots, v^m)^T, v^i \in \mathbb{L}^{1,4}_{loc} ([Nualart])$. 

27
$u$ and $v$ may not be adapted to the Brownian filtration $(\mathcal{F}_t)_{t \geq 0}$. Set $u(t) := 0$ for $t < 0$ or $t > a$,

$$v(t) := \begin{cases} 0, & t > a \\ \eta'(t), & -r \leq t \leq 0. \end{cases}$$

$W(t) := 0$ if $t < 0$ or $t > a$.

$$U(t) := \int_0^t u(s) dW(s), \quad V(t) := \begin{cases} \eta(0) + \int_0^t v(s) ds, & t > 0 \\ \eta(t), & -r \leq t \leq 0. \end{cases}$$

Then

$$D_s X_t = u_s 1_{[0,a]}(t-s) + D_s X_0 + \int_0^t D_s v_r dr + \int_0^t D_s u_r dW_r,$$

$T := [0, a], \ J := [-r, 0], \ C := C(J; , \mathbb{R}^m)$.

$\Pi :=$ projection associated with $s_1, \cdots, s_k \in J$.

Cannot apply multi-dimensional Itô formula to $\phi(t, \Pi(X_t))$ because $\Pi(U_t)$ is of the form

$$\left( \int_0^t u(s + s_1) dW(s + s_1), \cdots, \int_0^t u(s + s_k) dW(s + s_k) \right),$$
and the components \((W(t + s_1), \cdots, W(t + s_k))\) are not independent. Use anticipating calculus (Nualart-Pardoux) to derive an Itô formula for \(\phi(t, \Pi(X_t))\).

Assume \(\phi \in C^{1,2}(T \times \mathbb{R}^{mk}), \; \vec{x} = (\vec{x}_1, \cdots, \vec{x}_m), \; \vec{x}_i = (x_{i1}, \cdots, x_{ik}) \in \mathbb{R}^k\), Write

\[
\phi(t, \vec{x}) = \phi(t, \vec{x}_1, \cdots, \vec{x}_m). \tag{5.5}
\]

**Theorem 4.** (Itô’s formula).

Suppose \(X\) satisfies above conditions and let \(\phi \in C^{1,2}(T \times \mathbb{R}^{mk}, \mathbb{R})\). Then

\[
\phi(t, \Pi(X_t)) - \phi(0, \Pi(X_0)) = \int_0^t \frac{\partial \phi}{\partial s}(s, \Pi(X_s)) \, ds + \int_0^t \frac{\partial \phi}{\partial \vec{x}}(s, \Pi(X_s)) \, d(\Pi(X_s)) + \frac{1}{2} \sum_{i,j=1}^k \sum_{i_1,j_1=1}^m \int_0^t \frac{\partial^2 \phi}{\partial x_{i1} \partial x_{j1}}(s, \Pi(X_s)) u_{i_1}^i(s + s_i) D_{s+s_i} X_{j1}^j(s + s_j) \, ds
\]

a.s. for all \(t \in T\).
Example (Revisited)

\[ f(W(t - 1), W(t)) - f(W(-1), W(0)) = \int_0^t \frac{\partial f}{\partial x_1}(W(s - 1), W(s)) dW(s) + \int_0^t \frac{\partial f}{\partial x_2}(W(s - 1), W(s)) dW(s - 1) + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x_1^2}(W(s - 1), W(s)) ds + \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x_2^2}(W(s - 1), W(s)) 1_{(1, \infty)}(s) ds \]

\[ t > 0. \]
Weak differentiability of solutions of SDDE’s.

Cf. Bell and Mohammed, Nualart.

$$\mathbb{D}^{k,\infty}_m := \cap_{p \geq 2} \mathbb{D}^{k,p}_m, \ k \in \mathbb{N}.$$ 

$$D^l_r, 1 \leq l \leq d,$$ weak differentiation with respect to $$l$$-th component of $$W$$.

Proposition.

_In the Itô SDDE, assume that $$g \in C^0_1(T \times \mathbb{R}^{k_1 m}; L(\mathbb{R}^d, \mathbb{R}^m))$$ and $$h \in C^0_1(T \times \mathbb{R}^{k_2 m}; \mathbb{R}^m)$$. Let $$X$$ be the solution of (1.6). Then $$X(t) \in \mathbb{D}^{1,\infty}_m$$ for all $$t \in T$$, and

$$\sup_{0 \leq r \leq a} E(\sup_{r \leq s \leq a} |D_r X(s)|^p) < \infty$$

for all $$p \geq 2$$. Furthermore, the “partial” weak derivatives $$D^l_r X^j(t)$$ with respect to the $$l$$-th coordinate of $$W$$ satisfy_
the following linear SDDE's a.s.:

\[
D^l_rX^j(t) = g^{jl}(r, \Pi_1(X^j_r)) + \int_r^t \sum_{i=1}^{k_1} \frac{\partial g^{jl}}{\partial \vec{x}_i}(s, \Pi_1(X^j_s))D^l_rX^j(s + s_{1,i}) dW^l(s) \\
+ \int_0^t \sum_{i=1}^{k_2} \frac{\partial h^j}{\partial \vec{x}_i}(s, \Pi_2(X^j_s))D^l_rX^j(s + s_{2,i}) ds, \quad t \geq r, \\
= 0, \quad t < r, \ l = 1, \cdots, d, \ j = 1, \cdots, m
\]

\(g^{jl} = (j, l)\) entry of the \(m \times d\) matrix \(g\),

\(h^j = j\)-th coordinate of \(h\).
References


Hu, Y., *Strong and weak order of time discretization schemes of stochastic differential equations*, In Séminaire de Probabilités XXX, ed. by J. Azema, P.A. Meyer and


