II. MARKOV BEHAVIOR
AND THE WEAK GENERATOR

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II. MARKOV BEHAVIOR AND THE GENERATOR

Consider the sfde

\[ dx(t) = H(t, x_t) \, dt + G(t, x_t) \, dW(t), \quad t > 0 \]
\[ x_0 = \eta \in C := C([-r, 0], \mathbb{R}^d) \] (XIII)

with coefficients \( H : [0, T] \times C \to \mathbb{R}^d, \quad G : [0, T] \times C \to \mathbb{R}^{d \times m}, \) \( m \)-dimensional Brownian motion \( W \) and trajectory field \( \{^\eta x_t : t \geq 0, \eta \in C\} \).

1. Questions

(i) For the sfde (XIII) does the trajectory field \( x_t \) give a diffusion in \( C \) (or \( M_2 \))?

(ii) How does the trajectory \( x_t \) transform under smooth non-linear functionals \( \phi : C \to \mathbb{R} \)?

(iii) What “diffusions” on \( C \) (or \( M_2 \)) correspond to sfde’s on \( \mathbb{R}^d \)?

We will only answer the first two questions. More details in [Mo], Pitman Books, 1984, Chapter III, pp. 46-112. Third question is OPEN.
Difficulties

(i) Although the current state $x(t)$ is a semimartingale, the trajectory $x_t$ does not seem to possess any martingale properties when viewed as $C$- (or $M$2)-valued process: e.g. for Brownian motion $W$ ($H \equiv 0, G \equiv 1$):

$$[E(W_t|\mathcal{F}_{t_1})](s) = W(t_1) = W_{t_1}(0), \quad s \in [-r, 0]$$

whenever $t_1 \leq t - r$.

(ii) Lack of strong continuity leads to the use of weak limits in $C$ which tend to live outside $C$.

(iii) We will show that $x_t$ is a Markov process in $C$. However almost all tame functions lie outside the domain of the (weak) generator.

(iv) Lack of an Itô formula makes the computation of the generator hard.

Hypotheses ($M$)

(i) $\mathcal{F}_t :=$ completion of $\sigma\{W(u) : 0 \leq u \leq t\}$, $t \geq 0$.

(ii) $H, G$ are jointly continuous and globally Lipschitz in second variable uniformly wrt the first:

$$|H(t, \eta_1) - H(t, \eta_2)| + \|G(t, \eta_1) - G(t, \eta_2)\| \leq L\|\eta_1 - \eta_2\|_C$$

for all $t \in [0, T]$ and $\eta_1, \eta_2 \in C$.

2. The Markov Property

$\eta_{x^{t_1}} :=$ solution starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_{t_1})$ at $t = t_1$ for the sfde:

$$\eta_{x^{t_1}}(t) = \begin{cases} 
\eta(0) + \int_{t_1}^{t} H(u, x_u^{t_1}) \, du + \int_{t_1}^{t} G(u, x_u^{t_1}) \, dW(u), & t > t_1 \\
\eta(t - t_1), & t_1 - r \leq t \leq t_1.
\end{cases}$$
This gives a two-parameter family of mappings
\[ T_{t_2}^{t_1} : L^2(\Omega, C; F_{t_1}) \to L^2(\Omega, C; F_{t_2}), \ t_1 \leq t_2, \]
\[ T_{t_2}^{t_1}(\theta) := \theta x_{t_2}^{t_1}, \quad \theta \in L^2(\Omega, C; F_{t_1}). \]
\[ (1) \]

Uniqueness of solutions gives the \textit{two-parameter} semigroup property:
\[ T_{t_2}^{t_1} \circ T_{t_1}^0 = T_{t_2}^0, \quad t_1 \leq t_2. \]
\[ (2) \]

([Mo], Pitman Books, 1984, Theorem II (2.2), p. 40.)

\textbf{Theorem II.1} (Markov Property) ([Mo], 1984).

In (XIII) suppose Hypotheses (M) hold. Then the trajectory field \( \{ x_t : t \geq 0, \eta \in C \} \) is a Feller process on \( C \) with transition probabilities
\[ p(t_1, \eta, t_2, B) := P(\eta x_{t_2}^{t_1} \in B) \quad t_1 \leq t_2, \quad B \in \text{Borel} \ C, \quad \eta \in C. \]
\[ i.e. \]
\[ P(x_{t_2} \in B | F_{t_1}) = p(t_1, x_{t_1}^{t_2}, t_2, B) = P(x_{t_2} \in B | x_{t_1}) \text{ a.s.} \]

Further, if \( H \) and \( G \) do not depend on \( t \), then the trajectory is time-homogeneous:
\[ p(t_1, \eta, t_2, \cdot) = p(0, \eta, t_2 - t_1, \cdot), \quad 0 \leq t_1 \leq t_2, \quad \eta \in C. \]

\textbf{Proof.}

[Mo], 1984, Theorem III.1.1, pp. 51-58. [Mo], 1984, Theorem III.2.1, pp. 64-65. \( \Box \)
3. The Semigroup

In the autonomous sfde
\[
\begin{align*}
\frac{dx(t)}{dt} &= H(x_t)\, dt + G(x_t)\, dW(t) \quad t > 0 \\
x_0 &= \eta \in C 
\end{align*}
\]  
\text{(XIV)}

suppose the coefficients \( H : C \to \mathbb{R}^d, \ G : C \to \mathbb{R}^{d \times m} \) are \textit{globally bounded} and globally Lipschitz.

\( C_b := \) Banach space of all bounded uniformly continuous functions \( \phi : C \to \mathbb{R} \), with the sup norm
\[
\| \phi \|_{C_b} := \sup_{\eta \in C} |\phi(\eta)|, \quad \phi \in C_b.
\]

Define the operators \( P_t : C_b \hookrightarrow C_b, t \geq 0 \), on \( C_b \) by
\[
P_t(\phi)(\eta) := E\phi(\eta x_t) \quad t \geq 0, \ \eta \in C.
\]

A family \( \phi_t, t > 0, \) converges weakly to \( \phi \in C_b \) as \( t \to 0^+ \) if \( \lim_{t \to 0^+} < \phi_t, \mu > = < \phi, \mu > \) for all finite regular Borel measures \( \mu \) on \( C \). Write \( \phi := w - \lim_{t \to 0^+} \phi_t \). This is equivalent to
\[
\begin{align*}
\phi_t(\eta) &\to \phi(\eta) \text{ as } t \to 0^+, \text{ for all } \eta \in C \\
\{\|\phi_t\|_{C_b} : t \geq 0\} &\text{ is bounded}.
\end{align*}
\]

(Dynkin, [Dy], Vol. 1, p. 50). Proof uses uniform boundedness principle and dominated convergence theorem.

\textbf{Theorem II.2} ([Mo], Pitman Books, 1984)

(i) \( \{P_t\}_{t \geq 0} \) is a one-parameter contraction semigroup on \( C_b \).
(ii) \( \{P_t\}_{t \geq 0} \) is weakly continuous at \( t = 0 \):
\[
\begin{align*}
\{P_t(\phi)(\eta) \to \phi(\eta) \text{ as } t \to 0^+ \}
\{|P_t(\phi)(\eta)| : t \geq 0, \eta \in C \}\text{ is bounded by } \|\phi\|_{C_b}.
\end{align*}
\]

(iii) If \( r > 0 \), \( \{P_t\}_{t \geq 0} \) is never strongly continuous on \( C_b \) under the sup norm.

**Proof.**

(i) One parameter semigroup property
\[
P_{t_2} \circ P_{t_1} = P_{t_1+t_2}, \quad t_1, t_2 \geq 0
\]
follows from the continuation property (2) and time-homogeneity of the Feller process \( x_t \) (Theorem II.1).

(ii) Definition of \( P_t \), continuity and boundedness of \( \phi \) and sample-continuity of trajectory \( \eta x_t \) give weak continuity of \( \{P_t(\phi) : t > 0\} \) at \( t = 0 \) in \( C_b \).

(iii) Lack of strong continuity of semigroup:
Define the canonical shift (static) semigroup
\[
S_t : C_b \to C_b, \quad t \geq 0,
\]
by
\[
S_t(\phi)(\eta) := \phi(\tilde{\eta}_t), \quad \phi \in C_b, \quad \eta \in C,
\]
where \( \tilde{\eta} : [-r, \infty) \to \mathbb{R}^d \) is defined by
\[
\tilde{\eta}(t) = \begin{cases} 
\eta(0) & t \geq 0 \\
\eta(t) & t \in [-r, 0).
\end{cases}
\]
Then \( P_t \) is strongly continuous iff \( S_t \) is strongly continuous. \( P_t \) and \( S_t \) have the same “domain of strong continuity” independently of \( H, G, \) and \( W \). This follows from the global boundedness of \( H \) and \( G \). ([Mo], Theorem IV.2.1, pp. 72-73). Key relation is
\[
\lim_{t \to 0^+} E\|\eta x_t - \tilde{\eta}_t\|_{C_b}^2 = 0.
\]
uniformly in \( \eta \in C \). But \( \{S_t\} \) is strongly continuous on \( C_b \) iff \( C \) is locally compact iff \( r = 0 \) (no memory) ! ([Mo], Theorems IV.2.1 and IV.2.2, pp.72-73). Main idea is to pick any \( s_0 \in [-r,0) \) and consider the function \( \phi_0 : C \rightarrow \mathbb{R} \) defined by

\[
\phi_0(\eta) := \begin{cases} 
\eta(s_0) & \|\eta\|_C \leq 1 \\
\frac{\eta(s_0)}{\|\eta\|_C} & \|\eta\|_C > 1 
\end{cases}
\]

Let \( C_b^0 \) be the domain of strong continuity of \( P_t \), viz.

\[
C_b^0 := \{ \phi \in C_b : P_t(\phi) \rightarrow \phi \text{ as } t \rightarrow 0^+ \text{ in } C_b \}.
\]

Then \( \phi_0 \in C_b \), but \( \phi_0 \notin C_b^0 \) because \( r > 0 \).

4. The Generator

Define the weak generator \( A : D(A) \subset C_b \rightarrow C_b \) by the weak limit

\[
A(\phi)(\eta) := w - \lim_{t \rightarrow 0^+} \frac{P_t(\phi)(\eta) - \phi(\eta)}{t}
\]

where \( \phi \in D(A) \) iff the above weak limit exists. Hence \( D(A) \subset C_b^0 \) ([Dy], Vol. 1, Chapter I, pp. 36-43). Also \( D(A) \) is weakly dense in \( C_b \) and \( A \) is weakly closed. Further

\[
\frac{d}{dt} P_t(\phi) = A(P_t(\phi)) = P_t(A(\phi)), \quad t > 0
\]

for all \( \phi \in D(A) \) ([Dy], pp. 36-43).

Next objective is to derive a formula for the weak generator \( A \). We need to augment \( C \) by adjoining a canonical \( d \)-dimensional direction. The generator \( A \) will be equal to the weak generator of the shift semigroup \( \{S_t\} \) plus a second order linear partial differential operator along this new direction. Computation requires the following lemmas.

Let

\[
F_d = \{ v\chi_{\{0\}} : v \in \mathbb{R}^d \}
\]

\[
C \oplus F_d = \{ \eta + v\chi_{\{0\}} : \eta \in C, v \in \mathbb{R}^d \}, \quad \|\eta + v\chi_{\{0\}}\| = \|\eta\|_C + |v|
\]
Lemma II.1. ([Mo], Pitman Books, 1984)

Suppose \( \phi : C \to \mathbb{R} \) is \( C^2 \) and \( \eta \in C \). Then \( D\phi(\eta) \) and \( D^2\phi(\eta) \) have unique weakly continuous linear and bilinear extensions

\[
D\phi(\eta) : C \oplus F_d \to \mathbb{R}, \quad D^2\phi(\eta) : (C \oplus F_d) \times (C \oplus F_d) \to \mathbb{R}
\]

respectively.

**Proof.**

First reduce to the one-dimensional case \( d = 1 \) by using coordinates.

Let \( \alpha \in C^* = [C([-r, 0], \mathbb{R})]^* \). We will show that there is a weakly continuous linear extension \( \pi : C \oplus F_1 \to \mathbb{R} \) of \( \alpha \); viz. If \( \{\xi^k\} \) is a bounded sequence in \( C \) such that \( \xi^k(s) \to \xi(s) \) as \( k \to \infty \) for all \( s \in [-r, 0] \), where \( \xi \in C \oplus F_1 \), then \( \alpha(\xi^k) \to \pi(\xi) \) as \( k \to \infty \). By the Riesz representation theorem there is a unique finite regular Borel measure \( \mu \) on \( [-r, 0] \) such that

\[
\alpha(\eta) = \int_{-r}^{0} \eta(s) \, d\mu(s)
\]

for all \( \eta \in C \). Define \( \pi \in [C \oplus F_1]^* \) by

\[
\pi(\eta + v\chi_{\{0\}}) = \alpha(\eta) + v\mu(\{0\}), \quad \eta \in C, \quad v \in \mathbb{R}.
\]

Easy to check that \( \pi \) is weakly continuous. *(Exercise: Use Lebesgue dominated convergence theorem.)*

Weak extension \( \pi \) is unique because each function \( v\chi_{\{0\}} \) can be approximated weakly by a sequence of continuous functions \( \{\xi^k_0\} \):

\[
\xi^k_0(s) := \begin{cases} (ks + 1)v, & -\frac{1}{k} \leq s \leq 0 \\ 0 & -r \leq s < -\frac{1}{k}. \end{cases}
\]
Put $\alpha = D\phi(\eta)$ to get first assertion of lemma.

To construct a weakly continuous bilinear extension $\overline{\beta} : (C \oplus F_1) \times (C \oplus F_1) \to \mathbb{R}$ for any continuous bilinear form $\beta : C \times C \to \mathbb{R}$, use classical theory of vector measures (Dunford and Schwartz, [D-S], Vol. I, Section 6.3). Think of $\beta$ as a continuous linear map $C \to C^*$. Since $C^*$ is weakly complete ([D-S], I.13.22, p. 341), then $\beta$ is a weakly compact linear operator ([D-S], Theorem I.7.6, p. 494): i.e. it maps norm-bounded sets in $C$ into weakly sequentially compact sets in $C^*$. By the Riesz representation theorem (for vector measures), there is a unique $C^*$-valued Borel measure $\lambda$ on $[-r, 0]$ (of finite semi-variation) such that

$$\beta(\xi) = \int_{-r}^{0} \xi(s) d\lambda(s)$$

for all $\xi \in C$. ([D-S], Vol. I, Theorem VI.7.3, p. 493). By the dominated convergence theorem for vector measures ([D-S], Theorem IV.10.10, p. 328), one could reach elements in $F_1$ using weakly convergent sequences of type $\{\xi_k\}$. This gives a unique weakly continuous extension $\hat{\beta} : C \oplus F_1 \to C^*$. Next for each $\eta \in C$, $v \in \mathbb{R}$, extend $\hat{\beta}(\eta + v\chi_{\{0\}}) : C \to \mathbb{R}$ to a weakly continuous linear map $\overline{\beta}(\eta + v\chi_{\{0\}}) : C \oplus F_1 \to \mathbb{R}$. Thus $\overline{\beta}$ corresponds to the weakly continuous bilinear extension $\overline{\beta}(\cdot)(\cdot) : [C \oplus F_1] \times [C \oplus F_1] \to \mathbb{R}$ of $\beta$. (Check this as exercise).
Finally use $\beta = D^2\phi(\eta)$ for each fixed $\eta \in C$ to get the required bilinear extension $D^2\phi(\eta)$.

\[\square\]

**Lemma II.2.** ([Mo], Pitman Books, 1984)

For $t > 0$ define $W_t^* \in C$ by

\[
W_t^*(s) := \begin{cases} 
\frac{1}{\sqrt{t}}[W(t + s) - W(0)], & -t \leq s < 0, \\
0 & -r \leq s \leq -t.
\end{cases}
\]

Let $\beta$ be a continuous bilinear form on $C$. Then

\[
\lim_{t \to 0+} \frac{1}{t} E\beta(\eta x_t - \tilde{\eta}_t, \eta x_t - \tilde{\eta}_t) - E\beta(G(\eta) \circ W_t^*, G(\eta) \circ W_t^*) = 0
\]

**Proof.**

Use

\[
\lim_{t \to 0+} E\|\frac{1}{\sqrt{t}}(\eta x_t - \tilde{\eta}_t - G(\eta) \circ W_t^*)\|_C^2 = 0.
\]

The above limit follows from the Lipschitz continuity of $H$ and $G$ and the martingale properties of the Itô integral. Conclusion of lemma is obtained by a computation using the bilinearity of $\beta$, Hölder’s inequality and the above limit. ([Mo], Pitman Books, 1984, pp. 86-87.)

\[\square\]

**Lemma II.3.** ([Mo], Pitman Books, 1984)

Let $\beta$ be a continuous bilinear form on $C$ and $\{e_i\}_{i=1}^m$ be any basis for $\mathbb{R}^m$. Then

\[
\lim_{t \to 0+} \frac{1}{t} E\beta(\eta x_t - \tilde{\eta}_t, \eta x_t - \tilde{\eta}_t) = \sum_{i=1}^m \beta(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}})
\]

for each $\eta \in C$.

**Proof.**
By taking coordinates reduce to the one-dimensional case $d = m = 1$:

$$
\lim_{t \to 0+} E\beta(W_t^*, W_t^*) = \mathcal{B}(\chi_{\{0\}}, \chi_{\{0\}})
$$

with $W$ one-dimensional Brownian motion. The proof of the above relation is lengthy and difficult. A key idea is the use of the projective tensor product $C \otimes \pi C$ in order to view the continuous \textit{bilinear} form $\beta$ as a continuous \textit{linear} functional on $C \otimes \pi C$. At this level $\beta$ commutes with the (Bochner) expectation. Rest of computation is effected using Mercer’s theorem and some Fourier analysis. See [Mo], 1984, pp. 88-94. □

\textbf{Theorem II.3.} ([Mo], Pitman Books, 1984)

In (XIV) suppose $H$ and $G$ are globally bounded and Lipschitz. Let $S : D(S) \subset C_b \to C_b$ be the weak generator of $\{S_t\}$. Suppose $\phi \in D(S)$ is sufficiently smooth (e.g. $\phi$ is $C^2$, $D\phi$, $D^2\phi$ globally bounded and Lipschitz). Then $\phi \in D(A)$ and

$$
A(\phi)(\eta) = S(\phi)(\eta) + D\phi(\eta)\langle H(\eta)\chi_{\{0\}} \rangle
+ \frac{1}{2} \sum_{i=1}^{m} D^2\phi(\eta)\langle G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}} \rangle.
$$

where $\{e_i\}_{i=1}^{m}$ is any basis for $\mathbb{R}^m$.

\textbf{Proof.}

\textbf{Step 1.}

For fixed $\eta \in C$, use Taylor’s theorem:

$$
\phi(\eta x_t) - \phi(\eta) = \phi(\tilde{\eta}_t) - \phi(\eta) + D\phi(\tilde{\eta}_t)(\eta x_t - \tilde{\eta}_t) + R(t)
$$

a.s. for $t > 0$; where

$$
R(t) := \int_0^1 (1 - u) D^2\phi(\tilde{\eta}_t + u(\eta x_t - \tilde{\eta}_t))(\eta x_t - \tilde{\eta}_t, \eta x_t - \tilde{\eta}_t) du.
$$
Take expectations and divide by $t > 0$:

$$
\frac{1}{t} E[\phi(nx_t) - \phi(\eta)] = \frac{1}{t} \left[ S_t(\phi(\eta) - \phi(\eta)) + D\phi(\bar{\eta}_t) \left\{ E \left[ \frac{1}{t} (nx_t - \bar{\eta}_t) \right] \right\} \right] \\
+ \frac{1}{t} ER(t)
$$

(3)

for $t > 0$.

As $t \to 0+$, the first term on the RHS converges to $S(\phi)(\eta)$, because $\phi \in D(S)$.

**Step 2.**

Consider second term on the RHS of (3). Then

$$
\lim_{t \to 0^+} \left[ E \left\{ \frac{1}{t} (nx_t - \bar{\eta}_t) \right\} \right](s) = \begin{cases} 
\lim_{t \to 0^+} \frac{1}{t} \int_0^t E[H(nx_u)] \, du, & s = 0 \\
0 & -r \leq s < 0.
\end{cases}
$$

$$
= [H(\eta)\chi_{\{0\}}](s), & -r \leq s \leq 0.
\]

Since $H$ is bounded, then $\|E\{\frac{1}{t} (nx_t - \bar{\eta}_t)\}\|_C$ is bounded in $t > 0$ and $\eta \in C$ (**Exercise**). Hence

$$
w - \lim_{t \to 0^+} \left[ E \left\{ \frac{1}{t} (nx_t - \bar{\eta}_t) \right\} \right] = H(\eta)\chi_{\{0\}} (\notin C).
$$

Therefore by Lemma II.1 and the continuity of $D\phi$ at $\eta$:

$$
\lim_{t \to 0^+} D\phi(\bar{\eta}_t) \left\{ E \left[ \frac{1}{t} (nx_t - \bar{\eta}_t) \right] \right\} = \lim_{t \to 0^+} D\phi(\eta) \left\{ E \left[ \frac{1}{t} (nx_t - \bar{\eta}_t) \right] \right\} \\
= \overline{D\phi(\eta)}(H(\eta)\chi_{\{0\}})
$$

**Step 3.**
To compute limit of third term in RHS of (3), consider

$$
\left| \frac{1}{t} ED^2 \phi [\tilde{\eta}_t + u(\eta x_t - \tilde{\eta}_t)](\eta x_t - \tilde{\eta}_t, \eta x_t - \tilde{\eta}_t) 
- \frac{1}{t} ED^2 \phi (\eta)(\eta x_t - \tilde{\eta}_t, \eta x_t - \tilde{\eta}_t) \right|
$$

$$
\leq (E \|D^2 \phi [\tilde{\eta}_t + u(\eta x_t - \tilde{\eta}_t)] - D^2 \phi (\eta)\|^2)^{1/2} \left[ \frac{1}{t^2} E \|\eta x_t - \tilde{\eta}_t\|^4 \right]^{1/2}
$$

$$
\leq K(t^2 + 1)^{1/2} \left[ E \|D^2 \phi [\tilde{\eta}_t + u(\eta x_t - \tilde{\eta}_t)] - D^2 \phi (\eta)\|^2 \right]^{1/2}
\rightarrow 0
$$
as $t \to 0+$, uniformly for $u \in [0,1]$, by martingale properties of the Itô integral and the Lipschitz continuity of $D^2 \phi$. Therefore by Lemma II.3

$$
\lim_{t \to 0+} \frac{1}{t} ER(t) = \int_0^1 (1 - u) \lim_{t \to 0+} \frac{1}{t} ED^2 \phi (\eta)(\eta x_t - \tilde{\eta}_t, \eta x_t - \tilde{\eta}_t) \, du
$$

$$
= \frac{1}{2} \sum_{i=1}^m D^2 \phi (\eta)(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}). \)
$$

The above is a weak limit since $\phi \in D(S)$ and has first and second derivatives globally bounded on $C$. $\square$

5. Quasitame Functions

Recall that a function $\phi : C \to \mathbb{R}$ is tame (or a cylinder function) if there is a finite set $\{s_1 < s_2 < \cdots < s_k\}$ in $[-r,0]$ and a $C^\infty$-bounded function $f : (\mathbb{R}^d)^k \to \mathbb{R}$ such that

$$
\phi(\eta) = f(\eta(s_1), \cdots, \eta(s_k)), \quad \eta \in C.
$$

The set of all tame functions is a weakly dense subalgebra of $C_b$, invariant under the static shift $S_t$ and generates Borel $C$. For $k \geq 2$ the tame function $\phi$ lies outside the domain of strong continuity $C^0_b$ of $P_t$, and hence outside $D(A)$ ([Mo], Pitman Books, 1984, pp.98-103; see also proof of Theorem IV .2.2, pp. 73-76). To overcome this difficulty we introduce
Definition.

Say $\phi : C \to \mathbb{R}$ is quasitame if there are $C^\infty$-bounded maps $h : (\mathbb{R}^d)^k \to \mathbb{R}$, $f_j : \mathbb{R}^d \to \mathbb{R}^d$, and piecewise $C^1$ functions $g_j : [-r, 0] \to \mathbb{R}, 1 \leq j \leq k - 1$, such that

$$
\phi(\eta) = h \left( \int_{-r}^{0} f_1(\eta(s))g_1(s) ds, \ldots, \int_{-r}^{0} f_{k-1}(\eta(s))g_{k-1}(s) ds, \eta(0) \right)
$$

for all $\eta \in C$.

Theorem II.4. ([Mo], Pitman Books, 1984)

The set of all quasitame functions is a weakly dense subalgebra of $C_0^0$, invariant under $S_t$, generates Borel $C$ and belongs to $D(A)$. In particular, if $\phi$ is the quasitame function given by (4), then

$$
A(\phi)(\eta) = \sum_{j=1}^{k-1} D_j h(m(\eta))\{f_j(\eta(0))g_j(0) - f_j(\eta(-r))g_j(-r)
$$

$$
- \int_{-r}^{0} f_j(\eta(s))g_j'(s) ds \}
$$

$$
+ D_k h(m(\eta))(H(\eta)) + \frac{1}{2}\text{trace}[D_k^2 h(m(\eta)) \circ (G(\eta) \times G(\eta))].
$$

for all $\eta \in C$, where

$$
m(\eta) := \left( \int_{-r}^{0} f_1(\eta(s))g_1(s) ds, \ldots, \int_{-r}^{0} f_{k-1}(\eta(s))g_{k-1}(s) ds, \eta(0) \right).
$$

Remarks.

(i) Replace $C$ by the Hilbert space $M_2$. No need for the weak extensions because $M_2$ is weakly complete. Extensions of $D\phi(v, \eta)$ and $D^2\phi(v, \eta)$ correspond to partial derivatives in the $\mathbb{R}^d$-variable. Tame functions do not exist on $M_2$ but quasitame functions do! (with $\eta(0)$ replaced by $v \in \mathbb{R}^d$).
Analysis of supermartingale behavior and stability of $\phi^{(\eta x_t)}$ given in Kushner ([Ku], JDE, 1968). Infinite fading memory setting by Mizel and Trützer ([M-T], JIE, 1984) in the weighted state space $\mathbb{R}^d \times L^2((-\infty, 0], \mathbb{R}; \rho)$.

(ii) For each quasitame $\phi$ on $C$, $\phi^{(\eta x_t)}$ is a semimartingale, and the Itô formula holds:

$$d[\phi^{(\eta x_t)}] = A(\phi)(\eta x_t) \, dt + D\phi(\eta)(H(\eta)\chi_{\{0\}}) \, dW(t).$$