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Alison Watts
Southern Illinois University Carbondale

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Abstract: There are many situations where people join groups, the number of groups is fixed, and where a person can only join a new group if the new group approves the person's joining. We examine such situations where agents are concerned with either local status (each agent wants to be the highest status agent in his group) or global status (each agent wants to join the highest status group that she can join). For both cases, conditions are provided under which a segregated stable partition of groups form where similar people are grouped together and conditions are provided under which an integrated stable partition of groups form where dissimilar people are grouped together. We also show that the addition of an empty group (or location) to a segregated stable partition of groups may cause integration to occur.

Department of Economics
Southern Illinois University, Carbondale, IL 62901, USA
(wattsa@siu.edu)

1. Introduction

There are many situations where people join groups, the number of groups is fixed, and where a person can only join a new group if the new group approves the person's joining. Examples include academics joining an academic department where the number of schools (or departments) is fixed and where the current members of the department vote on whether or not to allow the new academic to join (or whether or not to make the new academic an offer). Other examples include athletes joining an athletic team and college students joining a sorority or fraternity. We examine such situations where agents are concerned with either local status (each agent wants to be the highest status agent in his group) or global status (each agent wants to join the highest status group that she can join).

Specifically, each person is endowed with a quality level and each person receives a payoff from the group he joins which depends only on who else is in the group (or only on the quality levels of the people in the group).[1] We consider two such possible payoff functions. The first is the average quality (or global status) payoff function where an agent's payoff is increasing in the average quality of the agents in his group. Such a person is concerned with global status since he will want to be a member of the most prestigious group (or the group with the highest average quality).[2] The second payoff function considered is the big fish (or local status) payoff function, here an agent prefers the group where he is the highest quality agent (or the "big fish")

[1]Thus payoffs are hedonic as in Bogomolnaia and Jackson [2002], Banerjee, Konishi and Sonmez [2001], and Milchtaich and Winter [2002].

[2]For simplicity, agents do not compare the average quality of their group to the average quality of other groups. Instead each agent simply prefers the group with the highest average quality. Thus an agent's ranking of the groups is identical to what it would be if an agent did compare his group's average quality to the average quality of other groups.

in the group. Such an agent is concerned with local status since he cares about his quality ranking within his group. For example, an academic concerned with local status may prefer a less prestigious job if he will be the "big fish" at the less prestigious university.[3]

For both types of payoff functions, conditions are provided under which a segregated stable partition of groups form where people of similar quality are grouped together and conditions are provided under which an integrated stable partition of groups form where people with dissimilar quality levels are grouped together. We also investigate what happens to a segregated stable partition of groups when a new empty group (or location) is added and show that the addition of such an empty location may cause integration to occur.

We define a match (or partition) of groups to be stable if whenever there exists an agent who would like to change groups (or locations), the non-strict majority of agents at the new location vetoes the move. Thus to move to a new location requires strict majority approval at the new location. This notion of stability is related to the concept of "individual stability" used in a hedonic coalition formation framework by Bogomolnaia and Jackson [2002].[4] (See also Greenberg [1978] and Drèze and Greenberg [1980].)

Next we give a brief overview of the average quality (or global status) payoff results.

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[3]A discussion of global versus local status and how a person may trade one for the other is given in Frank [1985], while an axiomatization of local status is given in Ok and Kockesen [2000].

[4] A coalition partition is individually stable if whenever there exists an agent who would like to change locations, at least one person at the new location would be made worse off by the move. Thus Bogomolnaia and Jackson [2002] require unanimous approval for a move where approval is granted as long as an agent is not made worse off by having the new agent join his group. In contrast we require strict majority approval for a move where approval is only given if an agent is made better off when the new agent joins his group. Thus individual stability does not imply the notion of stable match used here nor does our definition of stable match imply individual stability.

Proposition 1 shows that a segregated stable match of people to groups (or locations) always exists. An example of such a match is as follows. Place all agents of the highest quality level at one location, place all agents of the second highest quality at a second location, continue placing agents in this fashion until there are either no more locations or no more agents. If there are no more agents then some locations will remain empty, while if there are no more locations then place all remaining agents at the lowest quality location. This segregated match is stable because no agent will want to move to a lower quality group and even though an agent would like to move to a higher quality group, he will bring the average quality of this group down and so this group will refuse to let him join. Proposition 2 gives conditions under which there does not exist an integrated stable match. These conditions are roughly met if the agents are ranked according to quality and if for any agents i, j, k ranked next to each other, the difference in quality between agents j and k is quite a bit larger than the difference in quality between agents i and j. Thus the quality of the agents is decreasing quite rapidly. Here if agents are placed in an integrated match then there always exists at least one high quality agent who is in a low average quality group and who would increase the average quality of another higher quality group. Thus this agent will be able to leave for the high quality group and so the original match is not stable. Proposition 3 gives conditions under which there does exist an integrated stable match when there are only two possible locations. These conditions are roughly met if the quality of the top ranked agents is similar and if there is a big decrease in quality after that. Here an integrated stable match exists where all the high quality agents except one are grouped together and everyone else is grouped at the alternative location. (This match is integrated as long as the high quality agent chosen to be grouped with the low quality agents is not the "lowest" high quality agent.) Although the high quality agent who is not in the high quality group would like to join that group, the group will refuse to let him join since his quality is similar to theirs and so he will not increase the average quality of their group. Additionally, because of the sharp drop in quality, none of the agents in the high quality group will want to join the low quality group.

We also consider what happens to a segregated stable match when a new empty location is added. The following dynamic process is used to answer this question. Agents start in a match which may or may not be stable. Each period an agent is selected at random, this agent can change locations if he wants to and if he receives majority approval for the move from the agents at the new location. Proposition 4 gives conditions under which if agents start at a segregated stable match and if a new empty location is added, then the dynamic process leads to an integrated stable match. The conditions are met roughly if the highest quality agent is of significantly higher quality than the other agents and if the highest quality agent is initially in a group with quite a few other agents. Then if the new location is added and if a low quality agent who is the top ranked agent in his group is given the first chance to move to the new location he will since the average quality at the empty location after he moves will equal his own quality which is above the average quality of his current group. If the highest quality agent is given the next chance to move to the new location he will move as well, since there are a lot of low ranked agents at his current location, but there will only be one low ranked agent at the new location. Thus the average quality at the new location after the highest quality agent moves will be larger than the average quality at the top ranked agent's current location. This new match is now integrated and it remains integrated since the two agents at the new location will not allow all of the intermediate quality agents to join their group since these agents would decrease the average quality of the new location.

Next we give a brief overview of the big fish (or local status) payoff results. Proposition 5 shows that at least one stable segregated match always exists. An example of such a segregated match is if agents are segregated into groups of equal size, where if one group must be larger (because the number of agents does not divide evenly into the number of groups) then this will be the highest quality group. Such a match is stable since the only people who will want to leave their groups are those who are not the top ranked agents in their groups. Such an agent will want to leave for a lower quality group where he can improve his ranking, but since the original match is segregated an agent who leaves for a lower quality group would end up the top ranked agent at such a group and so the agents at the lower quality group will always veto such a move. Proposition 6 shows that if the number of agents is greater than the number of locations then an integrated stable match always exists as well. An example of such an integrated stable match is as follows. Consider the segregated match described above but switch the top ranked agent in the highest quality group with the top ranked agent in the second highest quality group. Similar reasoning to that above shows that this match is also stable. Thus it is easier to achieve integration with the big fish payoff than with the average quality payoff because with the big fish payoff people are only concerned about their ranking within their group. So a person is content to be in any group where the other agents are ranked below him even if the other agents are ranked significantly below him, whereas a person who is concerned with average quality would not be content to be in a group with significantly lower quality agents.

Proposition 7 gives conditions under which if agents are initially located in a segregated stable match and if a new empty location is added then the dynamic process will lead to either an integrated stable match or to a cycle of matches where at least one location remains integrated at all times. Such conditions are roughly met if agents are originally in a segregated stable match and if a new empty location is added and if the last ranked agents at two different locations are given the first chance to move. Both agents will agree to the move since their ranking will improve from last to first and last to second, respectively. This new location is now integrated and will remain integrated since the two agents will only allow others to join if they have quality levels which are less than both of the existing agents' quality levels.

The papers most closely related to the current one are Milchtaich and Winter [2002], Bogomolnaia and Jackson [2002] and Banerjee, Konishi, and Sonmez [2001]. Milchtaich and Winter [2002] also study group formation where the number of groups is fixed. Their model differs from ours in that agents want to join the group which has agents who are the most similar to them and agents do not need permission from the new group in order to join it. Milchtaich and Winter [2002] show that segregated, stable partitions of groups exist and that a dynamic model of group formation converges to a stable, segregated partition. Brams, Jones, and Kilgour [2002] study coalition formation when agents want to join a group which has others similar to them and show that disconnected (or integrated) coalitions may form. Bogomolnaia and Jackson [2002] and Banerjee, Konishi, and Sonmez [2001] study coalition formation when an agent's payoff is based only on who else is in his coalition. Bogomolnaia and Jackson [2002] give conditions under which individually stable coalition partitions form while Banerjee, Konishi, and Sonmez [2001] give conditions under which the core exists. Our model differs in that here the number of groups formed is fixed while in the coalition formation models, the number of coalitions formed is endogenous. Our focus is also different in that we are interested in conditions under which integrated and segregated partitions form and we also examine dynamic group formation in the context of adding a new empty location to a segregated, stable match or partition.

Additionally, there is a large literature of traditional coalition formation games where again the number of coalitions formed is endogenous. (See Guesnerie and Oddou [1981], Greenberg and Weber [1986] and [1993], Demange [1994], and Kaneko and Wooders [1982].) This literature is concerned with the stability of coalition partitions and the existence of the core.

The local public goods literature is also related to group formation since in these models agents join jurisdictions (or groups) which produce local public goods. Here agents prefer to join jurisdictions where the other people have preferences for how much of the local public good should be produced which are similar to their own preferences. See Wooders [1980], Bewley [1981], Greenberg and Weber [1986] and [1993], Jehiel and Scotchmer [1997], Guesnerie and Oddou [1981], and Konishi, Le Breton, and Weber [1998].

The formation of social networks is also related to group formation since a group can be thought of as a social network where every agent in a given group is linked to every other agent in the group. Aumann and Myerson [1988] and Jackson and Wolinsky [1996] were the first to look at network formation in a strategic context while Watts [2001] and Jackson and Watts [2002] examine the dynamic formation of social networks.

The paper proceeds as follows. The model is presented in section 2. The average quality payoff results are given in section 3 and the big fish payoff results are given in section 4.

2. Model

<u>Agents</u>

There are n *agents* who are represented by the set {1, 2,...,i,...,n}. Each agent i is endowed with *quality level* q_i. An agent's quality level may represent many different things such as an agent's athletic ability, academic ability, or publishing ability. Without loss of generality we assume that agents are indexed such that q_1 q_2 ... q_n.

Location

There are m *locations* which are represented by the set {A,B,...,G,...,M}. Each agent is positioned at exactly one location. An agent's location may represent many different things such as his athletic team or academic department. Thus an agent's location represents a group that he joins.

Match

A *match* is an assignment of each agent to exactly one location.

Payoffs

Consider the following general *payoff function*. Let agents {i...,j,...,k} be positioned at location C. Then agent j receives a payoff of u_j(i,...,j,...,k). For any agent not located at C, 's payoff if he moves to C is u (i,...,k,). Thus an agent's payoff depends only on the set of agents who are located with him.

We will focus our analysis on the following two specifications of u_j.

Average Quality Payoff Function

If each agent receives a payoff based on the *average quality* of his location, then u_j(i,..,j,...,k) is a strictly increasing function of $(q_i+...+q_k)/(k-i+1)$. Thus an agent's payoff is strictly increasing in the average quality of his location.

Big Fish Payoff Function (Quality Ranking Payoff Function)

Under the *big fish payoff function*, each agent prefers to be the highest quality agent (or the "big fish") in the group. Formally, let (X) represent the cardinality of integer set X. Here u_j(i,..,j,...,k) is strictly increasing in (n - ({i,...,k q q_j where j and {i,...,k})). We refer to (1+ ({i,...,k q q_j where j and {i,...,k}})) as j's *quality ranking* (or *rank*) at his current location. Thus j's quality ranking equals 1 if there are no agents ranked above him at his current location or if ({i,...,k q q_j where j and {i,...,k}})=0 and by definition j's most preferred rank is 1.

In addition, if agent j has the same quality ranking in groups C and D then j prefers to be a member of the larger group. Thus if the same number of people are ranked above j in groups C and D, then j prefers the group where more people are ranked below him. Formally, if agents {i,...,k} are located at C and if agents { ,...,m} are located at D and if ({ ,...,m q_i q_j where i j and i { ,...,m}}) = ({i,...,k q q_j where j and {i,...,k}}) then u_j(i,...,k,j) > u_j(,...,m,j) if and only if (k-i)>(m-).

<u>Stable Match</u>

Consider the match where agents {i_G,...,j_G} {1,...,n} are located at G, for all G {A,...,M}. Such a match is *stable* if and only if for any agent i located at G, if u_i(i_G,...,i,...,j_G) < u_i(i_D,...,j_D,i) for some location D G, then a non-strict majority of agents j {i_D,...,j_D} must have u_j(i_D,...,j_D) u_j(i_D,..,j_D,i).

Thus a match is stable if whenever there exists an agent i who would like to change locations, the non-strict majority of agents at the new location will veto the move and agent i will not be able to move to the new location. An agent can only change locations if the strict majority of agents at the new location approve such a change. Formally, agent i can only move to location

D if the strict majority of agents j $\{i_D,...,j_D\}$ at D have $u_j(i_D,...,j_D) < u_j(i_D,.,j_D,i)$. Additionally, we assume that if agent i is indifferent about changing locations, then he chooses to stay where he is.

Segregated Stable Match

A stable match is called a *segregated stable match* if anytime two agents i and j are both located at G, then all agents with quality levels in between q_i and q_j must also be located at G.

Integrated (Mixed) Stable Match

A stable match is called an *integrated stable match* if there exists agents i, j, and k such that $q_i>q_j>q_k$ and such that i and k are both located at some C $\{A,...,M\}$ but j is located at some D C. Thus agents of similar quality are not always located together.

3. Average Quality Payoff Results

In this section, we examine the type of stable matches that exist when players want to be in the group with the highest average quality. Specifically we are interested in whether or not segregated stable matches exist and whether or not integrated stable matches exist. We also define a dynamic process and use this process to look at what happens when a new empty location is added to a segregated stable match.

Proposition 1 and Corollary 1 show that a weakly Pareto efficient, segregated stable match always exists.

Proposition 1: A segregated stable match always exists.

Proof: We will show that the following segregated match is stable. Recall that q_1 q_2 ... q_n. Place agent 1 at location A. Place any agents i with $q_i=q_1$ also at A. Place all agents j with the second highest quality level at location B. Continue in this fashion until there are either no more agents or no more locations. If we run out of agents first then some locations (such as location M) will be empty. If we run out of locations first, then place all remaining agents at location M. Now each agent i located at G<M is at his own location (or is located with other agents of the exact same quality level). If there are more quality levels than locations then all low quality agents are located together at M. If there are strictly fewer quality levels than locations then some locations (such as M) are empty. Obviously such a match is segregated. Next we show that such a match is stable. First notice that no agent i located at I wants to move to a location G>I with lower quality agents since the average quality at G>I would be lower (even after i joins G) and thus i's payoff would be lower. However i would like to move to a location G<I with higher quality agents. But any agent g located at G<I will refuse to let i join, since doing so will lower the average quality at G and thus will lower g's payoff.

Corollary 1: The above segregated stable match is weakly Pareto efficient.

Proof: Consider the segregated stable match described in the proof of Proposition 1. If we move any agents not located at A to location A, then the average quality at A will fall and all agents originally located at A will be made worse off. (Similarly if we move any agents located at A to any other non-empty group then these A agents will be made worse off.) Thus the agents at A must remain by themselves and so we can effectively ignore them for the remainder of the proof. Next consider the agents located at B. If we move any agents not located at A or B to B then the average quality at B will fall (similarly, if we move any agents located at B to a non-empty lower ranked group then these B agents will be made worse off), thus the agents at B must also remain isolated. Continue in this fashion. All agents located at G<M must remain by themselves, otherwise at least one of them will be made worse off. Thus this match is weakly Pareto efficient. Note that if there exists agents i and j such that $q_i=q_j$ and if there exists an empty group, then we can move agent j to this empty location. The payoffs at this new match will be exactly the same as the payoffs at the old match, thus such a move is not Pareto improving.

Proposition 2 gives conditions under which there does not exist an integrated stable match. These conditions are illustrated in the example following Proposition 2. Proposition 3 gives conditions under which there does exist an integrated stable match.

Proposition 2: Assume $q_1>q_2>...>q_n$. There does not exist an integrated stable match if for all i $\{2,...,n-1\}$, $q_i > \max \{(q_1+q_{i+1})/2, (q_1+q_2+q_{i+1})/3,..., (q_1+q_2+...+q_{i-1}+q_{i+1})/i\}$.

Proof: We will prove Proposition 2 true by contradiction. Assume agents are in an integrated stable match and that for all i $\{2,...,n-1\}$, $q_i > \max \{(q_1+q_{i+1})/2, (q_1+q_2+q_{i+1})/3,..., (q_1+q_2+...+q_{i-1}+q_{i+1})/i\}$. First we show that there exists an agent who would like to change groups. Since the stable match is integrated there must exist at least two non-empty groups one of which, say C, is integrated. We consider two cases. In the first case, there exists a group D with average quality lower than or equal to C. In the second case, C has strictly lower average quality than all other groups. Consider the first case and let agents $\{i_C,...,j_C\}$ be located at C and let $\{i,...,j_D\}$ be located at D. Let agent i be the highest quality ranked agent in D. If there are at least two agents in D then agent i is pulling the average quality of D up (it is impossible for all agents in D to have the same quality level since $q_1>q_2>...>q_n$). Since $q_i > (q_i+...+q_{jD})/(j_D-i+1)$ and since $(q_{iC}+...+q_{jC})/(j_C-i_C+1)$ $(q_i+...+q_{jD})/(j_D-i+1)$, it must be that the average quality of C if i joins C is strictly greater than the average quality of D or $(q_{iC}+...+q_{jC}+q_i)/(j_C-i_C+2) > (q_i+...+q_{jD})/(j_D-i+1)$. Thus agent i would prefer to join group C. If, in the above analysis, agent i is the only member of D and if $(q_{iC}+...+q_{jC})/(j_C-i_C+1) > q_i$ then it is still the case that the average quality of C if i joins C is strictly greater than the average quality of D. Thus i wants to join C. (If agent i is the only member of D and if $(q_{iC}+...+q_{jC})/(j_C-i_C+1) = q_i$ then the highest quality ranked agent in C will want to join D and D will always agree to this, which contradicts our definition of stability.)

Next we show that the agents at C will let agent i join C, which contradicts the definition of stability. Since C is integrated the highest average quality that it is possible for group C to have is $\max\{(q_1+q_{i+1})/2, (q_1+q_2+q_{i+1})/3,..., (q_1+q_2+...+q_{i-1}+q_{i+1})/i\}$, thus $\max\{(q_1+q_{i+1})/2, (q_1+q_2+q_{i+1})/3,..., (q_1+q_2+...+q_{i-1}+q_{i+1})/i\} \quad (q_{iC}+...+q_{jC})/(j_C-i_C+1)$. Group C will let i join if $q_i > (q_{iC}+...+q_{jC})/(j_C-i_C+1)$. Since $q_i > \max\{(q_1+q_{i+1})/2, (q_1+q_2+q_{i+1})/3,..., (q_1+q_2+...+q_{i-1}+q_{i+1})/i\}$ the agents at C will let i join. Thus this integrated match is not stable. Contradiction.

Now consider the second case where C has strictly lower average quality than any other group. If there exists another integrated stable group of higher average quality, say B, then proceed as above where B replaces C in the analysis. If there is no such group then C is the only integrated group. Let j be the highest quality ranked agent in C. Since C is the only integrated group and since C has the lowest average quality, there must exist a group D with strictly higher average quality than C and such that q_j is strictly larger than the quality of every agent located at D. Analysis similar to that above can be used to show that j would like to join D and to show that the agents of D will let j join. Thus this match is not stable. Contradiction.

The conditions of Proposition 2 are met if (q_i-q_{i+1}) is increasing quite a bit as i increases. The intuition is as follows. If agents are in an integrated match, then the highest quality agent in any integrated group will be pulling up the average quality of his group quite a bit and will want to leave. For at least one such agent, say j, there should exist another integrated group of higher average quality where some agents are of lower quality than j. If (q_i-q_{i+1}) is significantly increasing then agent j should bring up the average quality of this group and will be allowed to join and so the original integrated match will not be stable. The following example illustrates this intuition.

<u>Example where No Integrated Stable Match Exists</u>

Let m=2 and n=4 with $q_1=10$, $q_2=9$, $q_3=7$, and $q_4=0$. Here (q_i-q_{i+1}) increases significantly as i increases and the conditions of Proposition 2 are met. To see that there is no integrated stable match consider the integrated match where agents 1 and 3 are located at A and agents 2 and 4 are located at B. It is easy to show that this match is not stable. Agent 2 would like to leave group B and join group A since the average quality at B is 4.5 but the average quality at A once 2 joins would be 8.7. Agents 1 and 3 will let 2 join A because he increases the average quality at A from 8.5 to 8.7. So the agents will end up in the segregated stable group where 1, 2, and 3 are at A and 4 is at B. One can easily show that all other integrated matches are not stable as well.

Proposition 3: Assume m=2 and that $q_1>q_2>...>q_n$. If there exists an agent i $\{2,...,n-1\}$ such that $(q_1+q_2+...+q_{i-1}+q_{i+1})/i > \max\ \{q_i,\ (q_1+q_i+q_{i+2}+q_{i+3}+...+q_n)/(n-i+1)\}$ then there exists an integrated stable match.

Proof: We assume that there exists an agent i $\{2,...,n-1\}$ such that $(q_1+q_2+...+q_{i-1}+q_{i+1})/i > \max\ \{q_i,\ (q_1+q_i+q_{i+2}+q_{i+3}+...+q_n)/(n-i+1)\}$ and show that this condition implies that there exists an integrated stable match. Consider the match where agents $\{q_1,q_2,...,q_{i-1},q_{i+1}\}$ are located at A and agents $\{q_i,q_{i+2},q_{i+3},...,q_n\}$ are located at B. First we check that even if an agent at B would like to move to A, he will not be able to do so. Since $q_1>q_2>...>q_n$ we know that q_i $(q_i+q_{i+2}+q_{i+3}+...+q_n)/(n-i)$ and by assumption $(q_1+q_2+...+q_{i-1}+q_{i+1})/i > q_i$, thus it must be that $(q_1+q_2+...+q_{i-1}+q_i+q_{i+1})/i > (q_i+q_{i+2}+q_{i+3}+...+q_n)/(n-i)$ which implies that i would like to leave B and join A. However since $(q_1+q_2+...+q_{i-1}+q_{i+1})/i > q_i$ the agents at A will refuse to let i join A. Similarly no agent j $\{i+2, i+3,...,n\}$ will be able to move to A. We must also check that no agent at A will move to B. Since $(q_1+q_2+...+q_{i-1}+q_{i+1})/i > (q_1+q_i+q_{i+2}+q_{i+3}+...+q_n)/(n-i+1)$, agent 1 prefers to stay at A than to go to B. Similarly, since $q_1>q_2>...>q_n$ no other agent located at A will want to move to B.

The conditions of Proposition 3 are met if there exists an agent (i+1) 3 such that agents i and (i+1) are of similar quality but agent (i+2) is of significantly lower quality than agent (i+1). Here an integrated stable match can exist where agents i and (i+2) are located together but (i+1) is located elsewhere. The intuition regarding the stability of such a match is as follows. Agent i will want to leave his location since (i+2) is substantially pulling down average quality, however i will not be able to join (i+1)'s group since i's quality is not high enough to bring up the average quality of (i+1)'s group. Additionally no one from (i+1)'s group will want to join i's group because (i+2) pulls down the average quality of that group too much. The following example illustrates this intuition.

Existence of Integrated Stable Match Example

Consider the previous example where we increase the quality of agent 3 so that $q_1=10$, $q_2=9$, $q_3=8.5$, and $q_4=0$. Here agents 2 and 3 are of similar quality, however agent 4 is of significantly lower quality and the conditions of Proposition 3 are met for i=2. We show that the following integrated match is stable. Let agents 1 and 3 be located at A and agents 2 and 4 be located at B. Agent 2 would like to leave B and join A, however agents 1 and 3 will refuse to let 2 join A since doing so will decrease the average quality at A from 9.25 to 9.2. Similarly agent 4 cannot leave B for A. Agent 1 does not want to leave A for B since A has average quality of 9.25 but the average quality at B after 1 joins is 5.8. Similarly, agent 3 does not want to leave A. Thus this integrated match is stable.

Dynamics

The following *dynamic process* is used in Propositions 4 and 7. Agents begin in a match which may or may not be stable. Time is divided into periods and in each period an agent is randomly identified. This agent is allowed to change locations, where an agent can only change locations if such a move strictly increases his payoff and if the agent has strict majority approval for the move by the agents at the new location. If after some time period no agents are able to change locations then the dynamic process has reached a stable match.

The following corollary is used in the proof of Proposition 4. This corollary shows that the dynamic process never becomes stuck in a cycle of matches.

Corollary 2: Assume agents are in an unstable match. The dynamic process leads to a stable match with probability approaching 1.

Proof: Assume to the contrary that the dynamic process does not lead to a stable match but instead leads to a series of matches in which one agent changes locations at each step in the series. (Since agents are randomly identified the probability that the same agent is chosen every time approaches 0 as the number of periods approaches infinity. Thus with probability approaching 1 the dynamic process cannot become stuck forever in an unstable match and must lead to either such a series of matches or to a stable match.) By assumption an agent can only change locations if the new location (with the addition of the agent making the change) has higher average quality than the old location. Thus for this series of matches there exists a series of locations for which the average quality is strictly increasing. Since there exists a fixed number of agents each of whom has a fixed quality level it is not possible for this series of matches to contain a series of locations which increases in average quality forever. Thus this series of matches must end and by definition the ending match is stable.

Next a new location is added to a segregated stable match and agents are given the opportunity to relocate. Proposition 4 provides conditions under which the dynamic process leads to an integrated stable match.

Proposition 4: Assume $q_1>q_2>...>q_n$. Assume agents are in a segregated stable match where agents $\{i_G,...,j_G\}$ $\{1,...,n\}$ are located at G for all G $\{A,...,M\}$. If there exists agent k located at A (the highest quality group) and agent located at some D A such that $q_k > (q_{iA}+...+q_{jA})/(j_A-i_A+1)$, $q > (q_{iD}+...+q_{jD})/(j_D-i_D+1)$, and $(q_k+q)/2 > (q_{iA}+...+q_{jA})/(j_A-i_A+1)$, then if a new location is added, and if agents and k are given the first opportunity to relocate, then agents will end up in an integrated stable match.

Proof: By Corollary 2 we know that with probability approaching 1, the dynamic process will end in a stable match, thus we only need to show that this match will be integrated. We show that if the assumptions of Proposition 4 are met, then agents k and prefer to go to the new location, and that at least one agent i {k+1,..., -1} does not go to the new location, thus agents end up in an integrated stable match. Note that since agents are originally in a segregated match and k is a member of the highest quality group, it must be that $q_k > q$.

Assume agents are initially in the segregated stable match described above. First we show that if is given the option to move to a new location he will. Since $q > (q_{iD}+...+q_{jD})/(j_D-i_D+1)$ we know that prefers the new location to location D. Next we show that k will also be willing to move to the new empty location. Since $(q_k+q)/2 > (q_{iA}+...+q_{jA})/(j_A-i_A+1)$ we know that k prefers to be at the new location with than to be at A. Since $q_k > q$ we know that will allow k to move to the new location. Next we check that agents k and do not allow all agents {k+1,..., -1} to join the new location. Since $(q_k+q)/2 > (q_{iA}+...+q_{jA})/(j_A-i_A+1)$ we know that k and will not allow all agents $\{q_{iA},...,q_{jA}\}$ to join the new location. Thus the new location will be integrated.

Next we check that agents k and do not leave the new location. The quality at the new location will always be greater than or equal to $(q_k+q)/2$, since k and will not allow anyone to join who lowers average quality. Agent k (or) will never want to leave for location A since some other high quality agents (agents with quality greater than or equal to $(q_k+q)/2$ may have left A and so the current average quality at A (after k or rejoins A) will be less than or equal to $(q_{iA}+...+q_{jA})/(j_A-i_A+1)$. However, agent k (or) may want to leave for a new location H with current quality level greater than $(q_k+q)/2$. Since $(q_k+q)/2$ is strictly greater than $(q_{iA}+...+q_{jA})/(j_A-i_A+1)$ and since A was initially the highest ranked group, no such group H exists. Thus if a new location is added agents will end up at a stable match where the new location is integrated.

The conditions of Proposition 4 are met if agent 1 is of significantly higher quality than the other agents and if the original segregated match has quite a few agents located with agent 1. The intuition for how adding a new location causes integration is as follows. Start at such a segregated match and give the top ranked agent at the second highest quality group the chance to move. He will leave for the new location since the others at his current location are pulling average quality down. Next give the top ranked agent (agent 1) at the top ranked group the opportunity to move. Agent 1 will also prefer to move to the new location, since his current location has a large number of agents who are bringing average quality down but the new location only has one agent who will bring average quality down. The new location is now integrated and will remain integrated since agent 1 will not allow everyone who was at his old location to join him at the new location. The following example illustrates this intuition.

Example where Adding a New Location Causes Integration

Let there be six agents and two initial locations where q_1=10, q_2=5, q_3=4, q_4=3, q_5=2 and q_6=1. Let the first three agents be located at A and the last three agents be located at B. It is easy to check that this segregated match is stable and that the inequalities of Proposition 4 are met where agents 1 and 4 correspond to agents k and in Proposition 4. Next we add a third location and show that if agents 1 and 4 are given the first chance to move to the new location they will, and that such a move leads to an integrated stable match. Add an empty location, C. If person 4 has the first opportunity to move to C he will since the average quality if 4 remains at B is 2 while the average quality if 4 moves to C is q_4=3. Next notice that person 1 will also move to C if given the opportunity. The average quality if 1 remains at A is 6.3 but the average quality if 1 moves to C is 6.5, so 1 is better off at C and agent 4 is better off if he allows agent 1 to join C. Next we show that the integrated match where 1 and 4 are located at C, 2 and 3 at A, and 5 and 6 at B is stable. Agent 1 does not want to move to A or B since doing so will give him a lower average quality. Agent 4 also does not want to move to A or B. Agent 2 would like to move to C, but he would lower the average quality at C so 1 and 4 will not let him join. Similarly 1 and 4 will not let anyone else join C. Likewise no other agent who wishes to change groups is able to. Thus this integrated group is stable.

4. Big Fish Payoff Results

In this section, we examine the type of stable matches that exist when each player wants to be the highest quality agent in the group. Specifically, we are interested in whether or not segregated stable matches exist and whether or not integrated stable matches exist. Additionally, we show in Proposition 7 that adding a new location to a segregated stable match may cause integration to occur.

Proposition 5 and Corollary 2 show that a segregated stable match always exists and give conditions under which a weakly Pareto efficient segregated stable match exists.

Proposition 5: Assume q_1 q_2 ... q_n. There exists at least one stable segregated match.

Proof: First assume N   M. Let (N/M)_ be the smallest integer closest to N/M. Let the remainder, r   N - M (N/M)_. We will show that the following segregated match is stable. Place agents $\{q_1, q_2, \ldots, q_{(N/M)\_+r}\}$ at location A. Place agents $\{q_{(N/M)\_+r+1}, q_{(N/M)\_+r+2, \ldots,}\ q_{2(N/M)\_+r}\}$ at location B. Continue placing the next (N/M)_ agents at the next location until the last (N/M)_ agents are placed at location M.

Next we show that this segregated match is stable. First notice that an agent who is not ranked first in his group would like to move to a lower quality group where his ranking would increase. (Note that if $q_1>q_2>...>q_n$ then moving to a lower quality group always increases an agent's ranking to first.) However, all agents in the lower quality group would veto such a move, since adding an agent whose quality is greater than or equal to all others in the group will decrease the quality ranking of all others in the group. Second, notice that no agent would like to move to a higher quality group since he would be ranked last in this group and this new group would have at least as many members as his current group. Thus such a move would not increase an agent's overall rank and thus would not increase his payoff.

Next assume $N<M$. Place each agent at a different location. An argument similar to that above shows that such a segregated group is stable.

Corollary 2: Assume $q_1>q_2>...>q_n$. The above segregated stable match is weakly Pareto efficient.

Proof: First assume N M and assume agents are in the segregated stable match described above. Since the number of locations and agents are fixed and since there are no empty locations, if one agent's ranking increases then there must exist another agent who's ranking decreases. Thus this match is weakly Pareto efficient.

Next assume $N<M$ and assume each agent is placed at a different location. Thus each agent is currently ranked first at his location. In order to increase any agent's payoff he must be ranked first in a larger group. However the other members of such a larger group would be made worse off since they no longer would be ranked first.

In the following example, we show that if some agents have identical quality levels then Corollary 2 may no longer hold true. Here the assumption that agents do not wish to share the limelight (i.e., agents prefer to have a quality level which is strictly above everyone else's in the group) prevents segregated matches from being Pareto optimal.

<u>Example where all weakly Pareto Optimal Matches are Integrated</u>

Let m=2 and n=4 and let agents 1 and 2 both have quality level q while agents 3 and 4 both have quality level q'<q. Here weak Pareto optimality requires that each location have exactly one high quality agent and one low quality agent. Thus each high quality agent will have rank 1 and each low quality agent will have rank 2. If agents are instead segregated then at least one agent's rank will decrease. For instance consider the match where agents 1 and 2 are at A and agents 3 and 4 are at B. Here everyone's rank equals two and so this match is not Pareto optimal. It is easy to show that no other segregated match is Pareto optimal as well.

Proposition 6 shows that an integrated stable match always exists as long as the number of agents is strictly greater than the number of locations. (If the number of agents is less than or equal to the number of locations then each agent prefers to be at his own location, thus all stable groups are trivially segregated.)

Proposition 6: Assume $q_1>q_2>...>q_n$. There exists at least one stable integrated match whenever N>M 2.

Proof: Let (N/M)_ be the smallest integer closest to N/M. Let the remainder, r N - M (N/M)_ and let s max{1, r}. We will show that the following integrated match is stable. Place agents $\{q_2, q_3,...,q_{(N/M)\_+s}\}$ at location A. Place agents $\{q_1, q_{(N/M)\_+s+1}, q_{(N/M)\_+s+2,....,} q_{2(N/M)\_+r}\}$ at location B.

(Note that since N>M 2, B always has at least two agents located at it and is thus integrated.) Place agents $\{q_{2(N/M)\_+r+1}, q_{2(N/M)\_+r+2,....,} q_{3(N/M)\_+r}\}$ at C. Continue by placing the next (N/M)_ agents at the next location until the last (N/M)_ agents are placed at location M.

Next we show that this integrated match is stable. First notice that if r 1 then agent 1 would like to move to location A, since agent 1 would also be ranked first at A and since there would be more agents ranked below 1 at A than there currently are at B. However, all agents currently located at A will veto this move. Next we check that no agent located at A would like to change locations. It is possible that an agent located at A (such as agent (n/m)_+s who is ranked last at A) may wish to move to location B where he would be ranked second. However, all agents currently at B except agent 1 will veto this move. Since N>M 2 there are at least 2 agents currently located at B, thus there will be at least one agent who will veto this move and so agent 1's vote for the move will be in the minority and the move will not occur. Notice that no agent currently located at A will be able to move to any other lower ranked group since all agents at such a location would veto this move. Lastly we must check that no other agent will change locations. An argument similar to that used in the proof of Proposition 5 shows that no other agent who wishes to change groups is able to do so.

Proposition 6 shows that integration is fairly easy to achieve with the big fish payoff function. Here each agent wants to be the highest quality agent in the group but does not care about the quality of the other agents in the group as long as their quality is below his. Thus integration can be achieved by placing agent 1 (the highest quality agent) with substantially lower quality agents while leaving other locations segregated. As long as the groups are of roughly the same size such an integrated match will be stable. (If one group is quite a bit larger than another, then the bottom ranked agent in the large group may prefer to move to the small group even if he will also be ranked last there simply because fewer people will be ranked above him. And if he will be ranked last in the small group the move will be approved.)

Proposition 7 shows that integration may occur when a new location is added to a segregated stable match. The dynamic process which was described prior to Proposition 4 will be used in Proposition 7 and the corresponding proof.

Proposition 7: Assume $q_1>q_2>...>q_n$. Assume agents are in a segregated stable match where agent 1 is located at A and where there exists at least one location other than A, say D, with at least three agents. If a new location is added and if the lowest ranked agents at A and D are given the first opportunity to relocate, then agents will end up in an integrated stable match or in a cycle of matches where at least one location remains integrated at all times.

Proof: After the new location is added, agents are given the opportunity to relocate according to the dynamic process described prior to Proposition 4. Since agents are randomly identified every period in the dynamic process, with probability approaching 1 the dynamic process cannot become stuck forever in an unstable match. Since there are a fixed number of agents and a fixed number of locations, the dynamic process must lead to either a stable match or to a cycle of matches. We show that in either case at least one location remains integrated at all times.

Assume agents are in a segregated stable match where agent 1 is located at A and where there exists another location D A with at least three agents. Let agent i be the lowest ranked agent at A and let k be the lowest ranked agent at D. Note that it must be that i is of higher quality than k, since the original match is segregated and both agents 1 and i are located at A.

Note also that stability requires that location A has at least two agents located at it (i.e. i   1). Any location, such as A, with higher quality agents than D must have at least two agents, otherwise agent k (who is ranked third or below at D) will want to move to this location and all agents at this location will agree (since they are of higher quality than k) which violates the definition of stable match.

Next we add a new location and show that both i and k will move to this location. If agent i is given the first opportunity to move to the new location he will since he is not the top ranked agent at location A but he would be the top ranked agent at the new empty location. Next we give agent k the opportunity to move to the new location. Since there are at least two agents ranked above k at location D, k would like to move to the new location where he would be ranked second. Since agent i is of higher quality than k, agent i will agree to let k move to the new location.

Now we check that no agent j   {i+1, i+2,...,k-1} is able to join the new location, thus the new location will be integrated. Since i and k are now both at the new location and since j would be ranked above k, k will always veto such a move and so the move will not occur since strict majority approval is needed. Notice that even if other agents join the new location they will be of lower quality than k (any other agent will not receive the strict majority approval necessary to join the new location) thus these agents will also veto an agent j   {i+1, i+2,...,k-1} from joining the new location.

Lastly we check that agent i and k will not leave the new location and thus the new location will remain integrated. Agent i is the top ranked agent at the new location. Agent i will only want to leave this location if he could be the top ranked agent at another location with more agents ranked below him. However, all the agents at this other location would veto the move. Next we check that agent k will not leave the new location. Agent k will always be ranked second at the new location, since he will never allow another higher ranked agent to join. So k will only want to leave this location if he could be the top ranked agent at another location or the second ranked agent with more people ranked below him at another location. However, agent k will not be able to move to a location where he is ranked first, since all agents at this location would veto the move (no location is empty since the initial match was stable with multiple agents at some locations). To move to a location where k is ranked second and there is at least one agent ranked below him is also not possible since all agents who would be ranked below him will veto the move.

Proposition 7 shows that adding a new location often causes integration to occur with the big fish payoff function. The rough intuition for the proof is as follows. If a new location is added to a segregated stable match and if the last ranked agent at two different locations are each given the chance to move to the new location they will (since the move will improve their rank from last to first and last to second, respectively). The new location will be integrated since these two agents will not allow anyone else to join the new location unless the new agent has strictly lower quality than both of these agents. (Any other agent who tries to join will be vetoed by at least one of these agents and so will not receive the strict majority approval required for such a move.) Lastly, the new location will remain integrated since neither of these agents will leave the new location. Each agent will only leave the new location for a higher ranked position at another location (for instance the top ranked position) and such an improved position will always be vetoed by those at the other location. Thus the dynamic process will lead to either an integrated stable match or to a cycle of matches where at least one location remains integrated at all times.

Integrated Cycle Example

In the following example a new location is added to a segregated stable match and agents end up in a cycle of matches where at least one location remains integrated at all times. Assume there are initially 4 locations and 15 agents with $q_1>q_2>...>q_{15}$. The first three agents are located at A, agents {4,5,6,7} at B, agents {8,9,10,11} at C, and agents {12,13,14,15} are located at D. It is easy to check that this segregated match is stable. Now we add a new empty location, G. If agent 3 is given the first chance to move to G he will since he is the third ranked agent at A but would be the top ranked agent at G. If agent 7 is given the next opportunity to move to G he will take it and agent 3 will approve this move. (In this example agent 3 represents agent i in the proof of Proposition 7 and agent 7 represents agent k.) Next we give agent 15 the chance to move to G and he will also move. There are currently three agents at locations G, B, and D, four agents at C and two agents at A. The following cycle may now occur. Agent 11 moves to A where he would be ranked third instead of fourth. Next agent 10 moves to A where he would be ranked three out of four instead of three out of three. Then agent 11 moves back to C and then 10 moves back to C. Agents 10 and 11 will continue to cycle between locations A and C. It is easy to check that no other agent who wishes to change locations is able to do so. The dynamic process will remain in this cycle and location G is integrated at all times.

Notice that if we eliminate locations C and D and agents 8 through 15 then this example can also be used to illustrated how the dynamic process may lead to an integrated stable state in Proposition 7. Here when the new location is added agents 3 and 7 move to the new location and this integrated match is stable.

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