SOME NUMERICS
OF STOCHASTIC SYSTEMS
WITH MEMORY

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Outline

• Strong Euler scheme for (general) SFDE’s. Order of convergence 0.5.

• Strong Milstein scheme for SDDE’s. Order of convergence 1.

• For Milstein scheme, use infinite dimensional Itô formula for “tame” functions acting on segment process of solution of SDDE. Presence of memory in SDDE requires use of Malliavin calculus + anticipating stochastic analysis of Nualart and Pardoux.

• Conjecture: Milstein scheme works for mixed discrete and continuous memory. Open: for general SFDE’s?
Types of SFDE’s

Suppose rate of change of physical system depends on *present state* and some noisy input. Model by SODE.

Rate of change depends on *present* and *past* states of the system: Model by SDDE or SFDE.

\[ \mathbb{R}^m := m \text{-dimensional Euclidean space.} \]

Euclidean norm:

\[ |x| := \sqrt{x_1^2 + \cdots + x_m^2}, \quad x = (x_1, \ldots, x_m) \in \mathbb{R}^m. \]

\[ T := [0, a], \quad J := [-r, 0], \quad r, a > 0. \]

\[ C := C(J; \mathbb{R}^m); \text{ sup norm:} \]

\[ \| \eta \|_C := \sup_{-r \leq s \leq 0} |\eta(s)|, \quad \eta \in C := C([-r, 0], \mathbb{R}^m). \]
\( W := d \)-dimensional Brownian motion.

**SDDE:**

\[
X(t) = \begin{cases}
\eta(0) + \int_{0}^{t} g(s, \Pi_1(X_s)) \, dW(s) \\
+ \int_{0}^{t} h(s, \Pi_2(X_s)) \, ds, & t \in [0, a] \\
\eta(t), & -r \leq t < 0.
\end{cases}
\]

\( \Pi_i : C \rightarrow \mathbb{R}^{mk_i}, i = 1, 2, \) two projections of discrete type based on \( s_{1,1}, \ldots, s_{1,k_1} \in [-r, 0] \) and \( s_{2,1}, \ldots, s_{2,k_2} \in [-r, 0] \):

\( \Pi_i(\eta) := (\eta(s_{i,1}), \ldots, \eta(s_{i,k_i})) \in \mathbb{R}^{mk_i}, \) \( \eta \in C, \) \( i = 1, 2. \)

**Segment process** \( X_t, t \in [0, a] : \)

\( X_t(s) = X(t+s), \) \( t \in [0, a], \) \( s \in [-r, 0]. \)

\( g : T \times \mathbb{R}^{mk_1} \rightarrow L(\mathbb{R}^d, \mathbb{R}^m), \) \( h : T \times \mathbb{R}^{mk_2} \rightarrow \mathbb{R}^m. \)
SFDE with mixed discrete and continuous memory:

\[ X(t) = \eta(t) + \int_0^t g(s, \Pi_1(X_s), Q_1(X_s)) \, dW(s) \]
\[ + \int_0^t h(s, \Pi_2(X_s), Q_2(X_s)) \, ds, \quad t \in [0, a], \]
\[ X_0 = \eta \in C = C(J; \mathbb{R}^m), J := [-r, 0]. \]

\[ g : T \times \mathbb{R}^{mk_1} \times \mathbb{R}^{m_1} \to L(\mathbb{R}^d, \mathbb{R}^m), \quad h : T \times \mathbb{R}^{mk_2} \times \mathbb{R}^{m_2} \to \mathbb{R}^m. \]
\(\Pi_1, \Pi_2\) two projections of discrete type;

\(Q_1, Q_2\) two projections of continuous type:

\[
Q_i(\eta) := (Q_{i,1}(\eta), \cdots, Q_{i,m_i}(\eta)), \quad i = 1, 2,
\]

\[
Q_{ij}(\eta) := \int_{-r}^{0} \phi_{ij}(\eta(s))a_{ij}(s) \, ds, \quad j = 1, \cdots, m_i.
\]

\(a_{ij} : J \to \mathbb{R}\) and \(\phi_{ij} : \mathbb{R}^m \to \mathbb{R}\) sufficiently regular,

\(i = 1, 2, j = 1, \cdots, m_i\).

General SFDE:

\[
X(t) = \begin{cases} 
\eta(0) + \int_{0}^{t} G(s, X_s) \, dW(s) \\
+ \int_{0}^{t} H(s, X_s) \, ds, & t \in [0, a] \\
\eta(t), & -r \leq t < 0.
\end{cases}
\]

\(G : T \times C \to L(\mathbb{R}^d, \mathbb{R}^m), \quad H : T \times C \to \mathbb{R}^m.\)
Numerical Schemes

SDDE’s and SFDE’s cannot be solved explicitly: Need effective numerical techniques.

Numerical methods for SODE’s: well developed; Kloeden and Platen, Kloeden, Platen and Schurz, McShane, Chapters 5 and 6), Hu, Talay, Protter, etc..

Aims.

- **Strong Euler schemes** for general SFDE’s. Allows for multiple delays and continuous memory. Estimates in supremum norm on $C([-r,0], \mathbb{R}^m)$ (cf. [A]).

- **Strong Milstein scheme** for SDDE’s. Solution of SDDE is non-anticipating. But need methods from *anticipating* stochastic analysis and Malliavin calculus to derive Itô’s formula for segment process. Itô’s formula needed for convergence of Milstein scheme.
Preliminaries

Recall *segment* process $X_t$, $t \in [0, a]$:

$$X_t(s) = X(t + s), \quad t \in [0, a], \quad s \in [-r, 0].$$

for continuous $m$-dimensional process $\{X(t)\}_{t \in [-r, a]}$.

$\{X_t\}$ is a $C$-valued or $L^2(J; \mathbb{R}^m)$-valued process.

*Distinguish between finite-dimensional current state $x(t)$ and infinite-dimensional segment $X_t$, $t \in [0, a]$.*

Itô SFDE:

$$X(t) = \begin{cases} 
\eta(0) + \int_0^t G(s, X_s) \, dW(s) \\
+ \int_0^t H(s, X_s) \, ds, & t \in [0, a] \\
\eta(t), & -r \leq t < 0.
\end{cases}$$
**Coefficients:** \( G : T \times C([-r, 0], \mathbb{R}^m) \to L(\mathbb{R}^d; \mathbb{R}^m) \) and \( H : T \times C([-r, 0], \mathbb{R}^m) \to \mathbb{R}^m. \)

\( \{ W(t) := (W^1(t), \cdots, W^d(t)) : t \geq 0 \} \), \( d \)-dimensional standard Brownian motion on \( (\Omega, \mathcal{F}, P) \).

\( (\mathcal{F}_t)_{t \geq 0} = \) Brownian filtration.

\( \eta \in C([-r, 0]; \mathbb{R}^m) = \) random initial path independent of \( \{ W(t) : t \geq 0 \} \).

**Lipschitz Condition:**

\[ \| G(t, \eta) - G(t, \xi) \| + | H(t, \eta) - H(t, \xi) | \leq L \| \eta - \xi \|_C \]

for all \( t \in T, \eta, \xi \in C; \ L > 0 \) constant.

**Boundedness Condition:**

\[ \sup_{0 \leq t \leq a} \left[ \| G(t, 0) \| + | H(t, 0) | \right] < \infty. \]
Lipschitz + bounded conditions imply SFDE has unique strong solution such that for each $q \geq 1$, there exists a constant $C = C(q, L, a) > 0$ with

$$E\|X_t\|_C^{2q} \leq C(1 + E\|\eta\|_C^{2q})$$

for all $\eta \in C, t \in [0, a]$ ([M], 1984).

Segment $X_t, t \geq 0$, is a $C$-valued Markov process.

Strong versus Weak:

SFDE’s do not lead to diffusions on Euclidean space. \textit{(Highly degenerate infinite-dimensional diffusions on C.)} Hence no natural link to deterministic PDE’s. Strong schemes give information on sample paths dynamics, a.s. financial option-pricing formulas with delays (Arriojas and Mohammed, 2001).
Strong Euler Scheme

Develop Euler scheme for general SFDE’s (include discrete and/or continuous memory).

\( n, l \) positive integers, \( T := [0, a], \ a > 0, \ J := [-r, 0]. \)

\( \pi_n : t_{-l} < t_{-l+1} < \cdots < 0 = t_0 < t_1 < t_2 < \cdots < t_n = a, \)

partition of \([-r, a].\)

\[ |\pi_n| := \max_{-l \leq i \leq n-1} (t_{i+1} - t_i), \text{ mesh of } \pi_n. \]

\( X^n := X_{\pi_n}. \)

SFDE:

\[
X(t) = \begin{cases} 
\eta(0) + \int_0^t G(s, X_s)) \, dW(s) \\
+ \int_0^t H(s, X_s) \, ds, & t \in [0, a] \\
\eta(t), & -r \leq t < 0.
\end{cases}
\]
Euler scheme for SFDE:

\[ X^n(t) = \begin{cases} 
  X^n(t_i) + G(t_i, X^n_{t_i})(W(t) - W(t_i)) \\
  + H(t_i, X^n_{t_i})(t - t_i), & t \in (t_i, t_{i+1}], \\ 
  \eta^n(t), & -r \leq t \leq 0 
\end{cases} \]

Approx. initial path \( \eta^n \in C(J, \mathbb{R}^m) \) is prescribed (e.g. a piece-wise linear approximation of \( \eta \) using partition points \( \{t_{-l}, \ldots, t_0\} \)).

Error function \( Z^n \):

\[ \begin{align*} 
  Z^n(t) &:= X^n(t) - X(t), & 0 \leq t \leq a, \\
  Z^n_0 &:= \eta^n - \eta. 
\end{align*} \]

Euler scheme for SFDE’s has strong order of convergence 0.5 (as in SODE).
Theorem 1.

Assume that the coefficients $G : T \times C([-r, 0], \mathbb{R}^m) \to L(\mathbb{R}^d; \mathbb{R}^m)$ and $H : T \times C([-r, 0], \mathbb{R}^m) \to \mathbb{R}^m$ in SFDE satisfy the following Lipschitz and regularity conditions:

$$
\|G(t, \eta) - G(t, \xi)\| + |H(t, \eta) - H(t, \xi)| \leq L\|\eta - \xi\|_C, \ t \in T
$$

$$
\sup_{0 \leq t \leq a} \left[\|G(t, 0)\| + |H(t, 0)|\right] < \infty
$$

$$
\|G(s, \eta) - G(t, \eta)\| \leq L_1 (1 + \|\eta\|_C)|s - t|^\gamma, \ s, t \in T
$$

$$
|H(s, \eta) - H(t, \eta)| \leq L_1 (1 + \|\eta\|_C)|s - t|^\gamma, \ s, t \in T
$$

for all $\eta, \xi \in C([-r, 0], \mathbb{R}^m)$, where $L$ and $L_1$ are positive constants. Fix any integer $q \geq 2$. Suppose that $\eta : [-r, 0] \to L^q(\Omega, \mathbb{R}^m)$ is independent of
and Hölder continuous with exponent $\gamma \in (0, 1]$, i.e., there is a positive constant $K$ such that

$$E|\eta(s) - \eta(t)|^q \leq K|s - t|^{\gamma q}$$

for all $s, t \in [-r, 0]$. Suppose also that there is a positive constant $C' := C'(q)$ such that

$$E\|\eta^n - \eta\|_C^q \leq C'|\pi_n|^{\gamma q}.$$

Then there is a constant $C'' := C''(q, a) > 0$, depending on $a$ and $q$, such that

$$E\sup_{0 \leq t \leq a} \|Z^n_t\|_C^q \leq C''|\pi_n|^{\tilde{\gamma} q}$$

where $\tilde{\gamma} := \gamma \wedge (1/2)$.
Proof of Theorem 1.

Based on moment estimates:

\[ E\|X_t\|^{2q}_C \leq C(1 + E\|\eta\|^{2q}_C), \quad q \geq 1 \]

for all \( \eta \in C, t \in [0, a] \) ([M], 1984), and Burkholder’s inequality. \( \square \)
Theorem 1 applies to SDDE’s under Lipschitz and boundedness conditions. Also to SFDE’s with mixed discrete and continuous memory:

\[ X(t) = \eta(0) + \int_0^t g(s, \Pi_1(X_s), Q_1(X_s)) \, dW(s) + \int_0^t h(s, \Pi_2(X_s), Q_2(X_s)) \, ds, \quad t \in [0, a], \]

\[ X_0 = \eta \in C = C(J; \mathbb{R}^m) \]

\( \Pi_1, \Pi_2 \) two projections of discrete type;

\( Q_1, Q_2 \) two projections of continuous type:

\[ Q_i(\eta) := (Q_{i,1}(\eta), \cdots, Q_{i,m_i}(\eta)), \quad i = 1, 2, \]

\[ Q_{ij}(\eta) := \int_{-1}^0 \phi_{ij}(\eta(s))a_{ij}(s) \, ds, \quad j = 1, \cdots, m_i. \]

\( a_{ij} \in C^{\frac{1}{2}}(J), \) and \( \phi_{ij} : \mathbb{R}^m \rightarrow \mathbb{R}, i = 1, 2, j = 1, \cdots, m_i, \) satisfy Lipschitz and linear growth conditions.
Euler scheme for SFDE with mixed discrete and continuous memory:

\[ X^n(t) = X^n(t_i) + g(t_i, \Pi_1(X^n_{t_i}), Q^n_1(X^n_{t_i}))(W(t) - W(t_i)) + h(t_i, \Pi_2(X^n_{t_i}), Q^n_2(X^n_{t_i}))(t - t_i), \quad t \in (t_i, t_{i+1}], \]

\[ X^n(t) = \eta^n(t), \quad -r \leq t \leq 0, \]

where \( Q^n_i(\eta), i = 1, 2, \) are approximations of \( Q_i(\eta) \) using partial sums of Riemann integral. Strong order of convergence 0.5 under Lipschitz and regularity conditions as in Theorem 1.
Example: Exact convergence rate.

One-dimensional SDDE:

\[
\begin{aligned}
   \left\{
      dX(t) &= b(t)X(t-1)\,dW(t), \quad 0 < t \leq a \\
      X(t) &= \eta(t), \quad -1 \leq t \leq 0.
   \end{aligned}
\]

Use partitions \(\{\pi_n(h)\}\) of \([-1, a]\) generated by a continuous (strictly positive) function \(h : [0, a] \rightarrow (0, \infty)\). For each integer \(n\), choose partition points \(t_{k,n} \equiv t_k\) of \(\pi_n(h)\) in \([0, a]\) such that

\[
t_0 = 0, \quad \int_{t_k}^{t_{k+1}} h(s) \, ds = \frac{1}{n}, \quad k = 0, 1, \cdots, n-1.
\]

i.e. subdivide interval in such a way that the areas under \(h\) over each subinterval are all equal to \(1/n\). Then

\[
\lim_{n \to \infty, t_k \to t} n(t_{k+1} - t_k) = 1/h(t).
\]
e.g. \( h(t) \equiv 1 \implies (t_{k+1} - t_k) = 1/n, \ k = 0, 1, \ldots, n-1. \)

Euler scheme gives

\[
X^\pi_n(t) = \begin{cases} 
X^\pi_n(t_k) + b(t_k)X^\pi_n(t_k - 1)(W(t) - W(t_k)), & t_k \leq t < t_{k+1}, \\
\eta(t), & t \in J := [-1, 0], 
\end{cases}
\]

for \( 0 \leq k \leq n-1. \) By Theorem 1, there is a positive constant \( C \) (independent of \( n \)) such that

\[
nE|X(t) - X^\pi_n(t)|^2 \leq C,
\]

for all \( n \geq 1, t \in [0, a]. \) Theorem 2 (below) shows that the left hand side of the above inequality has a limit (as \( n \to \infty \)) satisfying a deterministic DDE.
Theorem 2.

Suppose $\eta \in C^\gamma(J, \mathbb{R}^m), 1/2 < \gamma \leq 1$. Let $a \geq 1$. Suppose $b : [0, a] \to \mathbb{R}$ satisfies

$$|b(t) - b(s)| \leq K|t - s|^{(1/2)+\alpha}$$

for all $s, t \in [0, a]$ and some $K, \alpha > 0$. Let $X$ be the solution of the SDDE and $X^{\pi_n}$ its Euler approximation. Then $Z(t) := \lim_{n \to \infty} n E|X(t) - X^{\pi_n}(t)|^2$ exists for each $t \in [-1, a]$. Furthermore, $Z(t)$ satisfies the following deterministic linear DDE

$$Z'(t) = b^2(t)Z(t - 1) + b^2(t)b^2(t - 1)EX^2(t - 2)/h(t), \quad 1 < t < a,$$

$$Z(t) = 0, \quad -1 \leq t \leq 1,$$

where $EX^2(t)$ is given by the integral equation

$$EX^2(t) = \begin{cases} 
\eta(0)^2 + \int_0^t b^2(s)EX^2(s - 1) \, ds, & t \in [0, a], \\
\eta(t)^2, & t \in [-1, 0].
\end{cases}$$
Milstein Scheme

Strong second order scheme for SDDE:

\[ X(t) = \begin{cases} 
\eta(0) + \int_0^t g(s, \Pi_1(X_s)) \, dW(s) \\
+ \int_0^t h(s, \Pi_2(X_s)) \, ds, & t \in T := [0, a] \\
\eta(t), & -r \leq t < 0.
\end{cases} \]

\[ g : T \times \mathbb{R}^{mk_1} \to L(\mathbb{R}^d, \mathbb{R}^m), \quad h : T \times \mathbb{R}^{mk_2} \to \mathbb{R}^m. \]

Requires infinite-dimensional Itô formula for “tame” functions of segments of semimartingales or (anticipating) processes. Proof based on Nualart-Pardoux anticipating calculus techniques.
$l, n = \text{positive integers, } T := [0, a], \ a > 0, \ J := [-r, 0].$

Partitions: $\pi_n := \{ t_i : -l \leq i \leq n \}$ of $[-r, a]$, mesh $|\pi_n|$. $X^n := X^{\pi_n}; \ (x_{i_1,j_1}) \in \mathbb{R}^{mk_1}$.

**Milstein approximations for SDDE:**

$$X^{i,n}(t) = X^{i,n}(t_k) + h^i(t_k, \Pi_2(X^n_{t_k}))(t - t_k)$$

$$+ \sum_j g^{ij}(t_k, \Pi_1(X^n_{t_k}))(W^j(t) - W^j(t_k))$$

$$+ \sum_{i_1,j_1,j} \frac{\partial g^{ij}}{\partial x_{i_1,j_1}}(t_k, \Pi_1(X^n_{t_k}))g^{i_1j_1}(t_k + s_{1,j_1}, \Pi_1(X^n_{t_k+s_{1,j_1}})) \times$$

$$\times 1_{[0,T]}(t_k + s_{1,j_1}) \times I_{j,j_1}(t_k + s_{1,j_1}, t + s_{1,j_1}; s_{1,j_1}),$$

for $t_k < t \leq t_{k+1}, \ i, i_1 = 1, 2, \cdots, m, 1 \leq j \leq d,$

$1 \leq j_1 \leq k_1$, where

$$I_{j,j_1}(t_k + s_{1,j_1}, t + s_{1,j_1}; s_{1,j_1}) := \int_{t_k}^{t} \int_{t_k + s_{1,j_1}}^{t + s_{1,j_1}} \circ dW^j(t_2) \circ dW^{j_1}(t_1).$$

$X^i, h^i, g^{ij} = \text{coordinates of } X, h \text{ and } g \text{ with respect}$

**to standard bases in Euclidean space.**
Milstein scheme has strong order of convergence 1.

**Theorem 3.**

Consider the Milstein scheme for the SDDE. Let $0 < \gamma \leq 1$. Suppose that $\eta : [-r, 0] \to L^2(\Omega, \mathbb{R}^m)$ is Hölder continuous with exponent $\frac{\gamma}{2}$, i.e. there is a positive constant $K$ such that

$$E|\eta(s) - \eta(t)|^2 \leq K|s - t|^\gamma$$

for all $s, t \in J$. Suppose that $g \in C^{1,2}(T \times \mathbb{R}^{mk_1}, L(\mathbb{R}^d, \mathbb{R}^m)$, $h \in C^{1,2}(T \times \mathbb{R}^{mk_2}, \mathbb{R}^m)$ and have bounded first and second spatial derivatives. Let

$$\begin{cases} Z^n(t) := X^n(t) - X(t), & 0 \leq t \leq a, \\ Z^n_0 := \eta^n_0 - \eta. \end{cases}$$
Assume that

\[
\sup_{-r \leq s \leq 0} E(|Z^n(s)|^2) \leq C'|\pi_n|^{2\gamma}
\]

for some positive constant \(C'\). Then there exists a constant \(C > 0\) (depending on \(a\) and independent of \(\pi_n\)) such that

\[
\sup_{-r \leq t \leq a} E|Z^n(t)|^2 \leq C|\pi_n|^{2\gamma}
\]

for any \(n \geq 1\).
*Surprise!* Proof requires use of *anticipating* calculus techniques:

**Example:**

One-dimensional SDDE:

\[
dX(t) = g(X(t - 1), X(t)) \, dW(t), \quad t \geq 0
\]

\[
X(t) = W(t), \quad -1 \leq t < 0.
\]

\(g : \mathbb{R}^2 \to \mathbb{R}\) smooth function. For second-order scheme, formally seek a stochastic differential of the coefficient \(g(X(t - 1), X(t))\) on RHS of SDDE.
For $t \in (0, 1]$, formally expect something like:

$$
\begin{align*}
&dg(W(t - 1), W(t)) \\
&= \frac{\partial g}{\partial x_2}(W(t - 1), W(t)) \, dW(t) \\
&+ \frac{\partial g}{\partial x_1}(W(t - 1), W(t)) \, dW(t - 1) \quad (\text{anticipating!}) \\
&+ \frac{1}{2} \left( \frac{\partial^2 g}{\partial x_1^2}(W(t - 1), W(t)) \, dt + \frac{\partial^2 g}{\partial x_2^2}(W(t - 1), W(t)) \, dt \right) \\
&+ \frac{1}{2} \frac{\partial^2 g}{\partial x_1 \partial x_2}(W(t - 1), W(t)) \, dW(t - 1) \, dW(t) (= 0!)
\end{align*}
$$

- **LHS is adapted but anticipating integral on RHS.**
- $(\mathcal{F}_t)_{0 \leq t \leq 1}$-adapted process

$$
[0, 1] \ni t \to (X(t - 1), X(t)) \in \mathbb{R}^2
$$

is not a semimartingale with respect to any natural filtration.
• The components $X(t-1)$ and $X(t)$ are not independent. Existing anticipating versions of Itô’s formula do not apply (cf. [AN], [AP] and [NP]). Hence need new Itô formula for tame functions:

$$g(W(t-1), W(t)) = g(W_t(-1), W_t(0)).$$

• Last second-order Itô integral on RHS is zero:

Proof.

$(\Omega, \mathcal{F}, (\mathcal{F}_t), P) :=$ filtered probability space.

$\pi := \{t_i\}$ any partition of $[0,T]$, $f$ any $(\mathcal{F}_t)$-adapted (a.s. bounded) process on $[0,T]$. Then

$$\int_0^T f(t) \, dW(t-1) \, dW(t) = \lim_{|\pi| \to 0} \sum_i f(t_i) \Delta_i W(\cdot - 1) \Delta_i W$$
where

\[ \Delta_i W := W(t_{i+1}) - W(t_i), \]

\[ \Delta_i W(\cdot - 1) := W(t_{i+1} - 1) - W(t_i - 1) \]

\[ E| \sum_i f(t_i) \Delta_i W(\cdot - 1) \Delta_i W|^2 = \sum_{i,j} E X_{i,j} \]

\[ X_{i,j} := f(t_i) f(t_j) \Delta_i W(\cdot - 1) \Delta_j W(\cdot - 1) \Delta_i W \Delta_j W \]

For \( i < j \),

\[ E(X_{i,j}) = E\{E(X_{i,j}|\mathcal{F}_{t_j})\} \]

and

\[ E(X_{i,j}|\mathcal{F}_{t_j}) \]

\[ = f(t_i) f(t_j) \Delta_i W(\cdot - 1) \Delta_j W(\cdot - 1) \Delta_i W \cdot E(\Delta_j W|\mathcal{F}_{t_j}) \]

\[ = 0 \]
By symmetry,

\[
\sum_{i,j} E X_{i,j} = \sum_i E X_{i,i} = \sum_i E f(t_i)^2 [\Delta_i W(\cdot - 1)]^2 [\Delta_i W]^2 \leq K \sum_i E [\Delta_i W(\cdot - 1)]^2 \cdot E [\Delta_i W]^2 \leq KT |\pi|.
\]

Hence

\[
E \left| \int_0^T f(t) \, dW(t - 1) \, dW(t) \right|^2 = \lim_{|\pi| \to 0} \sum_{i,j} E X_{i,j} = 0. \quad \square
\]

**Shorthand:**

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**Proof.**

Exercise.
Projection $\Pi : C \to \mathbb{R}^{mk}$ associated with $s_1, \cdots, s_k \in [-r, 0]$:

$$\Pi(\eta) := (\eta(s_1), \cdots, \eta(s_k)) \in \mathbb{R}^{mk}, \quad \eta \in C$$

**Definition.**

$\Phi \in C(T \times C(J; \mathbb{R}^m); \mathbb{R})$ is *tame* if there exist $\phi \in C(T \times \mathbb{R}^{mk}, \mathbb{R})$ and a projection $\Pi$ such that

$$\Phi(t, \eta) = \phi(t, \Pi(\eta)).$$

for all $t \in T$ and $\eta \in C$.

**Proof (Milstein Scheme).**

Itô’s formula for “tame” functionals

$$T \times C(J, \mathbb{R}^m) \to \mathbb{R}$$
of the segment $X_t$. Use formula + moment estimates on weak derivatives of $X$ to get global error estimate for the Milstein approximations.

$$W(t) := (W^1(t), \cdots, W^d(t)), t \geq 0 := d\text{-dimensional standard Brownian motion on } (\Omega, \mathcal{F}, P).$$

$$D := (D_1, \cdots, D_d) := \text{Malliavin differentiation operator associated with } \{W(t) : t \geq 0\}.$$  

Pathwise-continuous process:

$$X(t) := \begin{cases} 
\eta(0) + \int_0^t u(s) \, dW(s) + \int_0^t v(s) \, ds, & t > 0, \\
\eta(t), & -r \leq t \leq 0,
\end{cases}$$

Skorohod integral. $\eta \in C, \ BV.$

$$u = (u^1, \cdots, u^m)^T, u^i \in L_{d, loc}^{2,4};$$

$$v = (v^1, \cdots, v^m)^T, v^i \in L_{loc}^{1,4} ([\text{Nualart}]).$$
$u$ and $v$ may not be adapted to the Brownian filtration $(\mathcal{F}_t)_{t \geq 0}$. Set $u(t) := 0$ for $t < 0$ or $t > a$,

$$v(t) := \begin{cases} 0, & t > a \\ \eta'(t), & -r \leq t \leq 0. \end{cases}$$

$W(t) := 0$ if $t < 0$ or $t > a$.

$$U(t) := \int_0^t u(s) \, dW(s), \quad V(t) := \begin{cases} \eta(0) + \int_0^t v(s) \, ds, & t > 0 \\ \eta(t), & -r \leq t \leq 0. \end{cases}$$

Then

$$D_s X(t) = u(s)1_{[0,a]}(t - s) + D_s \eta(0) + \int_0^t D_s v(r') \, dr'$$

$$+ \int_0^t D_s u(r') \, dW(r'), \quad t > 0$$

$\Pi :=$ projection associated with $s_1, \cdots, s_k \in J$.

Cannot apply multi-dimensional Itô formula to $\phi(t, \Pi(X_t))$ because $\Pi(U_t)$ is of the form

$$\left( \int_0^t u(s + s_1) \, dW(s + s_1), \cdots, \int_0^t u(s + s_k) \, dW(s + s_k) \right),$$
and the components \((W(t + s_1), \ldots, W(t + s_k))\) are not independent. Use anticipating calculus (Nualart-Pardoux) to derive an Itô formula for \(\phi(t, \Pi(X_t))\).

Assume \(\phi \in C^{1,2}(T \times \mathbb{R}^{mk})\), \(\vec{x} = (\vec{x}_1, \cdots, \vec{x}_m)\), \(\vec{x}_i = (x_{i1}, \cdots, x_{ik}) \in \mathbb{R}^k\). Write

\[\phi(t, \vec{x}) = \phi(t, \vec{x}_1, \cdots, \vec{x}_m).\]

**Theorem 4.** *(Itô’s formula).*

Suppose \(X\) satisfies above conditions and let \(\phi \in C^{1,2}(T \times \mathbb{R}^{mk}, \mathbb{R})\). Then

\[
\phi(t, \Pi(X_t)) - \phi(0, \Pi(X_0)) = \int_0^t \frac{\partial \phi}{\partial s}(s, \Pi(X_s)) \, ds + \int_0^t \frac{\partial \phi}{\partial \vec{x}}(s, \Pi(X_s)) \, d(\Pi(X_s)) + \\
\frac{1}{2} \sum_{i,j=1}^{k} \sum_{i_1,j_1=1}^{m} \int_0^t \frac{\partial^2 \phi}{\partial x_{i_1}\partial x_{j_1}}(s, \Pi(X_s)) u^{i_1}(s + s_i) D_{s+s_i} X^{j_1}(s + s_j) \, ds
\]

a.s. for all \(t \in T\).
Example (Revisited)

\[ g(W(t - 1), W(t)) - g(W(-1), W(0)) \]

\[ = \int_0^t \frac{\partial g}{\partial x_1}(W(s - 1), W(s)) \, dW(s) \]

\[ + \int_0^t \frac{\partial g}{\partial x_2}(W(s - 1), W(s)) \, dW(s - 1) \]

\[ + \frac{1}{2} \int_0^t \frac{\partial^2 g}{\partial x_1^2}(W(s - 1), W(s)) \, ds \]

\[ + \frac{1}{2} \int_0^t \frac{\partial^2 g}{\partial x_2^2}(W(s - 1), W(s)) \mathbf{1}_{(1,\infty)}(s) \, ds \]

for \( t > 0 \).
Weak differentiability of solutions of SDDE’s.

Cf. Bell and Mohammed, Nualart.

\[ \mathbb{D}_m^{k,\infty} := \cap_{p \geq 2} \mathbb{D}_m^{k,p}, \; k \in \mathbb{N}. \]

\[ D^l_u, \; 1 \leq l \leq d, \; 0 \leq u \leq a, \] weak differentiation with respect to \( l \)-th component of \( W \).

**Proposition.**

In the Itô SDDE, assume that \( g \in C_b^{0,1}(T \times \mathbb{R}^{k_1m}; L(\mathbb{R}^d, \mathbb{R}^m)) \) and \( h \in C_b^{0,1}(T \times \mathbb{R}^{k_2m}; \mathbb{R}^m) \). Let \( X \) be the solution of the SDDE. Then \( X(t) \in \mathbb{D}_m^{1,\infty} \) for all \( t \in T \), and

\[
\sup_{0 \leq u \leq a} E\left( \sup_{u \leq s \leq a} |D_u X(s)|^p \right) < \infty
\]

for all \( p \geq 2 \). Furthermore, the “partial” weak derivatives \( D^l_u X^j(t) \) with respect to the \( l \)-th coordinate of \( W \) satisfy
the following linear SDDE’s a.s.:

\[ D^l_t X^j(t) = g^{jl}(u, \Pi_1(X^j_u)) + \]

\[ \int_u^t \sum_{i=1}^{k_1} \frac{\partial g^{jl}}{\partial x_i}(s, \Pi_1(X^i_s)) D^l_u X^j(s + s_{1,i}) dW^l(s) \]

\[ + \int_0^t \sum_{i=1}^{k_2} \frac{\partial h^j}{\partial x_i}(s, \Pi_2(X^i_s)) D^l_u X^j(s + s_{2,i}) ds, \quad t \geq u, \]

\[ = 0, \quad t < u, \quad l = 1, \cdots, d, \quad j = 1, \cdots, m \]

\[ g^{jl} = (j, l) \text{ entry of the } m \times d \text{ matrix } g, \]

\[ h^j = j\text{-th coordinate of } h. \]
References


[C-H] Cambanis, S. and Hu, Y., *The exact convergence rate of Euler-Maruyama scheme and application to sample*


