On the Order Statistics of Standard Normal-Based Power Method Distributions

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Research Article

On the Order Statistics of Standard Normal-Based Power Method Distributions

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This paper derives a procedure for determining the expectations of order statistics associated with the standard normal distribution \( Z \) and its powers of order three and five \( Z^3 \) and \( Z^5 \). The procedure is demonstrated for sample sizes of \( n \leq 9 \). It is shown that \( Z^3 \) and \( Z^5 \) have expectations of order statistics that are functions of the expectations for \( Z \) and can be expressed in terms of explicit elementary functions for sample sizes of \( n \leq 5 \). For sample sizes of \( n = 6,7 \) the expectations of the order statistics for \( Z, Z^3 \), and \( Z^5 \) only require a single remainder term.

1. Introduction

Order statistics have played an important role in the development of techniques associated with estimation [1, 2], hypothesis testing [3, 4], and describing data in the context of \( L \)-moments [5, 6]. In terms of the latter, \( L \)-moments are based on the expectations of linear combinations of order statistics associated with a random variable \( X \). Specifically, the first four \( L \)-moments are expressed as

\[
\begin{align*}
\lambda_1 &= E[X_{1:1}], \\
\lambda_2 &= \frac{1}{2} E[X_{2:2} - X_{1:2}], \\
\lambda_3 &= \frac{1}{3} E[X_{3:3} - 2X_{2:3} + X_{1:3}], \\
\lambda_4 &= \frac{1}{4} E[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}] 
\end{align*}
\]
or more generally as
\[
\lambda_r = \frac{1}{r} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E[X_{r-j}r],
\]  
(1.2)

where the order statistics \( X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n} \) are drawn from the random variable \( X \). The values of \( \lambda_1 \) and \( \lambda_2 \) are measures of location and scale and are the arithmetic mean and one-half the coefficient of mean difference (or Gini’s index of spread), respectively. Higher-order \( L \)-moments are transformed to dimensionless quantities referred to as \( L \)-moment ratios defined as \( \tau_r = \lambda_r / \lambda_2 \) for \( r \geq 3 \), and where \( \tau_3 \) and \( \tau_4 \) are the analogs to the conventional measures of skew and kurtosis. In general, \( L \)-moment ratios are bounded in the interval \(-1 < \tau_r < 1\) as is the index of \( L \)-skew \( (\tau_3) \) where a symmetric distribution implies that all \( L \)-moment ratios with odd subscripts are zero. Other smaller boundaries can be found for more specific cases. For example, the index of \( L \)-kurtosis \( (\tau_4) \) has the boundary condition for continuous distributions of [7]
\[
\frac{5\tau_3^2 - 1}{4} < \tau_4 < 1.
\]  
(1.3)

Headrick [8] derived classes of standard normal-\( L \)-moment-based power method distributions using the polynomial transformation
\[
p(Z) = \sum_{i=1}^{m} c_i Z^{i-1},
\]  
(1.4)

where \( Z \sim \text{i.i.d. } N(0,1) \). Setting \( m = 4 \) \((m = 6)\) gives the third- \((\text{fifth-})\) order class of power method distributions. The shape of \( p(Z) \) in (1.4) is contingent on the values of the constant coefficients \( c_i \). For the larger class of nonnormal distributions associated with \( m = 6 \), the coefficients are computed from the system of equations given in Headrick ([8, Equations (2.8)–(2.13)] for specified values of \( L \)-moment ratios \((\tau_{3,6})\). In general, \( \lambda_1 \) and \( \lambda_2 \) are standardized to the unit normal distribution as
\[
\lambda_1 = c_1 + c_3 + 3c_5 = 0,
\]
\[
\lambda_2 = \frac{(4c_2 + 10c_4 + 43c_6)}{4\sqrt{\pi}} = \frac{1}{\sqrt{\pi}}.
\]  
(1.5)

The pdf and cdf associated with (1.4) are given in parametric form as in [8, Equations (1.3) and (1.4)]
\[
f_{p(z)}(p(z)) = \tilde{f}(z) = \left( p(z), \frac{\phi(z)}{p'(z)} \right),
\]
\[
F_{p(z)}(p(z)) = \tilde{F}(z) = (p(z), \Phi(z)),
\]  
(1.6)

where \( \tilde{f} : \mathbb{R} \mapsto \mathbb{R}^2 \) and \( \tilde{F} : \mathbb{R} \mapsto \mathbb{R}^2 \) are the parametric forms of the pdf and cdf with the mappings \( z \mapsto (x, y) \) and \( z \mapsto (x, v) \) with \( x = p(z), y = \phi(z) / p'(z), v = \Phi(z) \), and where \( \phi(z) \) and \( \Phi(z) \) are the standard normal pdf and cdf, respectively. For further details on the distributional properties associated with power method transformations see [9, pages 9–30] and [8] in terms of conventional moment and \( L \)-moment theory, respectively.
Of concern in this study are three power method distributions related to (1.4) and (1.5) as

$$p_t(Z) = c_{2t}Z^{2t-1},$$

where

$$\begin{cases} t = 1, & c_2 = 1, \ c_4 = 0, \ c_6 = 0, \\ t = 2, & c_2 = 0, \ c_4 = 2/5, \ c_6 = 0, \\ t = 3, & c_2 = 0, \ c_4 = 0, \ c_6 = 4/43, \end{cases}$$

and thus $p_1(Z) = Z$, $p_2(Z) = (2/5)Z^3$ and $p_3(Z) = (4/43)Z^5$. Note that these power method distributions are symmetric and imply that $c_{1,3,5} = 0$ in (1.4). The graphs of the pdfs associated with the distributions in (1.7) are given in Figure 1 along with their values of $L$-skew ($\tau_3$) and $L$-kurtosis ($\tau_4$).

**Figure 1:** Graphs of the three standard normal-based power method distributions $p_t(Z)$ in (1.7) and their values of $L$-skew ($\tau_3$) and $L$-kurtosis ($\tau_4$).
considered as functional transformations on random data, usually called Box-Cox transformations. Their importance in the area of statistics and its applications is well known.”

The standard normal distribution \( p_1(Z) \) in (1.7) is the only case of the three distributions considered that is moment determinant. That is, \( p_2(Z) \) and \( p_3(Z) \) have the so-called classical problem of moments insofar as their respective cdfs have nonunique solutions (i.e., they are moment indeterminant, see [10–12]). However, as pointed out by Huang [12], \( p_2(Z) \) and \( p_3(Z) \) are determinant in the context of order statistics moments.

The derivation of the expected values of single order statistics associated with \( p_1(Z) \) in terms of explicit elementary functions has been attempted by numerous authors (see [13–17]). As indicated by Johnson et al. [18, pages 93–94] these attempts fail to give explicit expressions in terms of elementary functions for the expected values of order statistics with sample sizes of \( n > 5 \). However, Renner [19] provides a technique for expressing the expected values of order statistics associated with \( p_1(Z) \) for \( n = 6, 7 \) based on a single power series.

There is a paucity of research on the expectations of order statistics associated with \( p_2(Z) \) and \( p_3(Z) \) in the context of explicit elementary functions. Thus, what follows in Section 2 is the development of an approach for determining the expected values of the order statistics for \( p_2(Z) \) and \( p_3(Z) \), which is based on a generalization of Renner’s [19] discussion in the context of \( p_1(Z) \). In Section 3, some specific evaluations of the generalization are provided to demonstrate the methodology.

## 2. Methodology

The expected values of the order statistics associated with (1.7) can be determined based on the following expression [20, page 34]:

\[
E[p(Z)_{j:n}] = n2^{-a} \binom{n-1}{j-1} \int_0^\infty p_i(z)q(z) \left( [1 + \Psi(z)]^{j-1} [1 - \Psi(z)]^{n-j} - [1 - \Psi(z)]^{j-1} [1 + \Psi(z)]^{n-j} \right) dz,
\]

(2.1)

where \( p_i(z) \) is defined as in (1.7) and \( q(z) = 2 \phi(z) \) and \( \Psi(z) = 2 \Phi(z) - 1 \) are the pdf and cdf of the folded unit normal distribution at \( z = 0 \). Table 1 gives a summary of some specific expansions of the polynomial in (2.1) for sample sizes of \( n = 1, \ldots, 9 \), which are applicable to all three distributions related to \( p_i(z) \). Inspection of Table 1 indicates that we have in general

(a) \( E[p(Z)_{j:n}] = -E[p(Z)_{n+1-j:n}] \),
(b) the median \( E[p(Z)_{j:n}] = -E[p(Z)_{j:n}] = 0 \),
(c) the\( E[p(Z)_{j:n}] \) are linear combinations of the integrals \( I_{2r-1} \) for \( r = 1, 2, \ldots \), with only odd subscripts appearing as only odd powers of \( \Psi(z) \) appear in the polynomial expansions associated with (2.1). As such, \( I_{2r-1} \) in (2.1) can be expressed as

\[
I_{2r-1} = \int_0^\infty p_i(z)q(z)[\Psi(z)]^{2r-1} dz.
\]

(2.2)

Equation (2.2) may be integrated by parts as

\[
I_{2r-1} = (2r - 1) \int_0^\infty q_i(z)q(z)^2[\Psi(z)]^{2r-2} dz,
\]

(2.3)
Table 1: General expressions for the expected values of the order statistics for $p_{r-1,2,3}(Z)$ in (1.7) and sample sizes of $n = 1, \ldots, 9$. $I_{2r-1}$ denotes an integral in (2.1) where $r = 1, \ldots, 4$.

<table>
<thead>
<tr>
<th>Sample size ($n$)</th>
<th>Expected value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$E[p_1(Z)<em>{11}] = -E[p_1(Z)</em>{12}] = 0$</td>
</tr>
<tr>
<td>2</td>
<td>$E[p_1(Z)<em>{22}] = -E[p_1(Z)</em>{12}] = I_1 = 1/\sqrt{\pi}$</td>
</tr>
<tr>
<td>3</td>
<td>$E[p_1(Z)<em>{23}] = -E[p_1(Z)</em>{32}] = 0$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{33}] = -E[p_1(Z)</em>{13}] = (3/2)I_1 = 3/(2\sqrt{\pi})$</td>
</tr>
<tr>
<td>4</td>
<td>$E[p_1(Z)<em>{34}] = -E[p_1(Z)</em>{24}] = (3/2)(I_1 - I_3)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{44}] = -E[p_1(Z)</em>{14}] = (1/2)(3I_1 + I_3)$</td>
</tr>
<tr>
<td>5</td>
<td>$E[p_1(Z)<em>{45}] = -E[p_1(Z)</em>{35}] = 0$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{55}] = -E[p_1(Z)</em>{25}] = (5/2)(I_1 - I_5)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{56}] = -E[p_1(Z)</em>{16}] = (5/4)(1 + I_5)$</td>
</tr>
<tr>
<td>6</td>
<td>$E[p_1(Z)<em>{46}] = -E[p_1(Z)</em>{36}] = (15/8)(I_1 - 2I_3 + I_5)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{56}] = -E[p_1(Z)</em>{26}] = (15/16)(3I_1 - 2I_3 - I_5)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{66}] = -E[p_1(Z)</em>{16}] = (3/16)(5I_1 + 10I_3 + I_5)$</td>
</tr>
<tr>
<td>7</td>
<td>$E[p_1(Z)<em>{57}] = -E[p_1(Z)</em>{47}] = 0$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{57}] = -E[p_1(Z)</em>{37}] = (105/32)(I_1 - 2I_3 + I_5)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{67}] = -E[p_1(Z)</em>{27}] = (21/8)(I_1 - I_5)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{77}] = -E[p_1(Z)</em>{17}] = (7/32)(3I_1 + 10I_3 + 3I_5)$</td>
</tr>
<tr>
<td>8</td>
<td>$E[p_1(Z)<em>{58}] = -E[p_1(Z)</em>{48}] = (35/16)(I_1 - 3I_3 + 3I_5 - I_7)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{68}] = -E[p_1(Z)</em>{38}] = (21/16)(3I_1 - 5I_3 + 3I_5 + I_7)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{78}] = -E[p_1(Z)</em>{28}] = (7/16)(5I_1 + 5I_3 - 9I_5 - I_7)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{88}] = -E[p_1(Z)</em>{18}] = (1/16)(7I_1 + 35I_3 + 21I_5 + I_7)$</td>
</tr>
<tr>
<td>9</td>
<td>$E[p_1(Z)<em>{69}] = -E[p_1(Z)</em>{59}] = 0$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{69}] = -E[p_1(Z)</em>{49}] = (63/16)(I_1 - 3I_3 + 3I_5 - I_7)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{79}] = -E[p_1(Z)</em>{39}] = (63/16)(I_1 - I_3 - I_5 + I_7)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{89}] = -E[p_1(Z)</em>{29}] = (9/16)(3I_1 + 7I_3 - 7I_5 + 3I_7)$</td>
</tr>
<tr>
<td></td>
<td>$E[p_1(Z)<em>{99}] = -E[p_1(Z)</em>{19}] = (9/32)(I_1 + 7I_3 + 7I_5 + I_7)$</td>
</tr>
</tbody>
</table>

where $q_1(z) = 1$, $q_2(z) = (2/5)(z^2 + 2)$ and $q_3(z) = (4/43)(z^4 + 4z^2 + 8)$, for $p_1(z)$, $p_2(z)$, and $p_3(z)$, respectively. Note that $\Psi(0) = 0$ and $\lim_{z \to +\infty} \varphi(z) = 0$. Evaluating (2.3) for $r = 1$ gives a coefficient of mean difference of

$$I_1 = \int_0^\infty q_1(z)\varphi(z)^2dz = \frac{1}{\sqrt{\pi}}$$

(2.4)

for all $p_r(z)$ in (1.7), which is consistent with the specification in (1.5) and given in Table 1.

The expression $[\Psi(z)]^{2r-2}$ in (2.3) can be expressed as

$$[\Psi(z)]^{2r-2} = \left(\frac{2}{\pi}\right)^{r-1} \left[\int_0^\infty \exp\left(-\frac{1}{2}u^2\right)du\right]^{2r-2}$$

(2.5)

or analogously as a double integral over $\mathbb{R}^2$ as

$$[\Psi(z)]^{2r-2} = \left(\frac{2}{\pi}\right)^{r-1} \left[\iint_{\mathbb{R}^2} \exp\left(-\frac{1}{2}(z_1^2 + z_2^2)\right)dz_1dz_2\right]^{r-1}.$$  

(2.6)
Using (2.6), let \( z_2 = z_1 \tan \theta_1 \) and thus \( dz_2 = z_1 \sec^2 \theta_1 d\theta_1 \). Further, let \( z_1^2 + z_2^2 = z_1^2 \sec^2 \theta_1 \). As such, the region of integration will be reduced to one-half of the area of the original rectangle associated with (2.6). Thus, we have

\[
[\Psi(z)]^{2r-2} = \left( \frac{2}{\pi} \right)^{r-1} \int_0^{\pi/4} \int_0^z \exp \left( -\frac{1}{2} \left( z_1^2 \sec^2 \theta_1 \right) \right) dz_1 \left( z_1 \sec^2 \theta_1 d\theta_1 \right) \right]^{r-1} \tag{2.7}
\]

Subsequently, setting \( z_1^2 = w \) in (2.7), where \( z_1 dz_1 = dw/2 \), gives

\[
[\Psi(z)]^{2r-2} = \left( \frac{4}{\pi} \right)^{r-1} \left[ \int_0^{\pi/4} \left( \int_0^z \exp \left( -\frac{1}{2} w \sec^2 \theta_1 \right) \frac{dw}{2} \right) \sec^2 \theta_1 d\theta_1 \right]^{r-1} \tag{2.8}
\]

and hence

\[
[\Psi(z)]^{2r-2} = \left( \frac{4}{\pi} \right)^{r-1} \left[ \int_0^{\pi/4} \left( 1 - \exp \left( -\frac{1}{2} \left( z^2 \sec^2 \theta_1 \right) \right) \right) d\theta_1 \right]^{r-1}. \tag{2.9}
\]

Expanding (2.9) yields

\[
[\Psi(z)]^{2r-2} = 1 + \sum_{k=1}^{r-1} (-1)^k \binom{r-1}{k} \left( \frac{4}{\pi} \right)^k \int_0^{\pi/4} \cdots \int_0^{\pi/4} \exp \left( -\frac{1}{2} z^2 \sum_{i=1}^{k} \sec^2 \theta_i \right) d\theta_1 \cdots d\theta_k, \tag{2.10}
\]

where the subscript \( i \) runs faster than \( k \). For example, if \( r = 4 \), then (2.10) would appear more specifically as

\[
[\Psi(z)]^{2r-2} = 1 - \binom{r-1}{1} \left( \frac{4}{\pi} \right)^{2} \int_0^{\pi/4} \exp \left( -\frac{1}{2} z^2 \sec^2 \theta_1 \right) d\theta_1 + \binom{r-1}{2} \left( \frac{4}{\pi} \right)^{3} \int_0^{\pi/4} \exp \left( -\frac{1}{2} z^2 \left( \sec^2 \theta_1 + \sec^2 \theta_2 \right) \right) d\theta_1 d\theta_2 - \binom{r-1}{3} \left( \frac{4}{\pi} \right)^{4} \int_0^{\pi/4} \exp \left( -\frac{1}{2} z^2 \left( \sec^2 \theta_1 + \sec^2 \theta_2 + \sec^2 \theta_3 \right) \right) d\theta_1 d\theta_2 d\theta_3. \tag{2.11}
\]
Substituting (2.10) into (2.3) and initially integrating with respect to \( z \) (Lichtenstein, [21]) yields

\[
\sqrt{\pi} \int_0^\infty q_t(z)\varphi(z)^2 \exp\left\{-\frac{1}{2}z^2 \sum_{i=1}^k \sec^2 \theta_i \right\} \, dz = g_l(\sec^2 \theta_i),
\]

where the specific forms of \( g_l(\sec^2 \theta_i) \), which are associated with \( p_t(z) \), are

\[
g_1(\sec^2 \theta_i) = \frac{\sqrt{2}}{(2 + \sum_{i=1}^k \sec^2 \theta_i)^{3/2}}, \\
g_2(\sec^2 \theta_i) = \frac{2\sqrt{2}(5 + 2 \sum_{i=1}^k \sec^2 \theta_i)}{5(2 + \sum_{i=1}^k \sec^2 \theta_i)^{3/2}}, \\
g_3(\sec^2 \theta_i) = \frac{4\sqrt{2}\left(3 + 4(2 + \sum_{i=1}^k \sec^2 \theta_i) + 8(2 + \sum_{i=1}^k \sec^2 \theta_i)^2\right)}{43(2 + \sum_{i=1}^k \sec^2 \theta_i)^{5/2}}.
\]

Equations (2.13) can be more conveniently expressed as

\[
g_l(\sec^2 \theta_i) = g_1(\sec^2 \theta_i) - h_l(\sec^2 \theta_i),
\]

where the specific forms of \( h_l(\sec^2 \theta_i) \) are

\[
h_1(\sec^2 \theta_i) = 0, \\
h_2(\sec^2 \theta_i) = \frac{\sqrt{2}(\sum_{i=1}^k \sec^2 \theta_i)}{5(2 + \sum_{i=1}^k \sec^2 \theta_i)^{3/2}}, \\
h_3(\sec^2 \theta_i) = \frac{\sqrt{2}\left(11 \sum_{i=1}^k \sec^4 \theta_i + 28 \sum_{i=1}^k \sec^2 \theta_i + 22 \sum_{i<j} \sec^2 \theta_i \sec^2 \theta_j\right)}{43(2 + \sum_{i=1}^k \sec^2 \theta_i)^{5/2}}.
\]

and where \( \sum_{i<j} \) in (2.17) indicates summing over all \( k(k-1)/2 \) pairwise combinations. Hence, the integral in (2.3) can be expressed as

\[
I_{2r-1} = \frac{2r-1}{\sqrt{\pi}} \left(1 + \left\{ \sum_{k=1}^{r-1} (-1)^k \binom{r-1}{k} \left(\frac{4}{\pi}\right)^k \int_0^{\pi/4} \cdots \int_0^{\pi/4} g_l(\sec^2 \theta_i) \, d\theta_1 \cdots d\theta_k \right\} \right),
\]
and subsequently substituting (2.14) into (2.18) gives

\[
I_{2r-1} = \frac{2r - 1}{\sqrt{\pi}} \left( 1 + \sum_{k=1}^{r-1} (-1)^k \binom{r-1}{k} \left( \frac{4}{\pi} \right)^k \int_0^{\pi/4} \cdots \int_0^{\pi/4} \left( g_1(\sec^2 \theta_1) - h_1(\sec^2 \theta_1) \right) d\theta_1 \cdots d\theta_k \right) \right)
\]

(2.19)

The integral associated with \( g_1(\sec^2 \theta_i) \) in (2.19) cannot be expressed in terms of explicit elementary functions for \( k > 1 \), which also implies \( r > 2 \) and sample sizes of \( n > 5 \) in Table 1. As such, we will consider the approximating function \( g_1^*(\sec^2 \theta_i) \) as

\[
g_1^*(\sec^2 \theta_i) = \left(2^{k/2}\int \frac{1}{(2 + \sec^2 \theta_i)^{1/2}} \right)
\]

(2.20)

where

\[
\int_0^{\pi/4} \cdots \int_0^{\pi/4} g_1(\sec^2 \theta_i) d\theta_1 \cdots d\theta_k = \int_0^{\pi/4} \cdots \int_0^{\pi/4} g_1^*(\sec^2 \theta_i) d\theta_1 \cdots d\theta_k
\]

\[
= \begin{cases} 
\tan^{-1}\left(1/\sqrt{2}\right), & k = 1, \\
0, & k \rightarrow \infty.
\end{cases}
\]

(2.21)

Thus, for finite \( k > 1 \) we have

\[
\int_0^{\pi/4} \cdots \int_0^{\pi/4} g_1(\sec^2 \theta_i) d\theta_1 \cdots d\theta_k = \int_0^{\pi/4} \cdots \int_0^{\pi/4} g_1^*(\sec^2 \theta_i) d\theta_1 \cdots d\theta_k + \varepsilon_k
\]

\[
= \left(\tan^{-1}\left(1/\sqrt{2}\right)\right)^k + \varepsilon_k,
\]

(2.22)

where \( \varepsilon_k \) is the remainder term required for \( k > 1 \) and where \( \varepsilon_1 = 0 \) for \( r = 1, 2 \) and \( n \leq 5 \). Thus, using (2.22), (2.19) can be expressed as

\[
I_{2r-1} = \frac{2r - 1}{\sqrt{\pi}} \left( 1 + \sum_{k=1}^{r-1} (-1)^k \binom{r-1}{k} \left( \frac{4}{\pi} \right)^k \times \left( \left(\tan^{-1}\left(1/\sqrt{2}\right)\right)^k + \varepsilon_k \right) - \int_0^{\pi/4} \cdots \int_0^{\pi/4} h_1(\sec^2 \theta_1) d\theta_1 \cdots d\theta_k \right) \right).
\]

(2.23)

The remainder terms \( \varepsilon_{k>1} \) in (2.23) can be solved by using (2.3), (2.15), (2.23), and the error function Erf [22], where Erf would replace \( \Phi(z) \) in (2.3) where \( \Psi(z) = 2\Phi(z) - 1 \). More specifically, Table 2 gives the values of \( \varepsilon_k \) for \( k = 1, \ldots, 12, 25, \) and 50 with 40-digit precision.
Table 2: Computed values of the remainder term $\varepsilon_k$ associated with (2.23). The values were computed with 40-digit precision.

<table>
<thead>
<tr>
<th>Sample size $(n)$</th>
<th>Integral</th>
<th>Remainder term $\varepsilon_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, \ldots, 5$</td>
<td>$I_1, I_5$</td>
<td>$\varepsilon_1 = 0.0$</td>
</tr>
<tr>
<td>$6, 7$</td>
<td>$I_5$</td>
<td>$\varepsilon_2 = 0.03140698829552010270731937950881276500595$</td>
</tr>
<tr>
<td>$8, 9$</td>
<td>$I_7$</td>
<td>$\varepsilon_3 = 0.05156068650031409787170392919312656858246$</td>
</tr>
<tr>
<td>$10, 11$</td>
<td>$I_9$</td>
<td>$\varepsilon_4 = 0.059001987103558171948642817928465212298$</td>
</tr>
<tr>
<td>$12, 13$</td>
<td>$I_{11}$</td>
<td>$\varepsilon_5 = 0.05808975458203638688882522593413660371348$</td>
</tr>
<tr>
<td>$14, 15$</td>
<td>$I_{13}$</td>
<td>$\varepsilon_6 = 0.05274763616761422221709626523935998463539$</td>
</tr>
<tr>
<td>$16, 17$</td>
<td>$I_{15}$</td>
<td>$\varepsilon_7 = 0.0455923657410464353074859375854475949676$</td>
</tr>
<tr>
<td>$18, 19$</td>
<td>$I_{17}$</td>
<td>$\varepsilon_8 = 0.03815223895234453779274127861572423887877$</td>
</tr>
<tr>
<td>$20, 21$</td>
<td>$I_{19}$</td>
<td>$\varepsilon_9 = 0.0312205691467168489718556870682270636055$</td>
</tr>
<tr>
<td>$22, 23$</td>
<td>$I_{21}$</td>
<td>$\varepsilon_{10} = 0.025148552525461486567020912288596241803047$</td>
</tr>
<tr>
<td>$24, 25$</td>
<td>$I_{23}$</td>
<td>$\varepsilon_{11} = 0.02002429921405354560405588075438666460570$</td>
</tr>
<tr>
<td>$26, 27$</td>
<td>$I_{25}$</td>
<td>$\varepsilon_{12} = 0.01580928681263632398753707685232879723154$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$52, 53$</td>
<td>$I_{51}$</td>
<td>$\varepsilon_{25} = 0.00057455597453332805073409074487236584232$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$102, 103$</td>
<td>$I_{101}$</td>
<td>$\varepsilon_{50} = 0.000000099193614769461065745252616987082859$</td>
</tr>
</tbody>
</table>

Table 3: Expected values of order statistics for $p_1(Z) = Z$ for $n = 4, 5$.

<table>
<thead>
<tr>
<th>$E[p_1(Z)]_{3,4}$</th>
<th>$-\frac{3}{\sqrt{\pi}} + \frac{18\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} = 0.29701138\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[p_1(Z)]_{4,4}$</td>
<td>$\frac{3}{\sqrt{\pi}} - \frac{6\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} = 1.02937537\ldots$</td>
</tr>
<tr>
<td>$E[p_1(Z)]_{3,5}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$E[p_1(Z)]_{4,5}$</td>
<td>$-\frac{5}{\sqrt{\pi}} + \frac{30\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} = 0.49501897\ldots$</td>
</tr>
<tr>
<td>$E[p_1(Z)]_{5,5}$</td>
<td>$\frac{5}{\sqrt{\pi}} - \frac{15\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} = 1.16296447\ldots$</td>
</tr>
</tbody>
</table>

Table 4: Expected values of order statistics for $p_2(Z) = (2/5)Z^2$ for $n = 4, 5$.

<table>
<thead>
<tr>
<th>$E[p_1(Z)]_{3,4}$</th>
<th>$-\frac{3\sqrt{7}}{5\pi^{3/2}} + E[p_1(Z)]_{3,4} = 0.14462665\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[p_2(Z)]_{3,4}$</td>
<td>$\frac{\sqrt{2}}{5\pi^{3/2}} + E[p_1(Z)]_{3,4} = 1.08017028\ldots$</td>
</tr>
<tr>
<td>$E[p_2(Z)]_{3,5}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$E[p_2(Z)]_{4,5}$</td>
<td>$-\frac{\sqrt{2}}{\pi^{3/2}} + E[p_1(Z)]_{4,5} = 0.24104442\ldots$</td>
</tr>
<tr>
<td>$E[p_2(Z)]_{5,5}$</td>
<td>$\frac{1}{\sqrt{2\pi^{3/2}}} + E[p_1(Z)]_{5,5} = 1.28995174\ldots$</td>
</tr>
</tbody>
</table>

Inspection of Table 2 indicates that the (positive) remainder term achieves a maximum at $\varepsilon_4$ and thereafter tends to zero as $k$ increases (i.e., $\varepsilon_k \to 0$ for $k > 4$).
Table 5: Expected values of order statistics for \( p_3(Z) = (4/43)Z^3 \) for \( n = 4, 5 \).

| \( E[p_3(Z)_{3.4}] = \frac{77}{43\sqrt{2\pi}} \approx 0.069615569 \ldots \) |
| \( E[p_3(Z)_{4.4}] = \frac{77}{129\sqrt{2\pi}} + E[p_1(Z)_{4.4}] = 1.10517397 \ldots \) |
| \( E[p_3(Z)_{3.5}] = 0 \) |
| \( E[p_3(Z)_{4.5}] = -\frac{385}{129\sqrt{2\pi}} + E[p_1(Z)_{4.5}] = 0.11602594 \ldots \) |
| \( E[p_3(Z)_{5.5}] = \frac{385}{258\sqrt{2\pi}} + E[p_1(Z)_{5.5}] = 1.35246098 \ldots \) |

Table 6: Expected values of order statistics for \( p_1(Z) = Z \) for \( n = 6, 7 \).

| \( E[p_1(Z)_{4.6}] = \frac{150\varepsilon_2}{\pi^{5/2}} - \frac{30\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} + \frac{150\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} = 0.20154683 \ldots \) |
| \( E[p_1(Z)_{5.6}] = -\frac{15}{2\sqrt{2\pi}} + \frac{75\varepsilon_2}{\pi^{5/2}} + \frac{60\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} - \frac{75\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} = 0.64177503 \ldots \) |
| \( E[p_1(Z)_{6.6}] = \frac{15}{2\sqrt{2\pi}} + \frac{15\varepsilon_2}{\pi^{5/2}} - \frac{30\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} + \frac{15\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} = 1.26720636 \ldots \) |
| \( E[p_1(Z)_{4.7}] = 0 \) |
| \( E[p_1(Z)_{5.7}] = \frac{525\varepsilon_2}{2\pi^{5/2}} - \frac{105\tan^{-1}(1/\sqrt{2})}{2\pi^{3/2}} + \frac{525\tan^{-1}(1/\sqrt{2})^2}{2\pi^{5/2}} = 0.35270695 \ldots \) |
| \( E[p_1(Z)_{6.7}] = -\frac{21}{2\sqrt{2\pi}} \frac{210\varepsilon_2}{\pi^{5/2}} + \frac{105\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} - \frac{210\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} = 0.75737427 \ldots \) |
| \( E[p_1(Z)_{7.7}] = \frac{21}{2\sqrt{2\pi}} \frac{105\varepsilon_2}{\pi^{5/2}} - \frac{105\tan^{-1}(1/\sqrt{2})}{\pi^{3/2}} + \frac{105\tan^{-1}(1/\sqrt{2})^2}{\pi^{5/2}} = 1.35217837 \ldots \) |

We would note that the approach taken here to determine \( \varepsilon_2 \) is analogous to Renner’s [19] approach of developing a power series for this value. That is, the remainder term \( \varepsilon_2 \) in Table 2 is also the value approximated in [19] for \( p_1(Z) \). Further, we would note that extending the approach in [19] for computing the remainder terms for \( k > 2 \) would become computationally burdensome.

To demonstrate (2.23) more specifically, if \( r = 4 \) and \( t = 2 \) in (1.7), then the integral \( I_7 \) associated with \( p_2(Z) \) would appear as

\[
\begin{align*}
I_7 &= \frac{2r - 1}{\sqrt{2\pi}} \left\{ 1 - \left( r - 1 \right) \frac{4}{\pi^2} \left( \tan^{-1} \left( \frac{1}{\sqrt{2}} \right) \right) - \int_0^{\pi/4} h_2 \left( \sec^2 \theta_1 \right) d\theta_1 \right. \\
& \quad \left. + \left( r - 1 \right) \frac{4}{\pi^2} \left( \tan^{-1} \left( \frac{1}{\sqrt{2}} \right) \right)^2 + \varepsilon_2 \right) - \int_0^{\pi/4} h_2 \left( \sec^2 \theta_1 \right) d\theta_1 d\theta_2 \\
& \quad - \left( r - 1 \right) \frac{4}{\pi^2} \left( \tan^{-1} \left( \frac{1}{\sqrt{2}} \right) \right)^3 + \varepsilon_3 \\
& \quad - \int_0^{\pi/4} h_2 \left( \sec^2 \theta_1 \right) d\theta_1 d\theta_2 d\theta_3 \right\}. \\
& \quad (2.24)
\end{align*}
\]
3. Evaluations

Tables 9 and 10 give the expected values of the order statistics associated with the standard $p$ for samples of sizes $E$. $E$ are all expressed in terms of explicit elementary functions and a single remainder term.

Table 7: Expected values of order statistics for $p_2(Z) = (2/5)Z^3$ for $n = 6, 7$.

<table>
<thead>
<tr>
<th>Table 7: Expected values of order statistics for $p_2(Z) = (2/5)Z^3$ for $n = 6, 7$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[p_2(Z)<em>{4,6}] = \frac{\sqrt{2}}{\pi^{3/2}} - \frac{10\sqrt{3}\tan^{-1}(3\sqrt{3}/2/7)}{\pi^{3/2}} + E[p_1(Z)</em>{4,6}] = 0.06475951\ldots$</td>
</tr>
<tr>
<td>$E[p_2(Z)<em>{5,6}] = \frac{2\sqrt{2}}{\pi^{3/2}} + \frac{5\sqrt{2}\tan^{-1}(3\sqrt{3}/2/7)}{\pi^{3/2}} + E[p_1(Z)</em>{5,6}] = 0.32918688\ldots$</td>
</tr>
<tr>
<td>$E[p_2(Z)<em>{6,6}] = \frac{2\sqrt{2}}{\pi^{3/2}} - \frac{\sqrt{2}\tan^{-1}(3\sqrt{3}/2/7)}{\pi^{3/2}} + E[p_1(Z)</em>{6,6}] = 1.48210471\ldots$</td>
</tr>
<tr>
<td>$E[p_2(Z)_{4,7}] = 0$</td>
</tr>
<tr>
<td>$E[p_2(Z)<em>{5,7}] = \frac{7}{2\sqrt{2}^{3/2}} - \frac{3\sqrt{2}\tan^{-1}(3\sqrt{3}/2/7)}{\sqrt{2}\pi^{5/2}} + E[p_1(Z)</em>{5,7}] = 0.11332914\ldots$</td>
</tr>
<tr>
<td>$E[p_2(Z)<em>{6,7}] = -\frac{7}{2\sqrt{2}^{3/2}} + \frac{14\sqrt{2}\tan^{-1}(3\sqrt{3}/2/7)}{\sqrt{2}\pi^{5/2}} + E[p_1(Z)</em>{6,7}] = 0.41552998\ldots$</td>
</tr>
<tr>
<td>$E[p_2(Z)<em>{7,7}] = \frac{7}{2\sqrt{2}^{3/2}} - \frac{7\tan^{-1}(3\sqrt{3}/2/7)}{\sqrt{2}\pi^{5/2}} + E[p_1(Z)</em>{7,7}] = 1.65986717\ldots$</td>
</tr>
</tbody>
</table>

Table 8: Expected values of order statistics for $p_3(Z) = (4/43)Z^5$ for $n = 6, 7$.

<table>
<thead>
<tr>
<th>Table 8: Expected values of order statistics for $p_3(Z) = (4/43)Z^5$ for $n = 6, 7$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[p_3(Z)<em>{4,6}] = \frac{10\sqrt{3}}{43\pi^{5/2}} + \frac{385}{129\sqrt{2}\pi^{5/2}} = \frac{1925\sqrt{2}\tan^{-1}(3\sqrt{3}/2/7)}{129\pi^{5/2}} + E[p_1(Z)</em>{4,6}] = 0.02045216\ldots$</td>
</tr>
<tr>
<td>$E[p_3(Z)<em>{5,6}] = -\frac{5\sqrt{3}}{43\pi^{5/2}} - \frac{385}{129\sqrt{2}\pi^{5/2}} + \frac{1925\sqrt{2}\tan^{-1}(3\sqrt{3}/2/7)}{129\pi^{5/2}} + E[p_1(Z)</em>{5,6}] = 0.16381284\ldots$</td>
</tr>
<tr>
<td>$E[p_3(Z)<em>{6,6}] = \frac{\sqrt{3}}{43\pi^{5/2}} + \frac{385}{129\sqrt{2}\pi^{5/2}} = \frac{385\tan^{-1}(3\sqrt{3}/2/7)}{129\sqrt{2}\pi^{5/2}} + E[p_1(Z)</em>{6,6}] = 1.59019061\ldots$</td>
</tr>
<tr>
<td>$E[p_3(Z)_{4,7}] = 0$</td>
</tr>
<tr>
<td>$E[p_3(Z)<em>{5,7}] = \frac{35\sqrt{3}}{86\pi^{5/2}} + \frac{2695}{516\sqrt{2}\pi^{5/2}} = \frac{1347\sqrt{2}\tan^{-1}(3\sqrt{3}/2/7)}{258\sqrt{2}\pi^{5/2}} + E[p_1(Z)</em>{5,7}] = 0.03579128\ldots$</td>
</tr>
<tr>
<td>$E[p_3(Z)<em>{6,7}] = -\frac{14\sqrt{3}}{43\pi^{5/2}} - \frac{2695}{516\sqrt{2}\pi^{5/2}} + \frac{2695\sqrt{2}\tan^{-1}(3\sqrt{3}/2/7)}{258\sqrt{2}\pi^{5/2}} + E[p_1(Z)</em>{6,7}] = 0.21502146\ldots$</td>
</tr>
<tr>
<td>$E[p_3(Z)<em>{7,7}] = \frac{7\sqrt{3}}{86\pi^{5/2}} + \frac{2695}{516\sqrt{2}\pi^{5/2}} = \frac{2695\tan^{-1}(3\sqrt{3}/2/7)}{258\sqrt{2}\pi^{5/2}} + E[p_1(Z)</em>{7,7}] = 1.81983546\ldots$</td>
</tr>
</tbody>
</table>

3. Evaluations

Tables 3–5 give evaluations for the expected values of the order statistics for $p_1(Z)$, $p_2(Z)$, and $p_3(Z)$ in (1.7), which are based on (2.23) and the general formulae given in Table 1 for sample sizes of $n = 4, 5$. Inspection of Tables 4 and 5 indicates that the expected values for $p_2(Z)$ and $p_3(Z)$ are all expressed in terms of elementary functions and are also functions of the expectations associated with $p_1(Z)$ in Table 3.

Presented in Tables 6, 7, and 8 are the evaluations for all three distributions in (1.7) for samples of sizes $n = 6, 7$ where the expectations of the order statistics for $p_1(Z)$, $p_2(Z)$, and $p_3(Z)$ are all expressed in terms of explicit elementary functions and a single remainder term. Tables 9 and 10 give the expected values of the order statistics associated with the standard
normal distribution $p_1(Z)$ for sample sizes of $n = 8$ and $n = 9$, respectively. We would also note that Mathematica [22] software is available from the authors for implementing the methodology.

### References


### Table 9: Expected values of order statistics for $p_1(Z) = Z$ for $n = 8$. 

<table>
<thead>
<tr>
<th>$E[p_1(Z)]_{5:8}$</th>
<th>$E[p_1(Z)]_{6:8}$</th>
<th>$E[p_1(Z)]_{7:8}$</th>
<th>$E[p_1(Z)]_{8:8}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{210\epsilon_2}{\pi^{3/2}} + \frac{980\epsilon_3}{\pi^{7/2}} - \frac{210\tan^{-1}(1/\sqrt{2})^2}{\pi^{3/2}} + \frac{980\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 0.15251439\ldots$</td>
<td>$\frac{546\epsilon_2}{\pi^{3/2}} - \frac{588\epsilon_3}{\pi^{7/2}} - \frac{84\tan^{-1}(1/\sqrt{2})^2}{\pi^{3/2}} - \frac{546\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 0.47282249\ldots$</td>
<td>$-\frac{14}{\sqrt{\pi}} + \frac{462\epsilon_2}{\pi^{7/2}} + \frac{168\tan^{-1}(1/\sqrt{2})^2}{\pi^{7/2}} + \frac{462\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 0.8522486\ldots$</td>
<td>$\frac{14}{\sqrt{\pi}} + \frac{126\epsilon_2}{\pi^{7/2}} - \frac{8\epsilon_3}{\pi^{7/2}} - \frac{84\tan^{-1}(1/\sqrt{2})^2}{\pi^{7/2}} + \frac{126\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} - \frac{28\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 1.42360030\ldots$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$E[p_1(Z)]_{5:9}$</th>
<th>$E[p_1(Z)]_{6:9}$</th>
<th>$E[p_1(Z)]_{7:9}$</th>
<th>$E[p_1(Z)]_{8:9}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{378\epsilon_2}{\pi^{3/2}} + \frac{1764\epsilon_3}{\pi^{7/2}} - \frac{378\tan^{-1}(1/\sqrt{2})^2}{\pi^{7/2}} + \frac{1764\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 0.27452591\ldots$</td>
<td>$-\frac{378\epsilon_2}{\pi^{3/2}} + \frac{1764\epsilon_3}{\pi^{7/2}} - \frac{378\tan^{-1}(1/\sqrt{2})^2}{\pi^{7/2}} + \frac{1764\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 0.27452591\ldots$</td>
<td>$-\frac{18}{\sqrt{\pi}} + \frac{882\epsilon_2}{\pi^{7/2}} + \frac{756\epsilon_3}{\pi^{7/2}} - \frac{252\tan^{-1}(1/\sqrt{2})^2}{\pi^{7/2}} - \frac{882\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} + \frac{756\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 0.93229745\ldots$</td>
<td>$-\frac{18}{\sqrt{\pi}} + \frac{252\epsilon_2}{\pi^{7/2}} + \frac{552\epsilon_3}{\pi^{7/2}} - \frac{126\tan^{-1}(1/\sqrt{2})^2}{\pi^{7/2}} - \frac{252\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} - \frac{126\tan^{-1}(1/\sqrt{2})^3}{\pi^{7/2}} = 1.48501316\ldots$</td>
</tr>
</tbody>
</table>


