2012

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Published in *ISRN Applied Mathematics,* Vol 2012, at doi:10.5402/2012/980153.

International Scholarly Research Network
ISRN Applied Mathematics
Volume 2012, Article ID 980153, 20 pages
doi:10.5402/2012/980153

Research Article

Characterizing Tukey h and hh-Distributions through L-Moments and the L-Correlation

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Received 10 October 2011; Accepted 31 October 2011

Academic Editors: M. Cho and K. Karamanos

This paper introduces the Tukey family of symmetric h and asymmetric hh-distributions in the contexts of univariate L-moments and the L-correlation. Included is the development of a procedure for specifying nonnormal distributions with controlled degrees of L-skew, L-kurtosis, and L-correlations. The procedure can be applied in a variety of settings such as modeling events (e.g., risk analysis, extreme events) and Monte Carlo or simulation studies. Further, it is demonstrated that estimates of L-skew, L-kurtosis, and L-correlation are substantially superior to conventional product-moment estimates of skew, kurtosis, and Pearson correlation in terms of both relative bias and efficiency when heavy-tailed distributions are of concern.

1. Introduction

The conventional moment-based Tukey families of h (or the g-and-h and Generalized Pareto) distributions (e.g., [1–3]) are often used in various applied mathematics contexts. Some examples include modeling events associated with operational risk [4], extreme oceanic wind speeds, [5], common stock returns [6], solar flare data [7], or in the context of Monte Carlo or simulation studies, for example, regression analysis [8].

The family of h-distributions is based on the transformation

$$q(Z) = Z \exp\left(\frac{hZ^2}{2}\right), \tag{1.1}$$

where $Z \sim i.i.d.\ N(0,1)$. Equation (1.1) produces symmetric h-distributions where the parameter h controls the tail weight or elongation of any particular distribution and is

positively related with kurtosis. The pdf and cdf associated with (1.1) are expressed as in [3, equations (12), (13)]

$$f_{q(Z)}(q(z)) = \overline{f}(z) = \left(q(z), \frac{\phi(z)}{q'(z)}\right), \tag{1.2}$$

$$F_{q(Z)}(q(z)) = \overline{F}(z) = (q(z), \Phi(z)), \tag{1.3}$$

where $\overline{f} : \Re \mapsto \Re^2$ and $\overline{F} : \Re \mapsto \Re^2$ are the parametric forms of the pdf and cdf with the mappings $z \mapsto (x, y)$ and $z \mapsto (x, v)$ with $x = q(z)$, $y = \phi(z)/q'(z)$, $v = \Phi(z)$, and where $\phi(z)$ and $\Phi(z)$ are the standard normal pdf and cdf, respectively. It is assumed that $q'(z) > 0$ in (1.2) to ensure a valid pdf, that is, the transformation in (1.1) is strictly increasing, which requires $h \geq 0$. Further, if $q(Z)$ in (1.1) has a valid pdf where the k-th order moment exists (for $k = 1, 2, \ldots$), then h must be bounded such that $0 \leq h < 1/k$. That is, a distribution will not have a first moment (or mean) for $h \geq 1$ [2, 3].

The variance (α_2^2) and kurtosis (α_4) of a distribution associated with (1.1) can be determined from [3, Equations (32), (36)]

$$\begin{aligned} \alpha_2^2 &= \frac{1}{(1-2h)^{3/2}}, \\ \alpha_4 &= 3(1-2h)^3\left(\frac{1}{(1-4h)^{5/2}} + \frac{1}{(2h-1)^3}\right), \end{aligned} \tag{1.4}$$

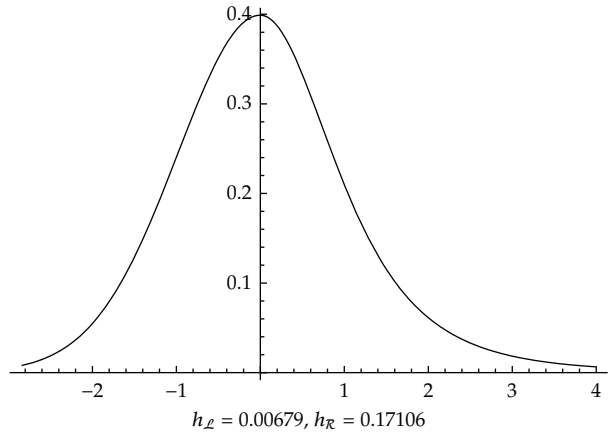
where the mean (α_1) and skew (α_3) of a distribution are both zero.

One of the extensions of (1.1)–(1.3) is the two parameter hh family of distributions introduced by Morgenthaler and Tukey [2]. More specifically, the hh family includes asymmetric distributions with heavy tails based on the transformation

$$q(Z) = \begin{cases} Z\exp\left(\dfrac{h_{\mathcal{L}} Z^2}{2}\right), & \text{for } Z \leq 0, \\ Z\exp\left(\dfrac{h_{\mathcal{R}} Z^2}{2}\right), & \text{for } Z \geq 0, \end{cases} \tag{1.5}$$

where $h_{\mathcal{L}}(h_{\mathcal{R}})$ is the parameter for the left (right) tail of a distribution. The properties of the transformation in (1.5) are the same as those associated with (1.1). However, the tails of hh distributions have to be considered separately as they are weighted differently that is, in general, $h_{\mathcal{L}} \neq h_{\mathcal{R}}$. For example, Figure 1 gives an example of an hh-distribution based on matching the values of α_3 and α_4 associated with a noncentral Student $t_{(\text{df}=5,\ \delta=1)}$ distribution. The values of $h_{\mathcal{L}}$ and $h_{\mathcal{R}}$ in Figure 1 were computed by simultaneously solving (A.3) and (A.4) in the appendix.

Conventional moment-based-estimators (e.g., $\widehat{\alpha}_3$, $\widehat{\alpha}_4$) have unfavorable attributes insofar as they can be substantially biased, have high variance, or can be influenced by outliers. For example, inspection of Figure 1 indicates, on average, that the estimates of $\widehat{\alpha}_3$ and $\widehat{\alpha}_4$ are only 78.83% and 42.81% of their associated population parameters. Note that each estimate of $\widehat{\alpha}_3$ and $\widehat{\alpha}_4$ in Figure 1 were calculated based on sample sizes of $n = 250$



Skew: $\alpha_3 = \mathbf{1.266}$			Kurtosis: $\alpha_4 = \mathbf{10.32}$		
$\widehat{\alpha}_3$	95% C.I.	SE	$\widehat{\alpha}_4$	95% C.I.	SE
0.9980	0.9877, 1.0083	0.0052	**4.418**	4.312, 4.518	0.0533
L-Skew: $\tau_3 = \mathbf{0.1110}$			L-Kurtosis: $\tau_4 = \mathbf{0.1937}$		
$\widehat{\tau}_3$	95% C.I.	SE	$\widehat{\tau}_4$	95% C.I.	SE
0.1096	0.1091, 0.1102	0.0003	**0.1930**	0.1926, 0.1934	0.0002

Figure 1: Graph of an hh-distribution based on matching the conventional moments of a noncentral Student $t_{(\text{df}=5,\ \delta=1)}$ distribution. The values of $h_{\mathcal{L}}$ and $h_{\mathcal{R}}$ were determined by solving equations (A.3) and (A.4) in the appendix. The estimates $(\widehat{\alpha}_{3,4}; \widehat{\tau}_{3,4})$ and bootstrap confidence intervals (C.I.s) were based on resampling 25,000 statistics. Each sample statistic was based on a sample size of $n = 250$.

and the formulae currently used by most commercial software packages such as SAS, SPSS, and Minitab for computing skew and kurtosis.

However, L-moment-based estimators such as L-skew and L-kurtosis have been introduced to address some of the limitations associated with conventional estimates of skew and kurtosis [9, 10]. Specifically, some of the advantages that L-moments (or their estimators) have over conventional moments are that they (a) exist whenever the mean of the distribution exists, (b) are nearly unbiased for all sample sizes and distributions, and (c) are more robust in the presence of outliers. For example, the estimates $\widehat{\tau}_3$ and $\widehat{\tau}_4$ in Figure 1 are relatively much closer to their respective parameters with much smaller standard errors than their corresponding conventional moment based analogs $(\widehat{\alpha}_3, \widehat{\alpha}_4)$. More specifically, the estimates of $\widehat{\tau}_3$ and $\widehat{\tau}_4$ that were simulated are, on average, 98.74% and 99.64% of their parameters.

In the context of multivariate data generation, the methodology has been developed for simulating h-(or g-and-h) distributions with specified Pearson correlation structures [11, pages 140–148] [12]. This methodology is based on conventional product moments and the popular NORTA [13] approach, which begins with generating multivariate standard normal deviates. However, the NORTA approach is not without its limitations. Specifically, one limitation arises because the Pearson correlation is not invariant under nonlinear strictly increasing transformations such as (1.1). As such, the NORTA approach must begin with the computation of an *intermediate correlation (IC)* matrix, which is different than the specified correlation matrix between the nonnormal h-distributions. The purpose of the IC matrix is to

adjust for the nonnormalization effect of the transformation in (1.1) such that the resulting nonnormal distributions have their specified skew, kurtosis, and specified correlation matrix.

Some additional consequences associated with NORTA in this context are that it (a) requires numerical integration to compute solutions to ICs between h-distributions—unlike the more popular power method [11] which has a straight-forward equation to solve for the ICs between distributions [11, page 30] and (b) may yield solutions to ICs that are not in the range of $[-1, +1]$ as the absolute values of ICs must be greater than (or equal to) their specified Pearson correlations [14]. Further, these two problems which can be exacerbated when h-distributions with heavy tails are used as functions performing numerical integration will more frequently either fail to converge or yield incorrect solutions to ICs.

In view of the above, the present aim is to derive the h and hh families of distributions in the contexts of L-moment and L-correlation theory. Specifically, the purpose of this paper is to develop the methodology and a procedure for simulating nonnormal symmetric h and asymmetric hh distributions with specified L-moments and L-correlations. Some of the advantages of the proposed procedure are that ICs (a) can be solved directly with a single equation, that is, numerical integration is not required and (b) cannot exist outside the range of $[-1, +1]$ as it is shown that the absolute value of an IC will be less than (or equal to) its associated specified L-correlation.

The remainder of the paper is outlined as follows. In Section 2, a summary of univariate L-moment theory is provided and the derivations of the systems of equations for the h and hh distributions are provided for modeling or simulating nonnormal distributions with specified values of L-skew and L-kurtosis. In Section 3, the coefficient of L-correlation is introduced and the equations are subsequently derived for determining ICs for specified L-correlations between nonnormal h or hh distributions. In Section 4, the steps for implementing the proposed L-moment procedure are described. A numerical example and results of a simulation are also provided to confirm the derivations and compare the new procedure with the traditional or conventional moment-based procedure. In Section 5, the results of the simulation are discussed.

2. Methodology

2.1. Preliminaries

Let $X_1, \ldots, X_j, \ldots, X_n$ be *iid* random variables each with continuous pdf $f(x)$, cdf $F(x)$, order statistics denoted as $X_{1:n} \leq \cdots \leq X_{j:n} \leq \cdots \leq X_{n:n}$, and L-moments defined in terms of either linear combinations of (a) expectations of order statistics or (b) probability-weighted moments (β_i). For the purposes considered herein, the first four L-moments associated with $X_{j:n}$ are expressed as [10, pages 20–22]

$$\begin{aligned} \lambda_1 &= E[X_{1:1}] = \beta_0, \\ \lambda_2 &= \frac{1}{2}E[X_{2:2} - X_{1:2}] = 2\beta_1 - \beta_0, \end{aligned} \tag{2.1}$$

$$\lambda_3 = \frac{1}{3}E[X_{3:3} - 2X_{2:3} + X_{1:3}] = 6\beta_2 - 6\beta_1 + \beta_0, \tag{2.2}$$

$$\lambda_4 = \frac{1}{4}E[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}] = 20\beta_3 - 30\beta_2 + 12\beta_1 - \beta_0, \tag{2.3}$$

where the β_i are determined from

$$\beta_i = \int x\{F(x)\}^i f(x)dx, \tag{2.4}$$

where $i = 0,\ldots,3$. The coefficients associated with β_i in (2.4) are obtained from shifted orthogonal Legendre polynomials and are computed as shown in [10, page 20] or in [15].

The L-moments λ_1 and λ_2 in (2.1) are measures of location and scale and are the arithmetic mean and one-half the coefficient of mean difference (or Gini's index of spread), respectively. Higher-order L-moments are transformed to dimensionless quantities referred to as L-moment ratios defined as $\tau_r = \lambda_r/\lambda_2$ for $r \geq 3$, and where τ_3 and τ_4 are the analogs to the conventional measures of skew and kurtosis. In general, L-moment ratios are bounded in the interval $-1 < \tau_r < 1$ as is the index of L-skew (τ_3) where a symmetric distribution implies that all L-moment ratios with odd subscripts are zero. Other smaller boundaries can be found for more specific cases. For example, the index of L-kurtosis (τ_4) has the boundary condition for continuous distributions of [16]

$$\frac{\left(5\tau_3^2 - 1\right)}{4} < \tau_4 < 1. \tag{2.5}$$

2.2. L-Moments for Symmetric h-Distributions

The family of h-distributions based on the method of L-moments is less restrictive than the family based on conventional method of moments as described in the previous section to the extent that we may consider the h parameter on the interval $0 \leq h < 1$ for any distribution with finite k-order L-moments rather than $0 \leq h < 1/k$ for the kth-order conventional moment to exist. This advantage is attributed to Hosking's Theorem 1 [9] which states that if the mean (λ_1) exists, then all other L-moments will have finite expectations.

We begin the derivation for symmetric h-distributions in the context of L-moments by defining the probability-weighted moments based on (2.4) in terms of $q(z)$ in (1.1) and the standard normal pdf and cdf as

$$\beta_i = \int_{-\infty}^{+\infty} q(z)\{\Phi(z)\}^i \phi(z)dz. \tag{2.6}$$

Integrating (2.6) for $i = 0, 1, 2$ gives

$$\lambda_1 = 0, \tag{2.7}$$

$$\lambda_2 = -\frac{\sqrt{2}}{\sqrt{\pi}(h-1)\sqrt{2-h}}, \tag{2.8}$$

$$\lambda_3 = \tau_3 = 0. \tag{2.9}$$

The fourth L-moment λ_4 (and τ_4) is subsequently derived in terms of the expectations of order statistics as in (2.3) by making use of the following expression for standard normal-based expectations and for $n = 4$ as [17]

$$E\left[q(Z)_{j:4}\right] = \frac{1}{4}\binom{3}{j-1}\int_0^\infty \Big(q(z)\varphi(z)[1+\Psi(z)]^{j-1}[1-\Psi(z)]^{4-j} - [1-\Psi(z)]^{j-1}[1+\Psi(z)]^{4-j}\Big)dz, \tag{2.10}$$

where $\varphi(z) = 2\phi(z)$ and $\Psi(z) = 2\Phi(z) - 1$ are the pdf and cdf of the folded unit normal distribution at $z = 0$, respectively. The relevant expansions of the polynomial in (2.10) are

$$\begin{aligned} E\left[q(Z)_{3:4}\right] &= -E\left[q(Z)_{2:4}\right] = \left(\frac{3}{2}\right)(I_1 - I_3), \\ E\left[q(Z)_{4:4}\right] &= -E\left[q(Z)_{1:4}\right] = \left(\frac{1}{2}\right)(3I_1 + I_3), \end{aligned} \tag{2.11}$$

where the expectations in (2.11) is linear combinations associated with the integrals denoted as I_1 and I_3. The specific expressions for I_1 and I_3 are

$$I_1 = \delta \cdot \lambda_2 = \left(-(h-1)\sqrt{\frac{1-h}{2}}\right) \cdot \left(-\frac{\sqrt{2}}{\sqrt{\pi}(h-1)\sqrt{2-h}}\right) = \frac{1}{\sqrt{\pi}}, \tag{2.12}$$

$$I_3 = \int_0^\infty q(z)\varphi(z)[\Psi(z)]^3 dz, \tag{2.13}$$

where it is convenient to use δ in (2.12) to standardize I_1 (λ_2) to the unit normal distribution. Equation (2.13) may be integrated by parts based on $\Psi'(z) = \varphi(z)$, $\varphi'(z) = -\varphi(z)z$, and noting that $\Psi(0) = 0$ and $\lim_{z\to+\infty}\varphi(z) = 0$. As such, we have

$$I_3 = 3\int_0^\infty \xi(z)\varphi(z)[\Psi(z)]^2 dz, \tag{2.14}$$

where the expression $\xi(z)$ is

$$\xi(z) = -\int \delta q(z)\varphi(z)dz = \frac{\exp\{(1/2)(h-1)z^2\}\sqrt{2-h}}{\sqrt{\pi}}. \tag{2.15}$$

Let us first consider the expression $[\Psi(z)]^2$ in (2.14), which can be expressed as

$$[\Psi(z)]^2 = \frac{2}{\pi}\left[\int_0^z \exp\left\{-\frac{1}{2}u^2\right\}du\right]^2 = \frac{2}{\pi}\int\int_0^z \exp\left\{-\frac{1}{2}\left(z_1^2+z_2^2\right)\right\}dz_1dz_2$$
$$= 1-\frac{4}{\pi}\int_0^{\pi/4}\exp\left\{-\frac{1}{2}z^2\sec^2\theta_1\right\}d\theta_1. \tag{2.16}$$

Substituting (2.16) into (2.14) and, using Lichtenstein's Theorem [18], and integrating first with respect to z yield

$$\sqrt{\pi}\int_0^{\infty}\xi(z)\varphi(z)^2\exp\left\{-\frac{1}{2}z^2\sec^2\theta_1\right\}dz = \frac{\sqrt{4-2h}}{\sqrt{2}\sqrt{2-h+\sec^2\theta_1}}. \tag{2.17}$$

Using (2.17), the integral in (2.14) is expressed as

$$I_3 = \frac{3}{\sqrt{\pi}}\left\{1-\frac{4}{\pi}\int_0^{\pi/4}\frac{\sqrt{4-2h}}{\sqrt{2}\sqrt{2-h+\sec^2\theta_1}}\right\}d\theta_1 = \frac{3}{\sqrt{\pi}}-\frac{12\tan^{-1}\left[(h-4/h-2)^{-1/2}\right]}{\pi^{3/2}}. \tag{2.18}$$

Hence, using (2.3) and (2.11) and (2.12), L-kurtosis can be expressed as

$$\tau_4 = 6-\frac{30\tan^{-1}\left[(h-4/h-2)^{-1/2}\right]}{\pi}, \tag{2.19}$$

where $0 \le h < 1$. Whence, it follows that we have a convenient closed formed solution for the parameter h as

$$h = 3-\sec\left[\frac{\pi}{15}(\tau_4-6)\right]. \tag{2.20}$$

Equation (2.19) has a lower limit of $\tau_4 \approx 0.1226$ ($h = 0$) that is equivalent to the normal distribution and an upper limit ($\tau_4 \to 1$; $h \to 1$) that is equivalent to the Cauchy or $t_{(\mathrm{df}=1)}$ distribution. Figure 2(d) gives an example of a symmetric h-distribution with L-kurtosis (τ_4) of a logistic distribution.

2.3. L-Moments for Asymmetric hh-Distributions

The derivation of the L-moments for asymmetric hh-distributions associated with (1.5) begins with determining the probability-weighted moments β_i in (2.6) by separately evaluating and summing two integrals as

$$\beta_i = I_{\mathcal{L}_i}(h_{\mathcal{L}}) + I_{\mathcal{R}_i}(h_{\mathcal{R}}) = \int_{-\infty}^{0} q(z,h_{\mathcal{L}})\{\Phi(z)\}^i\phi(z)dz + \int_0^{+\infty} q(z,h_{\mathcal{R}})\{\Phi(z)\}^i\phi(z)dz. \tag{2.21}$$

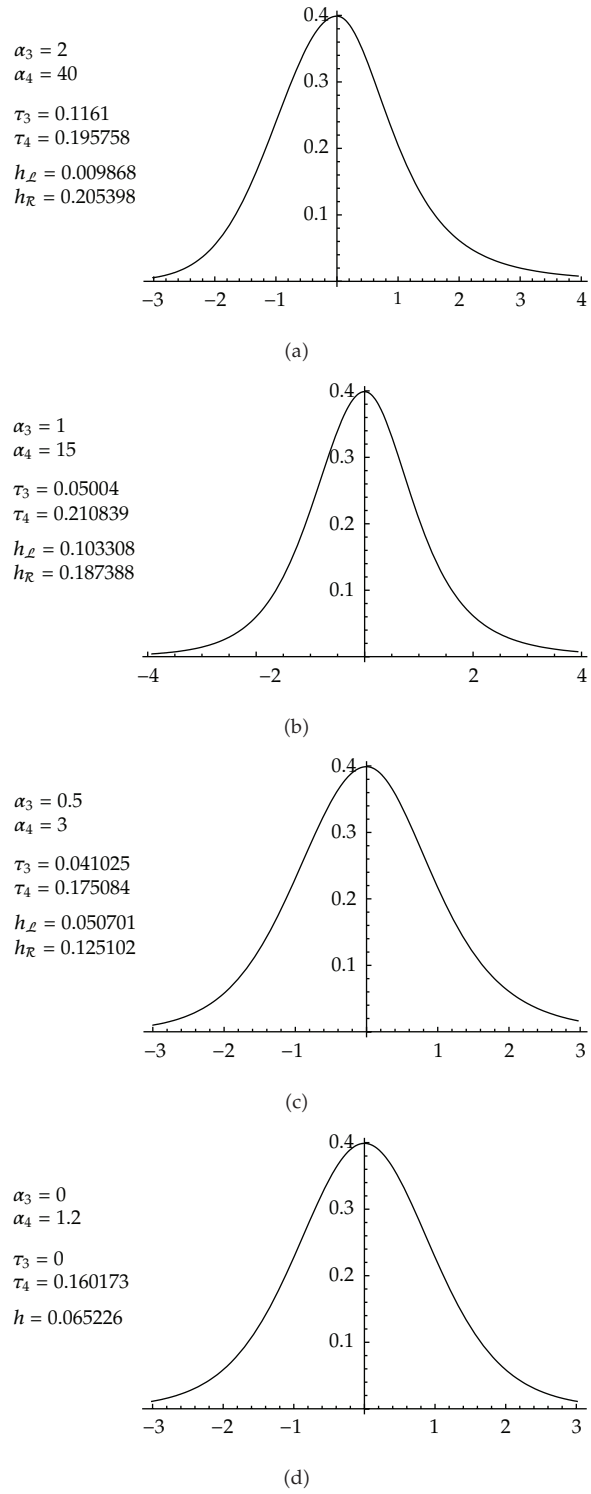


Figure 2: Three asymmetric hh-distributions (a)–(c) and one symmetric h distribution (d) with their conventional and L-moment parameters of skew (α_3) and L-skew (τ_3), kurtosis (α_4) and L-kurtosis (τ_4), and corresponding shape parameters for (1.5) and (1.1).

As such, using (2.1) and (2.21), it is straight-forward to obtain β_0, β_1, and the first two L-moments as

$$\lambda_1 = \frac{1}{\sqrt{2\pi}(h_{\mathcal{L}}-1)} + \frac{1}{\sqrt{2\pi}(1-h_{\mathcal{R}})},$$
$$\lambda_2 = \frac{\sqrt{2-h_{\mathcal{L}}} + \sqrt{2-h_{\mathcal{R}}} - h_{\mathcal{L}}\sqrt{2-h_{\mathcal{L}}} - h_{\mathcal{R}}\sqrt{2-h_{\mathcal{R}}}}{\sqrt{2\pi}(h_{\mathcal{L}}-1)(h_{\mathcal{R}}-1)\sqrt{(h_{\mathcal{L}}-2)(h_{\mathcal{R}}-2)}}. \tag{2.22}$$

In terms of deriving λ_3 and λ_4, it is convenient to consider β_2 in (2.2) as

$$\beta_2 = I_{\mathcal{L}_2}(h_{\mathcal{L}}) + I_{\mathcal{R}_2}(h_{\mathcal{R}}) = I_{\mathcal{L}_2}(h_{\mathcal{L}}) + \left(-I_{\mathcal{L}_2}(h_{\mathcal{R}}) + \frac{\lambda_2(h_{\mathcal{R}})}{2}\right), \tag{2.23}$$

where $\lambda_2(h_{\mathcal{R}})$ can be obtained from (2.8). Thus, it is only necessary to determine $I_{\mathcal{L}_2}(h_{\mathcal{L}})$ as

$$\begin{aligned} I_{\mathcal{L}_2}(h_{\mathcal{L}}) &= \int_{-\infty}^{0} q(z,h_{\mathcal{L}})\{\Phi(z)\}^2\phi(z)dz = \int_{-\infty}^{0} \frac{z}{\sqrt{2\pi}} \exp\left\{\frac{(h_{\mathcal{L}}-1)z^2}{2}\right\}\{\Phi(z)\}^2 dz \\ &= \frac{1}{\sqrt{2\pi}(h_{\mathcal{L}}-1)} \int_{-\infty}^{0} \{\Phi(z)\}^2 d\exp\left\{\frac{(h_{\mathcal{L}}-1)z^2}{2}\right\} \\ &= \frac{1}{\sqrt{2\pi}(h_{\mathcal{L}}-1)} \left[\{\Phi(z)\}^2 \exp\left\{\frac{(h_{\mathcal{L}}-1)z^2}{2}\right\}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{(h_{\mathcal{L}}-2)z^2}{2}\right\} 2\Phi(z)dz\right] \\ &= \frac{-1}{4\sqrt{2\pi}(1-h_{\mathcal{L}})} + \frac{2}{\sqrt{2\pi}\sqrt{2-h_{\mathcal{L}}}(1-h_{\mathcal{L}})} \int_{-\infty}^{0} \frac{\sqrt{2-h_{\mathcal{L}}}}{\sqrt{2\pi}} \exp\left\{-\frac{(2-h_{\mathcal{L}})z^2}{2}\right\}\Phi(z)dz. \end{aligned} \tag{2.24}$$

If we let $X \sim N(0,\ 1/(2-h_{\mathcal{L}}))$ and $\sim N(0,1)$, where X and Y are independent such that $(X/\sqrt{2-h_{\mathcal{L}}}, Y)$ jointly follow the standard bivariate normal distribution, then the integral in the last part of (2.24) is

$$\begin{aligned} &\int_{-\infty}^{0} \frac{\sqrt{2-h_{\mathcal{L}}}}{\sqrt{2\pi}} \exp\left\{-\frac{(2-h_{\mathcal{L}})z^2}{2}\right\}\Phi(z)dz \\ &\quad = \Pr\{X<0, Y<X\} \\ &\quad = \Pr\left\{\frac{X}{\sqrt{2-h_{\mathcal{L}}}} < 0, Y < \sqrt{2-h_{\mathcal{L}}}\frac{X}{\sqrt{2-h_{\mathcal{L}}}}\right\} \\ &\quad = \frac{1}{4}\left(1 - \frac{2\cot^{-1}\left(\sqrt{2-h_{\mathcal{L}}}\right)}{\pi}\right), \end{aligned} \tag{2.25}$$

where we are calculating the proportion of area between the y-axis and the line $y = x(\sqrt{2-h_{\mathcal{L}}})$ as a sector because the independent standard normal bivariate density has rotational symmetry about the origin. Combining terms from (2.24) and (2.25) yields

$$I_{\mathcal{L}_2}(h_{\mathcal{L}}) = \frac{1}{4(1-h_{\mathcal{L}})\sqrt{2\pi}}\left[\frac{2}{\sqrt{2-h_{\mathcal{L}}}}\left(1-\frac{2\cot^{-1}\left(\sqrt{2-h_{\mathcal{L}}}\right)}{\pi}\right)-1\right]. \tag{2.26}$$

Hence, given (2.26) and (2.19), we can solve for β_2, β_3, and subsequently obtain the expressions for L-skew (τ_3) and L-kurtosis (τ_4) as

$$\begin{aligned}
\tau_3 = {} & \left\{12\sqrt{2-h_{\mathcal{L}}}(h_{\mathcal{R}}-2)(h_{\mathcal{R}}-1)\cot^{-1}\left(\sqrt{2-h_{\mathcal{L}}}\right)-\pi(h_{\mathcal{L}}-h_{\mathcal{R}})(h_{\mathcal{L}}-2)(h_{\mathcal{R}}-2)\right.\\
& \left.-12\sqrt{2-h_{\mathcal{R}}}(h_{\mathcal{L}}-2)(h_{\mathcal{L}}-1)\cot^{-1}\left(\sqrt{2-h_{\mathcal{R}}}\right)\right\}\Big/\\
& \left\{2\pi\sqrt{(h_{\mathcal{L}}-2)(h_{\mathcal{R}}-2)}\left(h_{\mathcal{L}}\sqrt{2-h_{\mathcal{L}}}-\sqrt{2-h_{\mathcal{L}}}-\sqrt{2-h_{\mathcal{R}}}+h_{\mathcal{R}}\sqrt{2-h_{\mathcal{R}}}\right)\right\},\\
\tau_4 = {} & \left\{6\pi\left(h_{\mathcal{R}}\sqrt{(h_{\mathcal{L}}-4)(h_{\mathcal{L}}-2)(h_{\mathcal{L}}-1)(h_{\mathcal{R}}-2)}-2\sqrt{(h_{\mathcal{L}}-4)(h_{\mathcal{L}}-1)}\right.\right.\\
& \left.-h_{\mathcal{L}}(h_{\mathcal{L}}-3)\sqrt{(h_{\mathcal{L}}-4)(h_{\mathcal{L}}-1)}-\sqrt{(h_{\mathcal{L}}-4)(h_{\mathcal{L}}-2)(h_{\mathcal{L}}-1)(h_{\mathcal{R}}-2)}\right)\\
& +30(h_{\mathcal{L}}-1)\sqrt{\frac{(h_{\mathcal{L}}-4)(h_{\mathcal{L}}-2)(h_{\mathcal{R}}-2)}{h_{\mathcal{L}}-1}}(h_{\mathcal{R}}-1)\tan^{-1}\left(\sqrt{1+\frac{2}{h_{\mathcal{L}}-4}}\right)\\
& \left.+30(h_{\mathcal{L}}-2)\sqrt{(h_{\mathcal{L}}-4)(h_{\mathcal{L}}-1)}(h_{\mathcal{L}}-1)\tan^{-1}\left(\sqrt{1+\frac{2}{h_{\mathcal{R}}-4}}\right)\right\}\Big/\\
& \left\{\pi\sqrt{(4-h_{\mathcal{L}})(2-h_{\mathcal{L}})(1-h_{\mathcal{L}})}\left(h_{\mathcal{L}}\sqrt{2-h_{\mathcal{L}}}-\sqrt{2-h_{\mathcal{L}}}-\sqrt{2-h_{\mathcal{R}}}+h_{\mathcal{R}}\sqrt{2-h_{\mathcal{R}}}\right)\right\}.
\end{aligned} \tag{2.27}$$

Thus, given specified values of τ_3 and τ_4, (2.27) can be numerically solved for the corresponding values of $h_{\mathcal{L}}$ and $h_{\mathcal{R}}$. Figures 2(a), 2(b) and 2(c) provides some examples of various hh-distributions, which are used in the simulation portion of this study in Section 4.

3. L-Correlations for the h and hh-Distributions

The coefficient of L-correlation (see [19]) is introduced by considering two random variables Y_j and Y_k with distribution functions $F(Y_j)$ and $F(Y_k)$, respectively. The second L-moments of Y_j and Y_k can alternatively be expressed as

$$\lambda_2(Y_j) = 2\,\mathrm{Cov}(Y_j, F(Y_j)), \tag{3.1}$$

$$\lambda_2(Y_k) = 2\,\mathrm{Cov}(Y_k, F(Y_k)). \tag{3.2}$$

The second L-comoments of Y_j toward Y_k and Y_k toward Y_j are

$$\lambda_2(Y_j, Y_k) = 2\,\mathrm{Cov}(Y_j, F(Y_k)), \tag{3.3}$$

$$\lambda_2(Y_k, Y_j) = 2\,\mathrm{Cov}(Y_k, F(Y_j)). \tag{3.4}$$

As such, the L-correlations of Y_j toward Y_k and Y_k toward Y_j are expressed as

$$\eta_{jk} = \frac{\lambda_2(Y_j, Y_k)}{\lambda_2(Y_j)}, \tag{3.5}$$

$$\eta_{kj} = \frac{\lambda_2(Y_k, Y_j)}{\lambda_2(Y_k)}. \tag{3.6}$$

The L-correlation in (3.5) or (3.6) is bounded such that $-1 \le \eta_{jk} \le 1$, where a value of $\eta_{jk} = 1$ ($\eta_{jk} = -1$) indicates a strictly increasing (decreasing) *monotone* relationship between the two variables. In general, we would also note that $\eta_{jk} \ne \eta_{kj}$.

In the context of L-moment symmetric h-distributions ($0 \le h < 1$), suppose it is desired to simulate T distributions based on (1.1) with a specified L-correlation matrix and where each distribution has its own specified value of τ_4. Define $q(Z_j)$ and $q(Z_k)$ as in (1.1), where Z_j and Z_k have Pearson correlation ρ_{jk} and standard normal bivariate density of

$$f_{jk} = \left(2\pi\left(1-\rho_{jk}^2\right)^{1/2}\right)^{-1} \exp\left\{-\left(2\left(1-\rho_{jk}^2\right)\right)^{-1}\left(z_j^2 + z_k^2 - 2\rho_{jk} z_j z_k\right)\right\}. \tag{3.7}$$

Using (1.1), (1.3), and (3.5) with the denominator standardized to $\lambda_2 = 1/\sqrt{\pi}$ for the unit-normal distribution, and (3.7), the L-correlation of $q(Z_j)$ toward $q(Z_k)$ can be expressed as

$$\begin{aligned}
\eta_{jk} &= 2\sqrt{\pi}\ \mathrm{Cov}\left(q(z_j), F_{q(Z_k)}(q(z_k))\right) = 2\sqrt{\pi}\ \mathrm{Cov}\left(q(z_j), \Phi(z_k)\right) \\
&= 2\sqrt{\pi}\ E\left[q(z_j)\Phi(z_k)\right] - 2\sqrt{\pi}E\left[q(z_j)\right]\ E[\Phi(z_k)] \\
&= 2\sqrt{\pi}\int\int_{-\infty}^{+\infty} \delta q(z_j)\Phi(z_k) f_{jk}\, dz_j dz_k - 2\sqrt{\pi}E\left[q(z_j)\right]\ E[\Phi(z_k)],
\end{aligned} \tag{3.8}$$

where δ is the standardizing term in (2.12). Integrating (3.8) yields

$$\eta_{jk} = \rho_{jk}\sqrt{\frac{2-h_j}{2+h_j\left(\rho_{jk}^2 - 2\right)}} \tag{3.9}$$

given that $E[q(z_j)] = 0$ and $E[\Phi(z_k)] = 1/2$. Analogously, the L-correlation of $q(Z_k)$ toward $q(Z_j)$ is

$$\eta_{kj} = \rho_{jk}\sqrt{\frac{2-h_k}{2+h_k\left(\rho_{jk}^2 - 2\right)}}. \tag{3.10}$$

From (3.9), the intermediate correlation (IC) ρ_{jk} can be determined by simply evaluating

$$\rho_{jk} = \pm \frac{\sqrt{2}\sqrt{h_j \eta_{jk}^2 - \eta_{jk}^2}}{\sqrt{h_j \eta_{jk}^2 + h_j - 2}} \tag{3.11}$$

for a specified value of η_{jk} and a given value of h_j from (2.20). Given ρ_{jk} from (3.11), the L-correlation η_{kj} can be determined by evaluating (3.10) using the solved value of h_k. Note the special case of where $h_j = h_k$, in (3.9) and (3.10), then $\eta_{jk} = \eta_{kj}$.

Remark 3.1. Inspection of (3.9) indicates that $\eta_{jk} = \rho_{jk}$ when either (a) $q(Z_j)$ is standard normal that is, $h_j = 0$, (b) $\rho_{jk} = 0$, or (c) $\rho_{jk} = 1$.

Remark 3.2. If the IC is such that $0 < |\rho_{jk}| < 1$ and $0 < h_j < 1$ in (3.9), then we have the inequality

$$0 < \left|\rho_{jk}\right| < \left|\eta_{jk}\right| < 1, \tag{3.12}$$

as from inspecting (3.9) it is evident that $[(2 - h_j)/(2 + h_j(\rho_{jk}^2 - 2))]^{1/2} > 1$. Thus, solutions to ICs cannot exist outside the range of $[-1, +1]$.

The extension of determining ICs for asymmetric *hh*-distributions is analogous to the method described above for *h*-distributions, where (1.5) is standardized and subsequently integrated as in (3.8) to obtain

$$\eta_{jk} = \frac{1}{2}\rho_{jk}\sqrt{\frac{2 - h_{\mathcal{L}j}}{2 + h_{\mathcal{L}j}\left(\rho_{jk}^2 - 2\right)}} + \frac{1}{2}\rho_{jk}\sqrt{\frac{2 - h_{\mathcal{R}j}}{2 + h_{\mathcal{R}j}\left(\rho_{jk}^2 - 2\right)}}, \tag{3.13}$$

$$\eta_{kj} = \frac{1}{2}\rho_{jk}\sqrt{\frac{2 - h_{\mathcal{L}k}}{2 + h_{\mathcal{L}k}\left(\rho_{jk}^2 - 2\right)}} + \frac{1}{2}\rho_{jk}\sqrt{\frac{2 - h_{\mathcal{R}k}}{2 + h_{\mathcal{R}k}\left(\rho_{jk}^2 - 2\right)}}, \tag{3.14}$$

where it is straight-forward to see that if $h_{\mathcal{L}j} = h_{\mathcal{R}j}$ (or $h_{\mathcal{L}k} = h_{\mathcal{R}k}$), then (3.13) or (3.14) simplifies to (3.9) or (3.10) for the case of symmetric *h*-distributions. The IC in (3.13) is determined by substituting the specified L-correlation (η_{jk}) and the solved values of the parameters $h_{\mathcal{L}j}$ and $h_{\mathcal{R}j}$ (from (2.27)) into the left- and right-hand sides of (3.13), respectively, and then numerically solving for ρ_{jk}. We would also note that Remarks 3.1 and 3.2 also apply to (3.13) and (3.14).

4. The Procedure and Simulation Study

To implement the procedure for simulating *hh*- (or *h*-) distributions with specified L-moments and specified L-correlations, we suggest the following five steps.

(1) Specify the L-moments for T transformations of the form in (1.5), that is, $q(Z_1), \ldots, q(Z_T)$ and obtain the parameters of $h_{\mathcal{L}j}$ and $h_{\mathcal{R}j}$ by solving equations (2.27) using the specified values of L-skew (τ_3) and L-kurtosis (τ_4) for each distribution. Specify a $T \times T$ matrix of L-correlations (η_{jk}) for $q(Z_j)$ toward $q(Z_k)$, where $j < k \in \{1,2,\ldots,T\}$.

(2) Compute the (Pearson) intermediate correlations (ICs) ρ_{jk} by substituting the specified L-correlation η_{jk} and the parameters of $h_{\mathcal{L}j}$ and $h_{\mathcal{R}j}$ from step (1) into the left- and right-hand sides of (3.14), respectively, and then numerically solve for ρ_{jk}. Repeat this step separately for all $T(T-1)/2$ pairwise combinations of correlations.

(3) Assemble the ICs into a $T \times T$ matrix and decompose this matrix using a Cholesky factorization. Note that this step requires the IC matrix to be positive definite.

(4) Use the results of the Cholesky factorization from step (3) to generate T standard normal variables $(Z_1, \ldots, Z_T)$ correlated at the intermediate levels as follows:

$$\begin{aligned}
Z_1 &= a_{11}V_1, \\
Z_2 &= a_{12}V_1 + a_{22}V_2, \\
&\vdots \\
Z_j &= a_{1j}V_1 + a_{2j}V_2 + \cdots + a_{ij}V_i + \cdots + a_{jj}V_j, \\
&\vdots \\
Z_T &= a_{1T}V_1 + a_{2T}V_2 + \cdots + a_{iT}V_i + \cdots + a_{jT}V_j + \cdots + a_{TT}V_T,
\end{aligned} \tag{4.1}$$

where $V_1, \ldots, V_T$ are independent standard normal random variables and where a_{ij} represents the element in the i-th row and the j-th column of the matrix associated with the Cholesky factorization performed in step (3).

(5) Substitute $Z_1, \ldots, Z_T$ from step (4) into T equations of the form in (1.5), as noted in step (1), to generate the nonnormal hh-distributions with the specified L-moments and L-correlations.

To demonstrate the steps above and evaluate the proposed procedure, a comparison between the new L-moment and conventional moment-based procedures is subsequently described. Specifically, the distributions in Figure 2 are used as a basis for a comparison using the specified correlation matrices in Table 1 where both strong and moderate levels of correlation are considered. Tables 2 and 3 give the solved IC matrices for the L-moment and conventional moment-based procedures, respectively. Note that the ICs for the conventional procedure were computed by adapting the Mathematica source code in [12, Table 1] for (1.5). Tables 4 and 5 give the results of the Cholesky decompositions on the IC matrices, which are then used to create $Z_1, \ldots, Z_4$ with the specified ICs by making use of the formulae given in (4.1) of step 4 with $T = 4$. The values of $Z_1, \ldots, Z_4$ are subsequently substituted into equations of the form in (1.5) to produce $q(Z_1), \ldots, q(Z_4)$ for both procedures.

In terms of the simulation, a Fortran algorithm was written for both procedures to generate 25,000 independent sample estimates for the specified parameters of (a) conventional skew (α_3), kurtosis (α_4), and Pearson correlation (ρ^*_{jk}); (b) L-skew (τ_3), L-kurtosis (τ_4), and L-correlation (η_{jk}) based on samples of sizes $n = 25$ and $n = 1000$. The estimates for $\alpha_{3,4}$ were based on Fisher's k-statistics that is, the formulae currently used by

Table 1: Specified correlation matrices for the distributions in Figure 2.

(a)

	1	2	3	4
1	1			
2	0.70	1		
3	0.70	0.70	1	
4	0.85	0.70	0.70	1

(b)

	1	2	3	4
1	1			
2	0.40	1		
3	0.50	0.40	1	
4	0.60	0.50	0.40	1

most commercial software packages such as SAS, SPSS, and Minitab for computing indices of skew and kurtosis (where $\alpha_{3,4} = 0$ for the standard normal distribution). The formulae used for computing estimates for $\tau_{3,4}$ were Headrick's Equations (2.3) and (2.5) [15]. The estimate for ρ_{jk}^{*} was based on the usual formula for the Pearson product-moment of correlation statistic, and the estimate for η_{jk} was computed based on (3.13) using the empirical forms of the cdfs in (3.1) and (3.3). The estimates for ρ_{jk}^{*} and η_{jk} were both transformed using Fisher's z' transformation. Bias-corrected accelerated bootstrapped average estimates, confidence intervals (C.I.s), and standard errors were subsequently obtained for the estimates associated with the parameters $(\alpha_{3,4}, \tau_{3,4}, z'_{\rho_{jk}^{*}}, z'_{\eta_{jk}})$ using 10,000 resamples via the commercial software package Spotfire S+ [20]. The bootstrap results for the estimates of $z'_{\rho_{jk}^{*}}$ and $z'_{\eta_{jk}}$ were transformed back to their original metrics. Further, if a parameter (P) was outside its associated bootstrap C.I., then an index of relative bias (RB) was computed for the estimate (E) as RB $= ((E - P)/P) \times 100$. The results of the simulation are reported in Tables 6, 7, 8, 9, 10, and 11 and are discussed in the next section.

5. Discussion and Conclusion

One of the advantages that L-moment ratios have over conventional moment-based estimators is that they can be far less biased when sampling is from distributions with more severe departures from normality [10, 19]. And inspection of the simulation results in Tables 6 and 7 clearly indicates that this is the case. That is, the superiority that estimates of L-moment ratios (τ_3, τ_4) have over their corresponding conventional moment-based counterparts (α_3, α_4) is obvious. For example, with samples of size $n = 25$, the estimates of skew and kurtosis for Distribution 1 were, on average, only 32.4% and 4.2% of their associated population parameters, whereas the estimates of L-skew and L-kurtosis were 87.9% and 96.10% of their respective parameters. It is also evident from comparing Tables 6 and 7 that L-skew and L-kurtosis are more efficient estimators as their standard errors are significantly smaller than the conventional-moment-based estimators of skew and kurtosis.

Presented in Tables 8, 9, 10, and 11 are the results associated with the conventional Pearson and L-correlations. Overall inspection of these tables indicates that the L-correlation is superior to the Pearson correlation in terms of relative bias. For example, for moderate

Table 2: Intermediate correlations for the conventional moment procedure.

(a)

	1	2	3	4
1	1			
2	0.726	1		
3	0.722	0.715	1	
4	0.881	0.717	0.707	1

(b)

	1	2	3	4
1	1			
2	0.424	1		
3	0.521	0.414	1	
4	0.626	0.515	0.405	1

Table 3: Intermediate correlations for the *L*-moment procedure.

(a)

	1	2	3	4
1	1			
2	0.689	1		
3	0.689	0.685	1	
4	0.843	0.685	0.691	1

(b)

	1	2	3	4
1	1			
2	0.389	1		
3	0.488	0.386	1	
4	0.588	0.485	0.392	1

correlations ($n = 25$), the relative bias for the two heavy-tailed distributions (i.e., distributions 1 and 2) was 6% for the Pearson correlation compared with only 2.5% for the *L*-correlation. It is also noted that the variability of the *L*-correlation appears to be more stable than that of the Pearson correlation both within and across the different conditions.

In summary, the new *L*-moment-based procedure is an attractive alternative to the traditional conventional-moment-based procedure. In particular, the *L*-moment-based procedure has distinct advantages when distributions with large departures from normality are used. Finally, we note that Mathematica Version 8.0.1 [21] source code is available from the authors for implementing both the conventional and new *L*-moment-based procedures.

Appendix

System of Conventional Moment-Based Equations for Tukey *hh*-Distributions

The equations for the mean (α_1), variance (α_2^2), skew (α_3), and kurtosis (α_4) for conventional moment-based *hh*-distributions are given below without the details of their derivation. We

Table 4: Cholesky decompositions for the Conventional moment procedure.

(a)

$a_{11} = 1$	$a_{12} = 0.726$	$a_{13} = 0.722$	$a_{14} = 0.881$
0	$a_{22} = 0.687$	$a_{23} = 0.277$	$a_{24} = 0.112$
0	0	$a_{33} = 0.633$	$a_{34} = 0.062$
0	0	0	$a_{44} = 0.455$

(b)

$a_{11} = 1$	$a_{12} = 0.424$	$a_{13} = 0.521$	$a_{14} = 0.626$
0	$a_{22} = 0.906$	$a_{23} = 0.213$	$a_{24} = 0.275$
0	0	$a_{33} = 0.827$	$a_{34} = 0.025$
0	0	0	$a_{44} = 0.729$

Table 5: Cholesky decompositions for the *L*-moment procedure.

(a)

$a_{11} = 1$	$a_{12} = 0.689$	$a_{13} = 0.689$	$a_{14} = 0.843$
0	$a_{22} = 0.725$	$a_{23} = 0.291$	$a_{24} = 0.145$
0	0	$a_{33} = 0.664$	$a_{34} = 0.104$
0	0	0	$a_{44} = 0.508$

(b)

$a_{11} = 1$	$a_{12} = 0.389$	$a_{13} = 0.488$	$a_{14} = 0.588$
0	$a_{22} = 0.921$	$a_{23} = 0.213$	$a_{24} = 0.278$
0	0	$a_{33} = 0.846$	$a_{34} = 0.0543$
0	0	0	$a_{44} = 0.758$

Table 6: Skew (α_3) and Kurtosis (α_4) results for the conventional moment procedure.

(a) $n = 25$

Dist	Parameter	Estimate	95% Bootstrap C.I.	Standard error	Relative bias %
1	$\alpha_3 = 2$	0.649	0.638, 0.659	0.0056	−67.6
	$\alpha_4 = 40$	1.677	1.64, 1.72	0.0200	−95.8
2	$\alpha_3 = 1$	0.305	0.293, 0.317	0.0060	−69.5
	$\alpha_4 = 15$	1.73	1.69, 1.76	0.0175	−88.5
3	$\alpha_3 = 0.5$	0.248	0.239, 0.257	0.0048	−50.4
	$\alpha_4 = 3$	0.97	0.95, 1.00	0.0128	−67.7
4	$\alpha_3 = 0$	0.0056	−0.0024, 0.0140	0.0042	—
	$\alpha_4 = 1.2$	0.624	0.606, 0.643	0.0095	−48.0

(b) $n = 1000$

Dist	Parameter	Estimate	95% Bootstrap C.I.	Standard error	Relative bias %
1	$\alpha_3 = 2$	1.612	1.597, 1.627	0.0077	−19.4
	$\alpha_4 = 40$	11.41	11.11, 11.72	0.1542	−71.5
2	$\alpha_3 = 1$	0.834	0.823, 0.845	0.0057	−16.6
	$\alpha_4 = 15$	7.68	7.50, 7.87	0.0962	−48.8
3	$\alpha_3 = 0.5$	0.477	0.472, 0.481	0.0021	−4.6
	$\alpha_4 = 3$	2.65	2.61, 2.70	0.0223	−11.7
4	$\alpha_3 = 0$	−0.0003	−0.0023, 0.0018	0.0011	—
	$\alpha_4 = 1.2$	1.166	1.158, 1.175	0.0046	−2.83

Table 7: L-skew (τ_3) and L-kurtosis (τ_4) results.

(a) $n = 25$

Dist	Parameter	Estimate	95% Bootstrap C.I.	Stand. error	Relative bias %
1	$\tau_3 = 0.110610$	0.09724	0.09574, 0.09897	0.00082	−12.1
	$\tau_4 = 0.195758$	0.1882	0.1869, 0.1893	0.00061	−3.9
2	$\tau_3 = 0.050040$	0. 04361	0.04193, 0.04534	0.00086	−12.8
	$\tau_4 = 0.210839$	0.2038	0.2026, 0.2050	0.00060	−3.3
3	$\tau_3 = 0.041025$	0.03726	0.03592, 0.03883	0.00074	−9.2
	$\tau_4 = 0.175084$	0.1723	0.1713, 0.1734	0.00054	−1.6
4	$\tau_3 = 0$	0.0008	−0.0007, 0.0021	0.00070	—
	$\tau_4 = 0.160173$	0.1591	0.1582, 0.1602	0.00050	—

(b) $n = 1000$

Dist	Parameter	Estimate	95% Bootstrap C.I.	Stand. error	Relative bias %
1	$\tau_3 = 0.110610$	0.1101	0.1098, 0.1104	0.00014	−0.46
	$\tau_4 = 0.195758$	0. 1955	0. 1953, 0.1958	0.00010	—
2	$\tau_3 = 0.050040$	0.0498	0.04946, 0.050041	0.00015	—
	$\tau_4 = 0.210839$	0.2106	0.2104, 0.2109	0.00010	—
3	$\tau_3 = 0.041025$	0.04086	0.04062, 0.04109	0.00012	—
	$\tau_4 = 0.175084$	0.1750	0.1748, 0.1751	0.00010	—
4	$\tau_3 = 0$	−0.00003	−0.00027, 0.00015	0.00011	—
	$\tau_4 = 0.160173$	0.1602	0.1600, 0.1603	0.00010	—

Table 8: Correlation (strong) results for the conventional moment procedure.

(a) $n = 25$

Parameter	Estimate	95% Bootstrap C.I.	Standard error	Relative bias%
$\rho^*_{12} = 0.70$	0.725	0.724, 0.727	0.0015	3.57
$\rho^*_{13} = 0.70$	0.723	0.722, 0.724	0.0014	3.29
$\rho^*_{14} = 0.85$	0.873	0.873, 0.874	0.0013	2.71
$\rho^*_{23} = 0.70$	0.718	0.716, 0.719	0.0014	2.57
$\rho^*_{24} = 0.70$	0.718	0.717, 0.719	0.0014	2.57
$\rho^*_{34} = 0.70$	0.712	0.710, 0.713	0.0014	1.71

(b) $n = 1000$

Parameter	Estimate	95% Bootstrap C.I.	Standard error	Relative bias%
$\rho^*_{12} = 0.70$	0.7024	0.7021, 0.7027	0.00030	0.34
$\rho^*_{13} = 0.70$	0.7025	0.7023, 0.7028	0.00027	0.36
$\rho^*_{14} = 0.85$	0.8535	0.8532, 0.8536	0.00037	0.41
$\rho^*_{23} = 0.70$	0.7010	0.7008, 0.7013	0.00024	0.14
$\rho^*_{24} = 0.70$	0.7013	0.7011, 0.7015	0.00024	0.19
$\rho^*_{34} = 0.70$	0.7005	0.7003, 0.7007	0.00021	0.07

Table 9: Correlation (strong) results for the L-moment procedure.

(a) $n = 25$

Parameter	Estimate	95% Bootstrap C.I.	Standard error	Relative bias%
$\eta_{12} = 0.70$	0.710	0.708, 0.711	0.0015	1.40
$\eta_{13} = 0.70$	0.709	0.708, 0.711	0.0015	1.34
$\eta_{14} = 0.85$	0.856	0.855, 0.857	0.0015	0.71
$\eta_{23} = 0.70$	0.708	0.706, 0.709	0.0015	1.10
$\eta_{24} = 0.70$	0.707	0.706, 0.709	0.0015	1.01
$\eta_{34} = 0.70$	0.708	0.707, 0.710	0.0014	1.20

(b) $n = 1000$

Parameter	Estimate	95% Bootstrap C.I.	Standard error	Relative bias%
$\eta_{12} = 0.70$	0.7014	0.7012, 0.7016	0.00022	0.20
$\eta_{13} = 0.70$	0.7016	0.7014, 0.7018	0.00022	0.23
$\eta_{14} = 0.85$	0.8511	0.8509, 0.8512	0.00022	0.13
$\eta_{23} = 0.70$	0.7002	0.6999, 0.7004	0.00022	—
$\eta_{24} = 0.70$	0.7003	0.7001, 0.7005	0.00022	0.04
$\eta_{34} = 0.70$	0.7004	0.7002, 0.7006	0.00021	0.06

Table 10: Correlation (moderate) results for the conventional moment procedure.

(a) $n = 25$

Parameter	Estimate	95% Bootstrap C.I.	Standard error	Relative bias%
$\rho^*_{12} = 0.40$	0.424	0.422, 0.427	0.0014	6.00
$\rho^*_{13} = 0.50$	0.522	0.520, 0.524	0.0014	4.40
$\rho^*_{14} = 0.60$	0.624	0.622, 0.626	0.0013	4.00
$\rho^*_{23} = 0.40$	0.416	0.414, 0.418	0.0013	4.00
$\rho^*_{24} = 0.50$	0.516	0.514, 0.518	0.0013	3.20
$\rho^*_{34} = 0.40$	0.410	0.408, 0.412	0.0013	2.50

(b) $n = 1000$

Parameter	Estimate	95% Bootstrap C.I.	Standard Error	Relative Bias%
$\rho^*_{12} = 0.40$	0.4020	0.4016, 0.4023	0.00024	0.50
$\rho^*_{13} = 0.50$	0.5022	0.5018, 0.5025	0.00024	0.44
$\rho^*_{14} = 0.60$	0.6023	0.6020, 0.6026	0.00024	0.38
$\rho^*_{23} = 0.40$	0.4005	0.4001, 0.4008	0.00021	0.12
$\rho^*_{24} = 0.50$	0.5010	0.5007, 0.5013	0.00021	0.20
$\rho^*_{34} = 0.40$	0.4003	0.4000, 0.4006	0.00020	0.08

Table 11: Correlation (moderate) results for the L-moment procedure.

(a) $n = 25$

Parameter	Estimate	95% Bootstrap C.I.	Standard error	Relative bias%
$\eta_{12} = 0.40$	0.410	0.408, 0.412	0.0015	2.50
$\eta_{13} = 0.50$	0.509	0.507, 0.511	0.0015	1.80
$\eta_{14} = 0.60$	0.610	0.608, 0.611	0.0015	1.67
$\eta_{23} = 0.40$	0.407	0.404, 0.409	0.0015	1.75
$\eta_{24} = 0.50$	0.507	0.505, 0.509	0.0015	1.40
$\eta_{34} = 0.40$	0.408	0.406, 0.410	0.0014	2.00

(b) $n = 1000$

Parameter	Estimate	95% Bootstrap C.I.	Standard error	Relative bias%
$\eta_{12} = 0.40$	0.4013	0.4010, 0.4017	0.00022	0.32
$\eta_{13} = 0.50$	0.5017	0.5014, 0.5020	0.00022	0.34
$\eta_{14} = 0.60$	0.6016	0.6014, 0.6019	0.00022	0.27
$\eta_{23} = 0.40$	0.3999	0.3995, 0.4002	0.00022	—
$\eta_{24} = 0.50$	0.5004	0.5001, 0.5007	0.00022	0.08
$\eta_{34} = 0.40$	0.4003	0.3999, 0.4006	0.00021	—

would note that we derived (A.1)–(A.4) based on the formulae given in Headrick et al. [3, equations (16)–(18)].

$$\alpha_1 = \frac{1}{\sqrt{2\pi}(h_{\mathcal{L}} - 1)} + \frac{1}{\sqrt{2\pi}(h_{\mathcal{R}} - 1)}, \tag{A.1}$$

$$\alpha_2^2 = \frac{C}{2} = \frac{1/(1-2h_{\mathcal{R}})^{3/2} + 1/(1-2h_{\mathcal{L}})^{3/2} - (h_{\mathcal{L}} - h_{\mathcal{R}})^2/\pi(h_{\mathcal{L}} - 1)^2(h_{\mathcal{R}} - 1)^2}{2}, \tag{A.2}$$

$$\alpha_3 = \left\{ \frac{4\pi}{(1-3h_{\mathcal{R}})^2} - \frac{4\pi}{(1-3h_{\mathcal{L}})^2} - \frac{2(h_{\mathcal{L}} - h_{\mathcal{R}})^3}{(h_{\mathcal{L}} - 1)^3(h_{\mathcal{R}} - 1)^3} - \left[3\pi(h_{\mathcal{R}} - h_{\mathcal{L}})\left(\frac{1}{(1-2h_{\mathcal{L}})^{3/2}} + \frac{1}{(1-2h_{\mathcal{R}})^{3/2}}\right)\right] \Big/ \left[(h_{\mathcal{L}} - 1)(h_{\mathcal{R}} - 1)\right] \right\} \Big/ \left(\pi^{3/2} C^{3/2}\right), \tag{A.3}$$

$$\alpha_4 = -3 + \left\{ 6 \left(\frac{1}{(1-4h_{\mathcal{R}})^{5/2}} + \frac{1}{(1-4h_{\mathcal{L}})^{5/2}} - \frac{(h_{\mathcal{L}} - h_{\mathcal{R}})^4}{\pi^2(h_{\mathcal{L}} - 1)^4(h_{\mathcal{R}} - 1)^4} - \frac{(h_{\mathcal{L}} - h_{\mathcal{R}})^2}{\pi(h_{\mathcal{L}} - 1)^2(h_{\mathcal{R}} - 1)^2} + \left(2(h_{\mathcal{L}} - h_{\mathcal{R}})^2 \left(\frac{1}{(1-2h_{\mathcal{L}})^{3/2}} + \frac{1}{(1-2h_{\mathcal{R}})^{3/2}}\right)\right) \Big/ \left(\pi(h_{\mathcal{L}} - 1)^2(h_{\mathcal{R}} - 1)^2\right) + \frac{8(h_{\mathcal{L}} - h_{\mathcal{R}})^2(3h_{\mathcal{L}} + 3h_{\mathcal{R}} - 2)}{\left(\pi(1-3h_{\mathcal{L}})^2(1-3h_{\mathcal{R}})^2(h_{\mathcal{L}} - 1)(h_{\mathcal{R}} - 1)\right)} \right) \right\} \Big/ C^2. \tag{A.4}$$

References

[1] J. W. Tukey, "Modern techniques in data analysis," in *Proceedings of the NSF-Sponsored Regional Research Conference*, Southern Massachusetts University, North Dartmouth, Mass, USA, 1977.

[2] S. Morgenthaler and J. W. Tukey, "Fitting quantiles: doubling, HR, HQ, and HHH distributions," *Journal of Computational and Graphical Statistics*, vol. 9, no. 1, pp. 180–195, 2000.

[3] T. C. Headrick, R. K. Kowalchuk, and Y. Sheng, "Parametric probability densities and distribution functions for Tukey *g*-and-*h* transformations and their use for fitting data," *Applied Mathematical Sciences*, vol. 2, no. 9, pp. 449–462, 2008.

[4] D. Guegan and B. Hassani, "A modified Panjer algorithm for operational risk capital calculations," *Journal of Operational Risk*, vol. 4, no. 4, 26 pages, 2009.

[5] D. J. Dupuis and C. A. Field, "Large wind speeds: modeling and outlier detection," *Journal of Agricultural, Biological, and Environmental Statistics*, vol. 9, no. 1, pp. 105–121, 2004.

[6] S. G. Badrinath and S. Chatterjee, "A data-analytic look at skewness and elongation in common-stock return distributions," *Journal of Business and Economic Statistics*, vol. 9, no. 1, pp. 105–121, 1991.

[7] G. M. Goerg, *The Lambert way to Gaussianize Skewed, Heavy Tailed Data with the Inverse of Tukey's h Transformation as a Special Case*, Cornell University Library, 2011, arXiv:1010.2265v4 [math.ST].

[8] H. J. Keselman, R. K. Kowalchuk, and L. M. Lix, "Robust nonorthogonal analyses revisited: an update based on trimmed means," *Psychometrika*, vol. 63, no. 2, pp. 145–163, 1998.

[9] J. R. M. Hosking, "*L*-moments: analysis and estimation of distributions using linear combinations of order statistics," *Journal of the Royal Statistical Society B*, vol. 52, no. 1, pp. 105–124, 1990.

[10] J. R. M. Hosking and J. R. Wallis, *Regional Frequency Analysis: An Approach Based on L-Moments*, Cambridge University Press, Cambridge, UK, 1997.

[11] T. C. Headrick, *Statistical Simulation: Power Method Polynomials and other Tranformations*, Chapman and Hall/CRC, Boca Raton, Fla, USA, 2010.

[12] R. K. Kowalchuk and T. C. Headrick, "Simulating multivariate g-and-h distributions," *British Journal of Mathematical and Statistical Psychology*, vol. 63, no. 1, pp. 63–74, 2010.

[13] A. Nataf, "Determination des distribution de probabilities dont les marges sont donnees," *BComptes Rendus de L'Academie des Sciences*, vol. 225, pp. 42–43, 1962.

[14] M. Vorechovsky and D. Novák, "Correlation control in small-sample Monte Carlo type simulations I: a simulated annealing approach," *Probabilistic Engineering Mechanics*, vol. 24, no. 3, pp. 452–462, 2009.

[15] T. C. Headrick, "A characterization of power method transformations through *L*-moments," *Journal of Probability and Statistics*, vol. 2011, Article ID 497463, 22 pages, 2011.

[16] M. C. Jones, "On some expressions for variance, covariance, skewness and *L*-moments," *Journal of Statistical Planning and Inference*, vol. 126, no. 1, pp. 97–106, 2004.

[17] H. A. David and H. N. Nagaraja, *Order Statistics*, John Wiley & Sons, Hoboken, NJ, USA, 3rd edition, 2003.

[18] L. Lichtenstein, "Ueber die Integration eines bestimmten integrals in Bezug auf einen parameter," *Gottingen Nachrichten*, pp. 468–475, 1910.

[19] R. Serfling and P. Xiao, "A contribution to multivariate *L*-moments *L*-comoment matrices," *Journal of Multivariate Analysis*, vol. 98, no. 9, pp. 1765–1781, 2007.

[20] *TIBCO Spotfire S+ 8.1 for Windows*, TIBCO Software, Palo Alto, Calif, USA, 2008.

[21] S. Wolfram, *The Mathematica Book*, Wolfram Media, Champaign, Ill, USA, 5th edition, 1999.