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On Simulating Univariate and Multivariate Burr Type III and Type XII Distributions

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Abstract

This paper describes a method for simulating univariate and multivariate Burr Type III and Type XII distributions with specified correlation matrices. The methodology is based on the derivation of the parametric forms of a *pdf* and *cdf* for this family of distributions. The paper shows how shape parameters can be computed for specified values of skew and kurtosis. It is also demonstrated how to compute percentage points and other measures of central tendency such as the mode, median, and trimmed mean. Examples are provided to demonstrate how this Burr family can be used in the context of distribution fitting using real data sets. The results of a Monte Carlo simulation are provided to confirm that the proposed method generates distributions with user specified values of skew, kurtosis, and intercorrelation. Tabled values of shape parameters and boundary values of kurtosis are also provided in the appendices for the user.

Mathematics Subject Classification: 65C05, 65C10, 65C60

Keywords: Distribution fitting, Moments, Monte Carlo, Random variable generation, Simulation

1 Introduction

Burr [4] introduced a system of twelve cumulative distribution functions (cdfs) for the primary purpose of fitting data. Burr [5] and Tadikamalla [37] gave additional attention to the Type III and Type XII distributions because they include a variety of distributions with varying degrees of skew and kurtosis. For example, the Type XII distributions include characteristics of the normal, lognormal, gamma, logistic, and exponential distributions as well as other characteristics associated with the Pearson family of distributions [34, 37]. Further, the Type III or Type XII distributions have been used in a variety of settings for the purpose of statistical modeling. Some examples include such topics as forestry [8, 24], fracture roughness [28], life testing [42, 43], operational risk [6], option market price distributions [35], meteorology [26], modeling crop prices [38], and reliability [27].

Although the Type III and Type XII distributions have quantile (or percentile) functions available in closed-form, these distributions have received less attention or use in the context of Monte Carlo studies e.g. examining the robustness or power properties of competing parametric or nonparametric statistics. Other competing families of distributions that also possess closed-form quantile functions such as the power method [7, 13, 14], generalized lambda (GLD) [20, 30-32], or Tukey g-and-h distributions [15, 18, 19, 39] appear more often in the applied literature. One reason for this could be attributed to the computational difficulties associated with simulating Burr distributions with a specified correlation structure when juxtaposed to some of the other families of distributions. Specifically, and in terms of the power method, computationally efficient procedures for simulating correlated non-normal distributions have been provided (see [10, 13, 40, 41]). Some examples of where the power method has been used in this context include such topics or techniques as analysis of covariance [9, 11, 29], item response theory [36], logistic regression [16], regression [12], repeated measures [1, 23], and structural equation modeling [17, 33].

An additional concern to be made regarding the Type III and Type XII distributions is the paucity of research that considers these distributions as a single family in the context of simulating univariate and multivariate non-normal distributions. This concern is raised because one of the advantages that the Type III and Type XII family of distributions has over the other competing families of distributions mentioned above, is that it covers a larger region of kurtosis towards the lower end of the skew (α_1) and kurtosis (α_2) boundary i.e. $\alpha_2 \geq \alpha_1^2 - 2$. For example, there are some Type III distributions with values of $\alpha_1 = 1$ and $\alpha_2 < 0$ (see [5]). On the other hand, the power method, GLD, and Tukey g-and-h transformations are unable to produce distributions with such values of α_1 and α_2.

In view of the above, the present aim here is to develop the methodology for simulating univariate and multivariate Burr Type III and Type XII distributions. Specifically, we first consider the Type III and Type XII distributions together as a single family and derive general parametric forms of a probability density function (pdf) and a cdf for this family. As such, determining shape parameters, measures of central tendency, and fitting pdfs to empirical data can be done in a computationally efficient manner for either type of distribution. The methodology is subsequently presented for extending the Burr family from univariate to multivariate data generation. Numerical examples and the results from a Monte Carlo simulation are provided to demonstrate and confirm the methodology. Tabled values of shape parameters and boundary values of kurtosis are also provided.

2 Univariate Type III and Type XII Burr Distributions

The Burr family of distributions considered herein is based on transformations that produce non-normal distributions with defined mean, variance, skew, and kurtosis. These transformations are computationally efficient because they only require the knowledge of two shape parameters and an algorithm that generates regular uniform pseudo-random deviates. We begin the derivation of the general parametric forms of a pdf and a cdf for this family of distributions with the following definitions.

Definition 2.1 Let U be a uniformly distributed random variable with pdf and cdf expressed as

$$f_U(u) = 1 \tag{1}$$

$$F_U(u) = Pr(U \leq u) = \int_0^u 1du = u \tag{2}$$

where $0 \leq u \leq 1$. Let $u = (x, y)$ be the auxiliary variable that maps the parametric curves of (1) and (2) as

$$f : u \rightarrow \Re^2 := f_U(u) = f_U(x, y) = f_U(u, f_U(u)) = f_U(u, 1) \tag{3}$$

$$F : u \rightarrow \Re^2 := F_U(u) = F_U(x, y) = F_U(u, F_U(u)) = F_U(u, u). \tag{4}$$

Definition 2.2 Let the analytical forms of the quantile functions (for generating data) associated with Burr Type III and Type XII distributions be defined as

$$q(u) = q(u)_{\texttt{III}} = (u^{-\frac{1}{k}} - 1)^{\frac{1}{c}} \tag{5}$$

$$q(u) = q(u)_{\texttt{XII}} = ((1 - u)^{-\frac{1}{k}} - 1)^{\frac{1}{c}} \tag{6}$$

where $q(u)$ is said to be a strictly increasing monotonic function in u i.e. derivative $q'(u) > 0$ with real valued parameters c and k that control the shape of a distribution. For both Type III and Type XII distributions the value of k is positive. The value of c is negative (positive) for Type III (Type XII) distributions.

The derivatives of (5) and (6) are

$$q'(u) = q'(u)_{\mathtt{III}} = -(u^{-\frac{1}{k}-1})(u^{-\frac{1}{k}} - 1)^{\frac{1}{c}-1}/(ck) \quad (7)$$
$$q'(u) = q'(u)_{\mathtt{XII}} = (1-u)^{-\frac{1}{k}-1}((1-u)^{-\frac{1}{k}} - 1)^{\frac{1}{c}-1}/(ck). \quad (8)$$

Proposition 2.1 If the compositions $f \circ q$ and $F \circ q$ map the parametric curves of $f_{q(U)}(q(u))$ and $F_{q(U)}(q(u))$ where $q(u) = q(x, y)$ as

$$f \circ q : q(u) \rightarrow \Re^2 := f_{q(U)}(q(u)) = f_{q(U)}(q(x,y)) = f_{q(U)}(q(u), 1/q'(u)) \quad (9)$$

$$F \circ q : q(u) \rightarrow \Re^2 := F_{q(U)}(q(u)) = F_{q(U)}(q(x,y)) = F_{q(U)}(q(u), u) \quad (10)$$

then $f_{q(U)}(q(u), 1/q'(u))$ and $F_{q(U)}(q(u), u)$ in (9) and (10) are the pdf and cdf associated with the quantile function in either (5) or (6).

Proof: It is first shown that $f_{q(U)}(q(u), 1/q'(u))$ in (9) has the following properties:

Property 2.1 $\int_0^1 f_{q(U)}(q(u), 1/q'(u))du = 1$, and

Property 2.2 $f_{q(U)}(q(u), 1/q'(u)) \geq 0$, $0 \leq u \leq 1$.

To prove Property 2.1, let $y = f(x)$ be a function where $\int_0^1 f(x)dx = \int_0^1 ydx$. Thus, given that $x = q(u)$ and $y = 1/q'(u)$ in $f_{q(U)}(q(x, y))$ of (9) we have $\int_0^1 f_{q(U)}(q(u), 1/q'(u))du = \int_0^1 ydx = \int_0^1 (1/q'(u))dq(u) = \int_0^1 (1/q'(u))q'(u)$ $du = \int_0^1 du = 1$ which integrates to one because by Definition 2.1 $f_{(U)}(u) = 1$ is the regular uniform pdf.

To prove Property 2.2, it is given by definition that $q'(u) > 0$. Hence, $f_{q(U)}(q(u), 1/q'(u)) \geq 0$ because $1/q'(u)$ will be nonnegative on the support of u for all $u \in (0, 1)$ provided that $k > 0$ and $c < 0$ $(c > 0)$ for Type III (Type XII) distributions. In terms of Type III distributions, the limit conditions are:

(a) $\lim_{u\rightarrow 0} f_{q(U)}(q(u), 1/q'(u)) = +\infty$, for $0 < k < 1$ and $-1 < ck < 0$,

(b) $\lim_{u\rightarrow 0} f_{q(U)}(q(u), 1/q'(u)) = 0$, for $0 < k < 1$ and $ck < -1$,

(c) $\lim_{u\rightarrow 0} f_{q(U)}(q(u), 1/q'(u)) = 0$, for $k > 1$ and $c < -1$,

(d) $\lim_{u\rightarrow 1} f_{q(U)}(q(u), 1/q'(u)) = 0$, for $k > 0$ and $c < 0$.

In terms of Type XII distributions, the limit conditions are:

(a) $\lim_{u\rightarrow 0} f_{q(U)}(q(u), 1/q'(u)) = +\infty$, for $k > 0$ and $0 < c < 1$,

(b) $\lim_{u\rightarrow 0} f_{q(U)}(q(u), 1/q'(u)) = 0$, for $k > 0$ and $c > 1$,

(c) $\lim_{u\rightarrow 1} f_{q(U)}(q(u), 1/q'(u)) = 0$, for $k > 0$ and $c > 0$. □

See Figure 1 and Figure 2 for examples of the limit conditions for Type III and Type XII distributions, respectively.

A corollary to Property 2.1 is stated as follows

Corollary 2.1 The derivative of the cdf $F_{q(U)}(q(u), u)$ in (10) is the pdf $f_{q(U)}(q(u), 1/q'(u))$ in (9).

Proof: It follows from $x = q(u)$ and $y = u$ in $F_{q(U)}(q(u), u)$ of (10) that $dx = q'(u)du$ and $dy = 1du$. Hence, using the parametric form of the derivative we have $y = dy/dx = 1/q'(u)$ in $f_{q(U)}(q(x,y))$ of (9). Whence, $F'_{q(U)}(q(u),u) = F'_{q(U)}(q(x, dy/dx)) = f_{q(U)}(q(x,y)) = f_{q(U)}(q(u), 1/q'(u))$. Thus, $f_{q(U)}(q(u), 1/q'(u))$ in (9) and $F_{q(U)}(q(u), u)$ in (10) are the pdf and cdf associated with the quantile functions in (5) and (6). □

The moments associated with the Type III and Type XII family of distributions can be determined from

$$E[q(u)^r] = \int_0^1 q(u)^r du = \Gamma[(c+r)/c]\Gamma[k - r/c]/\Gamma[k]. \tag{11}$$

In terms of Type III distributions, the r-th moment exists if $c + r < 0$. For Type XII distributions, the r-th moment exists if $ck > r$. Given that the first $r = 4$ moments exist, the measures of skew (α_1) and kurtosis (α_2) can subsequently be obtained from [21]

$$\begin{aligned} \alpha_1 &= (E[q(u)^3] - 3E[q(u)^2]E[q(u)] + 2(E[q(u)])^3)/ \\ &\quad (E[q(u)^2] - (E[q(u)])^2)^{\frac{3}{2}} && (12) \\ \alpha_2 &= (E[q(u)^4] - 4E[q(u)^3]E[q(u)] - 3(E[q(u)^2])^2 + 12E[q(u)^2] \times \\ &\quad (E[q(u)])^2 - 6(E[q(u)])^4)/(E[q(u)^2] - (E[q(u)])^2)^2. && (13) \end{aligned}$$

Using (11), (12), and (13), the formulae for the mean, variance, skew, and kurtosis for the family of Burr distributions are

$$\begin{aligned} \mu &= (\Gamma[(c+1)/c]\Gamma[k - 1/c])/\Gamma[k] && (14) \\ \sigma^2 &= \Gamma[k]^{-2}(\Gamma[(2+c)/c]\Gamma[k]\Gamma[k-2/c] - \Gamma[1+1/c]^2\Gamma[k-1/c]^2) && (15) \\ \alpha_1 &= \{1/(\Gamma[(2+c)/c]\Gamma[k]\Gamma[k-2/c] - \Gamma[1+1/c]^2\Gamma[k-1/c]^2)\}^{\frac{3}{2}} \times \\ &\quad \{\Gamma[(3+c)/c]\Gamma[k]^2\Gamma[k-3/c] - \\ &\quad c^{-2}(6\Gamma[1/c]\Gamma[2/c]\Gamma[k]\Gamma[k-2/c]\Gamma[k-1/c]) + \\ &\quad 2\Gamma[1+1/c]^3\Gamma[k-1/c]^3\} && (16) \\ \alpha_2 &= \{\Gamma[(4+c)/c]\Gamma[k]^3\Gamma[k-4/c] - \\ &\quad c^{-3}(3\Gamma[k-1/c](4c\Gamma[1/c]\Gamma[3/c]\Gamma[k]^2\Gamma[k-3/c] - \\ &\quad 4\Gamma[1/c]^2\Gamma[2/c]\Gamma[k]\Gamma[k-2/c]\Gamma[k-1/c] + c^3\Gamma[1+1/c]^4 \times \\ &\quad \Gamma[k-1/c]^3))\}/ \\ &\quad \{\Gamma[(2+c)/c]\Gamma[k]\Gamma[k-2/c] - \Gamma[1+1/c]^2\Gamma[k-1/c]^2\}^2 - 3. && (17) \end{aligned}$$

Thus, given specified values of α_1 and α_2 associated with either Type III or Type XII distributions, equations (16) and (17) are used to simultaneously solve for the shape parameters of c and k. The solved values of c and k can then be used to evaluate (14) and (15) to determine the mean and variance. Appendix A gives solved values of c and k for various combinations of α_1 and α_2. Provided in Appendix B are (approximate) lower boundary (LB) and upper boundary (UB) values of α_2 for the given values of α_1 in Appendix A.

The other measures of central tendency considered are the mode(s), median, and trimmed mean. Specifically, if a mode associated with the pdf in (9) exists then it is located at $f_{(U)}(q(\tilde{u}), 1/q'(\tilde{u}))$, where $u = \tilde{u}$ is a critical number that solves $dy/du = d(1/(q'(u))/du = 0$ and maximizes (either locally or globally) $y = 1/q'(\tilde{u})$ at $x = q(\tilde{u})$. Type III distributions are unimodal if $c < 0$, $k > -1/c$, and $\tilde{u}_{\mathtt{III}} = ((1 + ck)/(c + ck))^k$. Type XII distributions are unimodal if $c > 1$, $k > 0$, and $\tilde{u}_{\mathtt{XII}} = 1 - ((1 + ck)/(c + ck))^k$. Substituting $\tilde{u}_{\mathtt{III}}$ into (5) and $\tilde{u}_{\mathtt{XII}}$ into (6) and simplifying locates the mode for either type of distribution at the point where $q(\tilde{u}) = ((c - 1)/(ck + 1))^{\frac{1}{c}}$.

The median associated with (9) is located at $q(u = 0.50) = (2^{\frac{1}{k}} - 1)^{\frac{1}{c}}$. This can be shown by letting $x_{0.50} = q(u)$ and $y_{0.50} = F_U(u) = Pr(U \leq u)$ denote the coordinates of the cdf in (10) that are associated with the 50th percentile. In general, we must have $u = 0.50$ such that $y_{0.50} = 0.50 = F_U(0.50) = Pr(U \leq 0.50)$ holds in (10) for the regular uniform distribution. As such, the limit of the quantile function locates the median at $\lim_{u \to 0.50} q(u) = (2^{\frac{1}{k}} - 1)^{\frac{1}{c}}$.

The 100γ percent symmetric trimmed mean (TM) can be obtained from (11) (with $r = 1$) and from the definition of a TM as ([2] p. 401)

$$\mathrm{TM} = (1 - 2\gamma)^{-1} \int_{\gamma}^{1-\gamma} q(u)du. \tag{18}$$

As $\gamma \to 0$ the TM will converge to the mean in (14). Conversely, as $\gamma \to 0.50$ then the TM will converge to the median.

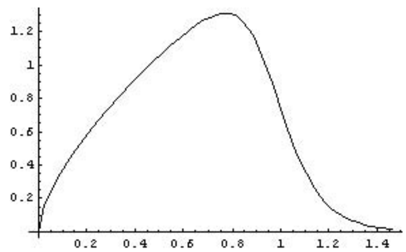
To demonstrate the use of the methodology above, presented in Figure 1 and Figure 2 are pdfs and cdfs from the Burr Type III and Type XII families, respectively. The values and graphs in these figures were obtained using various *Mathematica* [44] functions. More specifically, the values of c and k were obtained by solving equations (16) and (17) using the function FindRoot and the graphs of the pdfs and cdfs were determined using equations (9) and (10) and the function ParametricPlot. The modes were determined by evaluating $q(\tilde{u}) = ((c - 1)/(ck + 1))^{\frac{1}{c}}$ for the solved values of c and k. The heights of the pdfs were subsequently obtained by solving for the values of $\tilde{u}$ that yielded the modes and then evaluating for the heights using $1/q'(\tilde{u})$ in (9). The values that yielded the probabilities of obtaining values of $q(u)$ in the upper 5% of the tail regions were determined by evaluating the quantile functions in (5) and (6) for $u = 0.95$.

3 Fitting Burr Distributions to Data

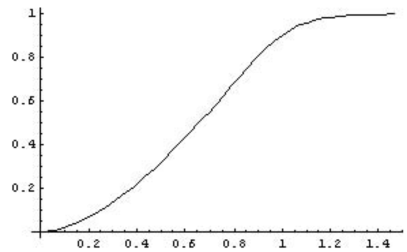
Presented in Figure 3 are Burr Type XII pdfs superimposed on histograms of circumference measures (in centimeters) taken from the abdomen, chest, forearm, and knee of 252 adult males. Inspection of Figure 3 indicates that

$\mu = 0.6419$
$\sigma^2 = 0.0806$
$\alpha_1 = 0$
$\alpha_2 = -0.4$

$c = -11.2519$
$k = 0.1463$



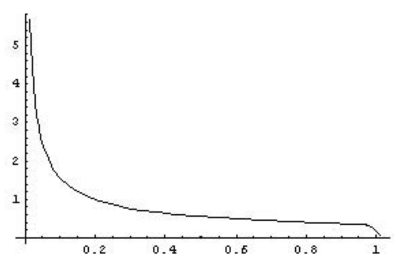
(A) Height = 1.3107
Mode = 0.7699



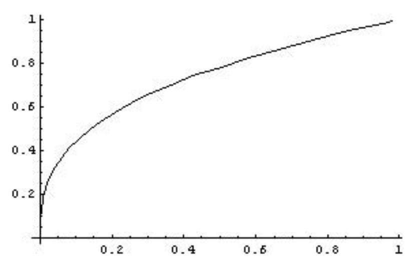
$\Pr\{q(U) \geq q(u) = 1.0802\} = 0.05$

$\mu = 0.2607$
$\sigma^2 = 0.0820$
$\alpha_1 = 1.0$
$\alpha_2 = -0.22$

$c = -97.7167$
$k = 0.0036$



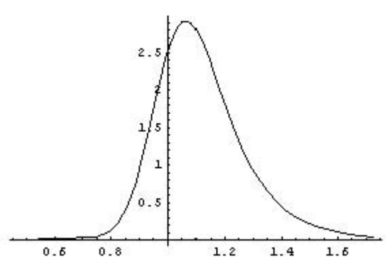
(B) Height = N/A
Mode = N/A



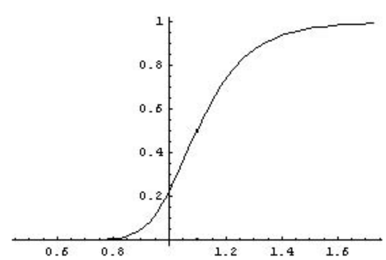
$\Pr\{q(U) \geq q(u) = 0.8646\} = 0.05$

$\mu = 1.1244$
$\sigma^2 = 0.0292$
$\alpha_1 = 1.4$
$\alpha_2 = 5.2$

$c = -10.3939$
$k = 2.1775$



(C) Height = 2.9217
Mode = 1.0636

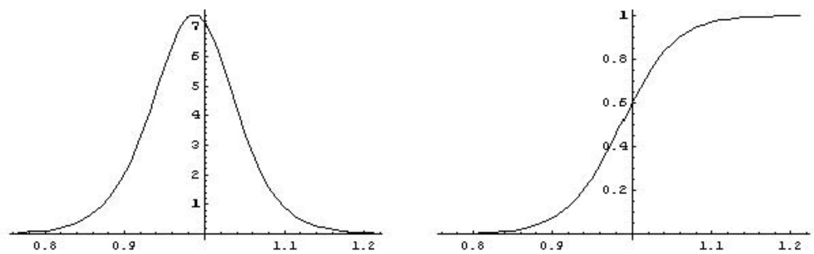


$\Pr\{q(U) \geq q(u) = 1.4326\} = 0.05$

Figure 1: Examples of c and k parameters for Type III distributions and their associated pdfs and cdfs. These distributions can also be empirically simulated by using the quantile function in Equation (5).

$\mu = 0.9858$
$\sigma^2 = 0.0036$
$\alpha_1 = 0$
$\alpha_2 = 1$

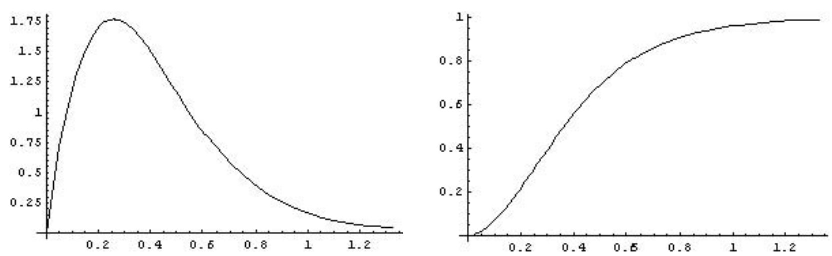
$c = 27.0729$
$k = 1.3257$



(A) Height = 7.411 $\Pr\{q(U) \geq q(u) = 1.0826\} = 0.05$
Mode = 0.9873

$\mu = 0.4186$
$\sigma^2 = 0.0772$
$\alpha_1 = 1.5$
$\alpha_2 = 4.5$

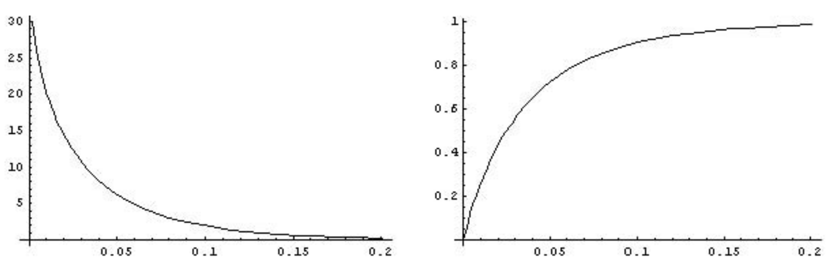
$c = 1.8149$
$k = 4.6909$



(B) Height = 1.769 $\Pr\{q(U) \geq q(u) = 0.9401\} = 0.05$
Mode = 0.2582

$\mu = 0.0408$
$\sigma^2 = 0.0023$
$\alpha_1 = 2.75$
$\alpha_2 = 13$

$c = 0.9066$
$k = 20.0039$



(C) Height = N/A $\Pr\{q(U) \geq q(u) = 0.1339\} = 0.05$
Mode = N/A

Figure 2: Examples of c and k parameters for Type XII distributions and their associated pdfs and cdfs. These distributions can also be empirically simulated by using the quantile function in Equation (6).

the Burr pdfs provide good approximations to the empirical data. We note that to fit the Burr pdfs to the data, the following transformation had to be imposed on the quantile function $q(u) : (M\sigma - \mu S + Sq(u))/\sigma$. The values of the means (M, μ) and standard deviations (S, σ) for the data and Burr pdfs are given in Figure 3.

One way of determining how well a Burr pdf models a set of data is to compute a chi-square goodness of fit statistic. For example, listed in Table 1 below are the cumulative percentages and class intervals based on the Burr pdf for the chest data in Panel B of Figure 3. The asymptotic value of $p = 0.290$ indicates that the Burr pdf provides a good fit to the data. We note that the degrees of freedom for this test were computed as $df = 5 = 10$(class intervals) -4(parameter estimates) -1(sample size). Further, the Burr TMs given in Table 2 also indicate a good fit as the TMs are all within the 95% bootstrap confidence intervals based on the data. The confidence intervals are based on 25000 bootstrap samples.

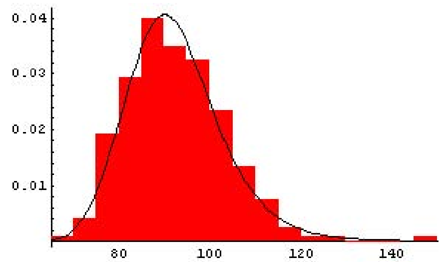
Cumulative %	Burr Class Intervals	Observed Freq	Expected Freq
5	< 88.52	10	12.6
10	88.52 – 90.67	14	12.6
15	90.67 – 92.26	11	12.6
30	92.26 – 95.90	38	37.8
50	95.90 – 100.02	57	50.4
70	100.02 – 104.52	48	50.4
85	104.52 – 109.35	38	37.8
90	109.35 – 111.85	9	12.6
95	111.85 – 115.83	13	12.6
100	> 115.83	13	12.6
$\chi^2 = 2.938$	$\Pr\{\chi_5^2 \leq 2.938\} = 0.292$	$n = 252$	

Table 1: Observed and expected frequencies and chi-square test based on the Burr Type XII approximation to the chest data in Figure 3.

Empirical Distribution	20% TM	Burr TM
Abdomen	91.808 (91.030, 92.605)	91.721
Chest	100.128 (99.546, 100.736)	100.172
Forearm	28.699 (28.539, 28.845)	28.608
Knee	38.491 (38.320, 38.637)	38.472

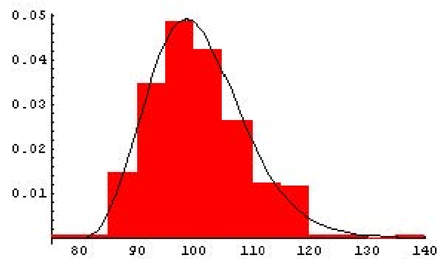
Table 2: Examples of Burr Type XII trimmed means (TMs). Each TM is based on a sample size of $n = 152$ and has a 95% bootstrap confidence interval enclosed in parentheses.

Data	Burr Distribution
$M = 92.5560$	$\mu = 0.793968$
$S = 10.7831$	$\sigma = 0.287478$
$\alpha_1 = 0.833419$	$c = 3.872133$
$\alpha_2 = 2.18074$	$k = 2.264652$



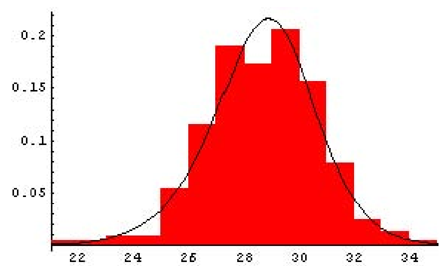
(A) Abdomen

Data	Burr Distribution
$M = 100.824$	$\mu = 0.559304$
$S = 8.43048$	$\sigma = 0.238712$
$\alpha_1 = 0.677492$	$c = 2.867086$
$\alpha_2 = 0.944087$	$k = 4.468442$



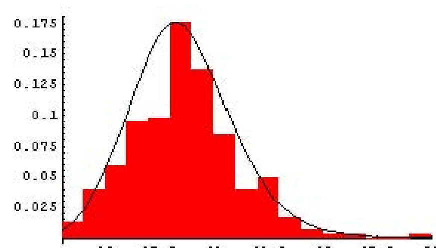
(B) Chest

Data	Burr Distribution
$M = 28.6639$	$\mu = 0.966818$
$S = 2.02069$	$\sigma = 0.0626464$
$\alpha_1 = -0.218025$	$c = 23.543917$
$\alpha_2 = 0.825501$	$k = 1.7765786$



(C) Forearm)

Data	Burr Distribution
$M = 38.5905$	$\mu = 0.781809$
$S = 2.41180$	$\sigma = 0.239507$
$\alpha_1 = 0.513663$	$c = 4.446853$
$\alpha_2 = 1.01687$	$k = 2.576122$



(D) Knee

Figure 3: Examples of Burr Type XII pdfs' approximations to empirical data using measures of circumference (in centimeters) taken from $n = 252$ males. To fit the Burr distributions to the data the quantile functions $q(u)$ were transformed as $(M\sigma - \mu S + Sq(u))/\sigma$.

4 Multivariate Data Generation

The family of Burr distributions we are considering can be extended from univariate to multivariate data generation by specifying T quantile functions $q(u)$ of the form of either (5) or (6). Specifically, let $Z_1, \ldots, Z_T$ denote standard normal variables where the distribution functions and bivariate density function associated with Z_j and Z_k are expressed as

$$\Phi(z_j) = \Pr\{Z_j \le z_j\} = \int_{-\infty}^{z_j} (2\pi)^{-\frac{1}{2}} \exp\{-u_j^2/2\} du_j \tag{19}$$

$$\Phi(z_k) = \Pr\{Z_k \le z_k\} = \int_{-\infty}^{z_k} (2\pi)^{-\frac{1}{2}} \exp\{-u_k^2/2\} du_k \tag{20}$$

$$\begin{aligned} f_{jk} &:= f_{z_j z_k}(z_j, z_k, \rho_{z_j z_k}) = \left(2\pi(1-\rho_{z_j z_k}^2)^{\frac{1}{2}}\right)^{-1} \times \\ &\exp\left\{-\left(2(1-\rho_{z_j z_k}^2)\right)^{-1} (z_j^2 + z_k^2 - 2 z_j z_k \rho_{z_j z_k})\right\}. \end{aligned} \tag{21}$$

Using (19), it follows that a quantile function of the form of either (5) or (6) can be expressed as $q_j(\Phi(z_j))$ because $\Phi(z_j) \sim U[0,1]$. As such, the bivariate correlation between two standardized Burr distributions, denoted as $x_j(q_j(\Phi(z_j)))$ and $x_k(q_k(\Phi(z_k)))$, can be determined as

$$\rho_{x_j(q_j(\Phi(z_j))), x_k(q_k(\Phi(z_k)))} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_j(q_j(\Phi(z_j))) x_k(q_k(\Phi(z_k))) f_{jk} dz_j dz_k \tag{22}$$

for $j \ne k$ and where the correlation $\rho_{z_j z_k}$ in f_{jk} is referred to as an intermediate correlation. Thus, the objective is to use equation (22) to solve for the values of the $T(T-1)/2$ intermediate correlations $\rho_{z_j z_k}$ such that T specified Burr distributions also have a specified correlation matrix.

To generate multivariate Burr distributions, the process begins by assembling the solved intermediate correlations into a $T \times T$ matrix and subsequently factoring this matrix (e.g., a Cholesky factorization). The results from the factorization are then used to produce standard normal deviates that are correlated at the intermediate levels as follows

$$\begin{aligned} &Z_1 = a_{11} V_1 \\ &Z_2 = a_{12} V_1 + a_{22} V_2 \\ &\vdots \\ &Z_j = a_{1j} V_1 + a_{2j} V_2 + \cdots + a_{ij} V_i + \cdots + a_{jj} V_j \\ &\vdots \\ &Z_T = a_{1T} V_1 + a_{2T} V_2 + \cdots + a_{iT} V_i + \cdots + a_{TT} V_T \end{aligned} \tag{23}$$

where $V_1, \ldots, V_T$ are independent standard normal random deviates and where a_{ij} represents the element in the i-th row and j-th column of the matrix associated with the Cholesky factorization.

To generate the regular uniform deviates required for the quantile functions in (5) or (6), it is suggested that the following series expansion for the unit normal cdf [25] be used on the Z_j in (23) as

$$\Phi(z_j) = .5+\varphi(z_j)\{z_j+z_j^3/3+z_j^5/(3\cdot5)+z_j^7/(3\cdot5\cdot7)+z_j^9/(3\cdot5\cdot7\cdot9)+\cdots\} \quad (24)$$

where $\varphi(z_j)$is the standard normal pdf and where the absolute error associated with (24) is less than 8×10^{-16}.

5 Numerical Example and Monte Carlo Simulation

Suppose we desire to generate correlated data associated with the following Type III and Type XII Burr distributions: (1) Panel A in Figure 1, (2) Panel A in Figure 2, and (3) Panel B in Figure 2 with the specified correlation matrix given in Table 3. Figure 4 gives *Mathematica* source code for numerically solving equation (22) for the three intermediate correlations that are given in Table 4. The results of a Cholesky factorization on the intermediate correlation matrix are given in Table 5. These results are then used in (23) to create Z_1, Z_2, and Z_3 with the specified intermediate correlations.

	$q_1(\Phi(Z_1))$	$q_2(\Phi(Z_2))$	$q_3(\Phi(Z_3))$
$q_1(\Phi(Z_1))$	1	0.50	0.60
$q_2(\Phi(Z_2))$	0.50	1	0.70
$q_3(\Phi(Z_3))$	0.60	0.70	1

Table 3: Specified correlations between three Burr distributions. See Figure 1, Panel A (q_1); Figure 2, Panels A and B (q_2 and q_3).

	Z_1	Z_2	Z_3
Z_1	1	0.504372	0.633945
Z_2	0.504372	1	0.736946
Z_3	0.633945	0.736946	1

Table 4: Solved intermediate correlations.

To empirically demonstrate the procedure described above, the three selected Burr distributions in Figures 1 and 2 were simulated in accordance to

```
(* c and k Parameters *)
c1 = -11.25186;
k1 = 0.146295;
c2 = 27.072953;
k2 = 1.325711;
c3 = 1.814856;
k3 = 4.690922;

(* Intermediate Correlations *)
ρ12 = .504372;
ρ13 = .633945;
ρ23 = .736946;

(* Standard Normal CDFs *)
Φ1 = ∫_{-∞}^{z1} Exp[-u1^2/2]/Sqrt[2*Pi] ⅆu1;
Φ2 = ∫_{-∞}^{z2} Exp[-u2^2/2]/Sqrt[2*Pi] ⅆu2;
Φ3 = ∫_{-∞}^{z3} Exp[-u3^2/2]/Sqrt[2*Pi] ⅆu3;

(* Quantile Functions *)
q1 = ((Φ1)^(-1/k1) - 1)^(1/c1);
q2 = ((1 - Φ2)^(-1/k2) - 1)^(1/c2);
q3 = ((1 - Φ3)^(-1/k3) - 1)^(1/c3);

(* Means and Standard Deviations *)
m1 = 0.641872;
m2 = 0.985833;
m3 = 0.418600;
s1 = 0.283920;
s2 = 0.059991;
s3 = 0.277863;

(* Standardized Quantile Functions *)
x1 = (q1 - m1)/s1;
x2 = (q2 - m2)/s2;
x3 = (q3 - m3)/s3;

(* Standard Normal Bivariate PDFs *)
f12 = (Exp[(-1/(2*(1 - ρ12^2)))*((z1^2) - 2*ρ12*z1*z2 + z2^2)])/
  (2*Pi*Sqrt[1 - (ρ12^2)]);
f13 = (Exp[(-1/(2*(1 - ρ13^2)))*((z1^2) - 2*ρ13*z1*z3 + z3^2)])/
  (2*Pi*Sqrt[1 - (ρ13^2)]);
f23 = (Exp[(-1/(2*(1 - ρ23^2)))*((z2^2) - 2*ρ23*z2*z3 + z3^2)])/
  (2*Pi*Sqrt[1 - (ρ23^2)]);

(* Integrals to Compute the Specified Correlations. See Equation 22 *)
int1 = NIntegrate[x1*x2*f12, {z1, -8, 8}, {z2, -8, 8}, Method → Trapezoidal]
int2 = NIntegrate[x1*x3*f13, {z1, -8, 8}, {z3, -8, 8}, Method → Trapezoidal]
int3 = NIntegrate[x2*x3*f23, {z2, -8, 8}, {z3, -8, 8}, Method → Trapezoidal]
```

Figure 4: Mathematica code for computing the intermediate correlations listed in Table 4.

$a_{11} = 1$	$a_{12} = 0.504372$	$a_{13} = 0.633945$
0	$a_{22} = 0.863486$	$a_{23} = 0.483160$
0	0	$a_{33} = 0.603878$

Table 5: Cholesky factorization on the intermediate correlations in Table 4.

the specified correlation matrix in Table 3 using an algorithm coded in Fortran. The algorithm employed the use of subroutines UNI1 and NORMB1 [3] to generate pseudo-random uniform and standard normal deviates. Samples of size $N = 10, 250$, and 1000000 were drawn for each of the three selected distributions using the specified values of c and k given in Figures 1 and 2.

For the samples of size $N = 10$ and 250 the empirical estimates of the correlations, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ were computed using an averaging procedure across 25000 replications and are reported in Tables 6 and 7 along with their root mean squared errors (RMSEs). In terms of the samples of size $N = 1000000$, the statistics were computed directly on these single samples and are reported in Table 8. Inspection of Tables 6-8 indicates that the procedure produced excellent agreement between the empirical estimates and the specified parameters even for samples sizes as small as $N = 10$.

	$q_1(\Phi(Z_1))$	$q_2(\Phi(Z_2))$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
$q_1(\Phi(Z_1))$	1		0.006(1.157)	−0.389(4.200)
$q_2(\Phi(Z_2))$	0.504(0.359)	1	−0.002(1.896)	1.031(7.495)
$q_3(\Phi(Z_3))$	0.601(0.402)	0.702(0.475)	1.500(5.972)	4.700(68.93)

Table 6: Estimated values of correlation, skew ($\hat{\alpha}_1$), and kurtosis($\hat{\alpha}_2$) between three Burr distributions. See Figure 1, Panel A (q_1); Figure 2, Panels A, B (q_2, q_3). The estimates are based on 25,000 samples of size 10. The RMSEs are in parentheses.

	$q_1(\Phi(Z_1))$	$q_2(\Phi(Z_2))$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
$q_1(\Phi(Z_1))$	1		0.001(0.229)	−0.395(0.848)
$q_2(\Phi(Z_2))$	0.500(0.071)	1	0.000(0.377)	1.002(1.487)
$q_3(\Phi(Z_3))$	0.601(0.080)	0.700(0.095)	1.501(1.102)	4.510(12.81)

Table 7: Estimated values of correlation, skew ($\hat{\alpha}_1$), and kurtosis($\hat{\alpha}_2$) between three Burr distributions. See Figure 1, Panel A (q_1); Figure 2, Panels A, B (q_2, q_3). The estimates are based on 25,000 samples of size 250. The RMSEs are in parentheses.

	$q_1(\Phi(Z_1))$	$q_2(\Phi(Z_2))$	$\hat{\alpha}_1$	$\hat{\alpha}_2$
$q_1(\Phi(Z_1))$	1		0	−0.4
$q_2(\Phi(Z_2))$	0.5	1	−0.005	1.013
$q_3(\Phi(Z_3))$	0.6	0.7	1.493	4.444

Table 8: Estimated values of correlation, skew ($\hat{\alpha}_1$), and kurtosis($\hat{\alpha}_2$) between three Burr distributions. See Figure 1, Panel A (q_1); Figure 2, Panels A, B (q_2, q_3). The estimates are based on single samples of size 1,000,000.

6 Comments

Monte Carlo and simulation techniques are widely used in statistical research. Since real-world data sets can often be radically non-normal, it is essential that statisticians have a variety of techniques available for univariate or multivariate non-normal data generation. As mentioned above, the Burr distributions have not been as popular as some other competing methods such as power method polynomials [7, 10, 13, 40] for simulating multivariate non-normal distributions. More specifically, Kotz et al. [22] noted "this [power] method does provide a way of generating bivariate non-normal random variables. Simple extensions of other univariate methods are not available yet" (p. 37). However, the extension of the Burr family of distributions from univariate to multivariate data generation, as presented above, makes this family a viable competitor to the power method because of its simplicity and ease of execution.

Provided in Appendix A are values of the shape parameters c and k to assist the user of the methodology. These values can be used directly or can be used as initial guesses in an equation solver if other shape parameters are desired but not listed in the Tables. It is also important to point out that the solutions to the shape parameters are not unique and that the Type III and Type XII distributions are not mutually exclusive. For example, consider the Type III distribution depicted in Panel C of Figure 1. Given the values of $\alpha_1 = 1.4$ and $\alpha_2 = 5.2$ there also exists a Type XII distribution with these values of skew and kurtosis. The values of the shape parameters are $c = 2.62418$ and $k = 2.57947$.

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Appendix A. Tabled values of skew (α_1), kurtosis (α_2), and the shape parameters c and k for Burr Type III and Type XII distributions. The notation LB and UB indicate lower and upper boundary approximations of kurtosis. See Appendix B for these values.

	$\alpha_1 = 0.00$			$\alpha_1 = 0.25$	
α_2	c	k	α_2	c	k
LB(III)	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**
−1.10	−24.6697	0.043379	−1.10	−23.1436	0.034529
−1.00	−18.1484	0.062932	−1.00	−16.4642	0.051653
−0.900	−15.3988	0.078999	−0.900	−13.8078	0.065363
−0.800	−13.8406	0.093482	−0.800	−12.3253	0.077547
−0.700	−12.8336	0.107125	−0.700	−11.3679	0.088897
−0.600	−12.1341	0.120321	−0.600	−10.6970	0.099767
−0.500	−11.6274	0.133318	−0.500	−10.2021	0.110371
−0.400	−11.2519	0.146295	−0.400	−9.82484	0.120860
−0.300	−10.9717	0.159396	−0.300	−9.53083	0.131347
LB(XII)	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**
−0.200	3.93894	19.8648	−0.200	2.84626	58.5708
−0.100	4.38133	9.28559	−0.100	3.12218	14.1224
0.000	4.87370	6.15784	0.000	3.41714	8.22985
0.100	5.42896	4.64573	0.100	3.73499	5.88534
0.200	6.06515	3.74501	0.200	4.08089	4.61469
0.300	6.80812	3.14027	0.300	4.46168	3.80955
0.400	7.69615	2.70070	0.400	4.88667	3.24748
0.500	8.78846	2.36221	0.500	5.36873	2.82780
0.600	10.1821	2.08956	0.600	5.92622	2.49817
0.700	12.0484	1.86157	0.700	6.58636	2.22859
0.800	14.7238	1.66448	0.800	7.39162	2.00040
0.900	18.9777	1.48851	0.900	8.41285	1.80112
1.00	27.0729	1.32571	1.00	9.77959	1.62162
1.10	50.1060	1.16772	1.10	11.7626	1.45424
1.14	79.8942	1.10318	1.20	15.0647	1.29071
1.18	226.282	1.03572	1.30	22.4992	1.11650
1.185	299.147	1.02695	1.31	23.8764	1.09721
1.186	319.912	1.02519	1.32	25.5078	1.07728
1.187	343.946	1.02342	1.33	27.4835	1.05652
1.188	372.019	1.02164	1.34	29.9459	1.03474
1.189	405.158	1.01986	1.35	33.1373	1.01159
UB(XII)	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**
1.21	−16.1163	0.549522	1.36	−10.1125	0.434188
1.22	−16.5021	0.559659	1.38	−10.2509	0.445136
1.23	−16.9469	0.570742	1.40	−10.4093	0.457149
1.24	−17.4699	0.583025	1.42	−10.5938	0.470512
1.25	−18.1019	0.596899	1.44	−10.8137	0.485656
1.26	−18.8975	0.613024	1.46	−11.0846	0.503296
1.27	−19.9674	0.632684	1.48	−11.4371	0.524778
1.28	−21.6129	0.659161	1.50	−11.9460	0.553312
1.29	−29.1338	0.742939	1.52	−12.9726	0.603780
UB(III)	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**

$\alpha_1 = 0.50$			$\alpha_1 = 0.75$		
α_2	c	k	α_2	c	k
LB(III)	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**
−1.00	−46.0059	0.012682	−0.600	−19.5672	0.024130
−0.900	−18.6204	0.033295	−0.500	−14.1760	0.035118
−0.800	−14.2225	0.046148	−0.400	−11.9317	0.043850
−0.700	−12.1994	0.056789	−0.300	−10.6498	0.051497
−0.600	−11.0003	0.066322	−0.200	−9.80584	0.058498
−0.500	−10.1968	0.075201	−0.100	−9.20256	0.065078
−0.400	−9.61780	0.083673	0.000	−8.74752	0.071368
−0.300	−9.17993	0.091888	0.100	−8.39106	0.077457
−0.200	−8.83754	0.099951	0.200	−8.10388	0.083407
−0.100	−8.56321	0.107940	0.300	−7.86753	0.089265
0.000	−8.33949	0.115918	0.400	−7.66971	0.095068
LB(XII)	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**
0.050	2.25567	93.1850	0.500	1.847858	260.250
0.100	2.34793	28.9670	0.600	1.97615	23.1788
0.200	2.53777	12.5235	0.700	2.10690	12.4888
0.250	2.63565	9.84825	0.800	2.24057	8.69337
0.300	2.73572	8.15116	0.900	2.37770	6.73989
0.400	2.94329	6.11386	1.00	2.51898	5.54297
0.500	3.16241	4.92667	1.10	2.66525	4.72961
0.600	3.39555	4.14290	1.20	2.81751	4.13709
0.700	3.64589	3.58172	1.30	2.97700	3.68309
0.750	3.77874	3.35546	1.40	3.14523	3.32145
0.800	3.91752	3.15591	1.50	3.32409	3.02425
0.900	4.21588	2.81820	1.60	3.51594	2.77359
1.00	4.54829	2.54063	1.70	3.72382	2.55738
1.10	4.92495	2.30552	1.80	3.95173	2.36713
1.20	5.36067	2.10099	1.90	4.20512	2.19660
1.25	5.60729	2.00743	2.00	4.49162	2.04099
1.30	5.87801	1.91851	2.10	4.82254	1.89637
1.40	6.51378	1.75152	2.20	5.21560	1.75924
1.50	7.33386	1.59425	2.30	5.70106	1.62599
1.60	8.47384	1.44049	2.40	6.33749	1.49206
1.70	10.2892	1.28064	2.50	7.26553	1.34941
1.75	11.7908	1.19174	2.60	9.04255	1.17248
1.80	14.4575	1.08486	2.64	14.6528	0.920288
UB(XII)	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**
1.85	−17.0877	0.975710	2.65	−12.1441	0.996805
1.90	−15.6339	0.921065	2.75	−11.1867	0.915262
1.95	−14.3478	0.865594	2.80	−10.7539	0.875494
2.00	−13.1773	0.807813	2.85	−10.3452	0.836030
2.02	−12.7292	0.783482	2.90	−9.95588	0.796518
2.04	−12.2859	0.758017	2.95	−9.58086	0.756484
2.05	−12.0640	0.744706	3.00	−9.21402	0.715209
2.06	−11.8405	0.730889	3.05	−8.84631	0.671434
2.08	−11.3818	0.701146	3.10	−8.45981	0.622385
2.10	−10.8864	0.666678	3.15	−7.99014	0.557593
2.12	−10.2715	0.619925	3.16	−7.85578	0.537766
UB(III)	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**

$\alpha_1 = 1.00$			$\alpha_1 = 1.25$		
α_2	c	k	α_2	c	k
LB(III)	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**
−0.220	−97.7167	0.003607	0.400	−77.8369	0.003615
−0.200	−37.5921	0.009476	0.500	−18.2498	0.016146
−0.150	−21.6247	0.016903	0.600	−13.5111	0.022759
−0.100	−16.9845	0.022060	0.700	−11.4263	0.028006
0.000	−13.0680	0.030050	0.800	−10.2063	0.032551
0.100	−11.2171	0.036584	0.900	−9.39039	0.036657
0.200	−10.1028	0.042342	1.00	−8.80014	0.040457
0.300	−9.34648	0.047612	1.10	−8.35022	0.044033
0.400	−8.79449	0.052544	1.20	−7.99422	0.047435
0.500	−8.37147	0.057231	1.30	−7.70452	0.050701
0.600	−8.03568	0.061736	1.40	−7.46358	0.053857
0.700	−7.76194	0.066102	1.50	−7.25964	0.056924
0.800	−7.53413	0.070362	1.60	−7.08452	0.059916
0.900	−7.34135	0.074540	1.70	−6.93234	0.062848
1.00	−7.17600	0.078658	1.80	−6.79875	0.065728
1.10	−7.03257	0.082732	1.90	−6.68046	0.068567
1.15	−6.96775	0.084757	2.00	−6.57492	0.071371
LB(XII)	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**
1.16	1.56469	3513.13	2.05	1.37086	319.543
1.20	1.60059	75.1183	2.10	1.40279	62.8350
1.40	1.78118	13.5242	2.25	1.49820	19.1527
1.60	1.96453	7.78095	2.50	1.65651	9.34831
1.80	2.15238	5.59624	2.75	1.81495	6.40101
2.00	2.34709	4.42865	3.00	1.97511	4.96274
2.20	2.55183	3.68996	3.25	2.13898	4.09786
2.40	2.77083	3.17099	3.50	2.30914	3.51055
2.60	3.00996	2.77837	3.75	2.48885	3.07761
2.80	3.27769	2.46377	4.00	2.68248	2.73830
3.00	3.58715	2.19903	4.25	2.89620	2.45885
3.20	3.96069	1.96562	4.50	3.13933	2.21837
3.40	4.44205	1.74877	4.75	3.42752	2.00226
3.60	5.13985	1.53136	5.00	3.79092	1.79800
3.80	6.54824	1.26600	5.25	4.30327	1.58963
3.85	7.50529	1.15231	5.50	5.29664	1.32907
UB(XII)	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**
3.90	−9.28606	0.973971	5.55	−7.84676	0.993742
4.00	−8.91221	0.914589	5.60	−7.75894	0.971292
4.10	−8.56773	0.858028	5.75	−7.51053	0.907459
4.20	−8.24735	0.803609	6.00	−7.13903	0.810745
4.30	−7.94610	0.750605	6.25	−6.80899	0.722917
4.40	−7.65883	0.698120	6.50	−6.50828	0.640466
4.50	−7.37913	0.644846	6.75	−6.22275	0.558828
4.60	−7.09644	0.588318	7.00	−5.91943	0.465971
4.70	−6.78186	0.521220	7.05	−5.84426	0.441358
4.75	−6.56210	0.470669	7.10	−5.74138	0.406026
4.76	−6.33284	0.412921	7.11	−5.70470	0.392824
UB(III)	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**

$\alpha_1 = 1.50$			$\alpha_1 = 1.75$		
α_2	c	k	α_2	c	k
LB(III)	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**
1.20	−28.4928	0.008116	2.10	−31.5356	0.005988
1.40	−12.7255	0.019623	2.25	−14.6230	0.013607
1.60	−9.94774	0.026856	2.50	−10.1955	0.021074
1.80	−8.66271	0.032764	2.75	−8.59534	0.026733
2.00	−7.89914	0.037968	3.00	−7.72986	0.031559
2.20	−7.38556	0.042725	3.25	−7.17664	0.035886
2.40	−7.01321	0.047172	3.50	−6.78822	0.039872
2.60	−6.72926	0.051392	3.75	−6.49845	0.043611
2.80	−6.50468	0.055441	4.00	−6.27288	0.047162
3.00	−6.32208	0.059357	4.25	−6.09166	0.050564
3.10	−6.24290	0.061275	4.40	−5.99889	0.052546
LB(XII)	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**
3.15	1.21895	289.115	4.50	1.10967	129.675
3.25	1.26480	43.0357	4.75	1.19231	22.3478
3.50	1.37774	14.5521	5.00	1.27275	12.7864
3.75	1.48863	9.12892	5.25	1.35120	9.19642
4.00	1.59801	6.81583	5.50	1.42794	7.30711
4.25	1.70650	5.52348	5.75	1.50325	6.13589
4.50	1.81486	4.69092	6.00	1.57743	5.33461
4.75	1.92391	4.10410	6.50	1.72372	4.29890
5.00	2.03463	3.66358	7.00	1.86959	3.64658
5.50	2.26565	3.03351	7.50	2.01814	3.18718
6.00	2.51961	2.58775	8.00	2.17302	2.83727
6.25	2.66066	2.40412	8.50	2.33892	2.55407
6.50	2.81562	2.23726	9.00	2.52255	2.31265
6.75	2.98995	2.08191	9.50	2.73486	2.09622
7.00	3.19243	1.93327	10.00	2.99698	1.89054
7.25	3.43899	1.78589	10.25	3.16056	1.78637
7.50	3.76445	1.63134	10.50	3.36294	1.67636
7.75	4.28226	1.44729	10.75	3.63962	1.55161
UB(XII)	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**
7.80	−6.84565	0.996351	11.00	−6.06218	0.949150
7.90	−6.76206	0.964324	11.25	−5.96551	0.898282
8.00	−6.68187	0.933769	11.50	−5.87537	0.851632
8.25	−6.49480	0.863015	11.75	−5.79105	0.808602
8.50	−6.32445	0.799035	12.00	−5.71197	0.768698
8.75	−6.16819	0.740504	12.50	−5.56748	0.696660
9.00	−6.02379	0.686324	13.00	−5.43833	0.632844
9.25	−5.88921	0.635526	13.50	−5.32159	0.575179
9.50	−5.76252	0.587174	14.00	−5.21463	0.521893
9.75	−5.64158	0.540229	14.50	−5.11483	0.471196
10.00	−5.52356	0.493250	15.00	−5.01871	0.420622
10.25	−5.40321	0.443457	15.50	−4.91822	0.364132
10.50	−5.26067	0.380088	15.75	−4.85273	0.323226
10.55	−5.21731	0.359173	15.80	−4.82985	0.307473
UB(III)	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**

$\alpha_1 = 2.00$			$\alpha_1 = 2.25$		
α_2	c	k	α_2	c	k
LB(III)	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**
3.15	−32.0866	0.004883	4.35	−30.0285	0.004390
3.25	−17.3387	0.009324	4.50	−15.2841	0.008986
3.50	−10.9929	0.015781	4.75	−10.7165	0.013622
3.75	−9.03971	0.020413	5.00	−9.00243	0.017116
4.00	−8.03115	0.024275	5.50	−7.45386	0.022677
4.25	−7.40078	0.027685	6.00	−6.70323	0.027258
4.50	−6.96399	0.030789	6.50	−6.24933	0.031275
4.75	−6.64091	0.033670	6.75	−6.08207	0.033133
5.00	−6.39089	0.036377	7.00	−5.94131	0.034912
5.25	−6.19087	0.038946	7.25	−5.82098	0.036622
5.50	−6.02670	0.041401	7.50	−5.71677	0.038272
5.75	−5.88922	0.043761	7.75	−5.62553	0.039869
LB(XII)	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**
6.25	1.06154	33.5651	8.00	0.964341	49.5091
6.50	1.12124	17.5502	9.00	1.13706	10.0656
6.75	1.17922	12.2017	10.00	1.29191	6.22760
7.00	1.23560	9.51942	11.00	1.43359	4.74548
7.50	1.34414	6.82032	12.00	1.56643	3.93934
8.00	1.44800	5.45282	13.00	1.69422	3.41857
9.00	1.64627	4.04593	14.00	1.82042	3.04406
10.00	1.83962	3.29892	15.00	1.94843	2.75358
11.00	2.03757	2.81150	16.00	2.08202	2.51464
12.00	2.25179	2.44854	17.00	2.22612	2.30797
13.00	2.50080	2.14834	18.00	2.38824	2.12023
14.00	2.82573	1.87044	19.00	2.58223	1.93965
14.50	3.05263	1.72397	19.90	2.81100	1.77000
UB(XII)	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**
14.75	−5.64482	0.993900	20.00	−5.27064	0.995077
15.00	−5.59150	0.955744	22.00	−5.07179	0.820197
16.00	−5.40343	0.826892	24.00	−4.91989	0.697793
17.00	−5.24779	0.725734	26.00	−4.80024	0.606045
18.00	−5.11667	0.643254	28.00	−4.70356	0.533685
19.00	−5.00445	0.573819	30.00	−4.62371	0.474168
20.00	−4.90688	0.513603	32.00	−4.55635	0.423269
21.00	−4.82057	0.459713	34.00	−4.49822	0.377847
22.00	−4.74246	0.409513	36.00	−4.44643	0.334838
23.00	−4.66879	0.359467	38.00	−4.39683	0.288513
24.00	−4.58814	0.297340	39.00	−4.36660	0.253827
24.20	−4.55908	0.270118	39.10	−4.36154	0.246848
UB(III)	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**

	$\alpha_1 = 2.50$			$\alpha_1 = 2.75$	
α_2	c	k	α_2	c	k
LB(III)	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**
5.75	−20.4751	0.005554	7.25	−19.4164	0.005038
6.00	−12.1101	0.009942	7.50	−12.2294	0.008414
6.25	−9.76591	0.012969	7.75	−9.95846	0.010811
6.50	−8.57999	0.015449	8.00	−8.77172	0.012787
6.75	−7.84497	0.017610	8.50	−7.50321	0.016065
7.00	−7.33785	0.019555	9.00	−6.81567	0.018815
7.50	−6.67457	0.023005	9.50	−6.37617	0.021234
7.75	−6.44350	0.024568	10.00	−6.06772	0.023419
8.00	−6.25397	0.026048	10.25	−5.94499	0.024442
8.25	−6.09531	0.027457	10.50	−5.83769	0.025426
8.50	−5.96026	0.028805	10.75	−5.74299	0.026375
8.75	−5.84373	0.030100	11.00	−5.65869	0.027292
9.00	−5.74200	0.031347	11.25	−5.58310	0.028179
9.25	−5.65232	0.032552	11.50	−5.51489	0.029040
9.50	−5.57257	0.033719	11.75	−5.45299	0.029876
9.75	−5.50115	0.034851	12.00	−5.39652	0.030690
LB(XII)	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**
10.00	0.888728	73.3269	12.25	0.827666	114.505
10.25	0.923385	31.7492	12.50	0.854700	42.9179
10.50	0.957042	20.7510	12.75	0.880999	27.0076
11.00	1.02153	12.7285	13.00	0.906591	20.0039
12.00	1.14045	7.70136	14.00	1.00244	10.4735
13.00	1.24819	5.81321	15.00	1.08911	7.51456
14.00	1.34720	4.80981	20.00	1.43178	3.86055
15.00	1.43957	4.17793	25.00	1.70083	2.92542
20.00	1.85511	2.74960	30.00	1.95505	2.42521
25.00	2.30914	2.08508	35.00	2.24103	2.05949
26.00	2.42577	1.97165	36.00	2.30882	1.99133
26.50	2.49130	1.91432	36.25	2.32672	1.97424
26.75	2.52650	1.88519	36.50	2.34506	1.95710
27.00	2.56367	1.85559	36.75	2.36388	1.93988
UB(XII)	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**
27.50	−4.95742	0.968543	37.00	−4.74999	0.993670
27.75	−4.94319	0.952446	38.00	−4.72001	0.952112
28.00	−4.92933	0.936953	39.00	−4.69216	0.914749
29.00	−4.87733	0.880358	40.00	−4.66623	0.880957
30.00	−4.83026	0.831055	45.00	−4.55955	0.750753
35.00	−4.64941	0.655365	50.00	−4.48057	0.661829
40.00	−4.52768	0.545758	75.00	−4.27484	0.448076
50.00	−4.37525	0.411441	100.00	−4.18834	0.358616
60.00	−4.28335	0.325092	125.00	−4.14128	0.305665
70.00	−4.21722	0.246717	150.00	−4.11167	0.267360
71.00	−4.21007	0.234738	175.00	−4.09087	0.233401
71.25	−4.20799	0.230845	180.00	−4.08728	0.225938
71.50	−4.20560	0.225994	185.00	−4.08373	0.217403
71.75	−4.20192	0.217432	190.00	−4.07987	0.205368
UB(III)	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**

$\alpha_1 = 3.00$			$\alpha_1 = 3.25$		
α_2	c	k	α_2	c	k
LB(III)	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**
8.75	−45.6292	0.001809	10.50	−54.0743	0.001330
9.00	−14.8737	0.005821	10.75	−15.6863	0.004788
9.25	−11.2192	0.008055	11.00	−11.7330	0.006655
9.50	−9.58007	0.009808	11.25	−9.97315	0.008113
9.75	−8.61484	0.011303	11.50	−8.93898	0.009354
10.00	−7.96754	0.012631	11.75	−8.24591	0.010453
11.00	−6.62622	0.016971	12.00	−7.74378	0.011451
12.00	−6.00926	0.020432	13.00	−6.60393	0.014798
13.00	−5.64564	0.023386	14.00	−6.03400	0.017518
14.00	−5.40276	0.025992	15.00	−5.68480	0.019855
14.25	−5.35392	0.026600	16.00	−5.44620	0.021924
14.50	−5.30871	0.027194	17.00	−5.27158	0.023789
14.60	−5.29155	0.027428	17.25	−5.23490	0.024228
14.65	−5.28316	0.027543	17.45	−5.20719	0.024573
LB(XII)	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**
14.75	0.776878	212.175	17.50	0.733666	859.874
15.00	0.798279	59.9268	17.75	0.750845	90.1411
15.25	0.819140	35.6489	18.00	0.767628	48.5952
15.50	0.839482	25.7367	19.00	0.831033	18.3024
15.75	0.859320	20.3492	20.00	0.889009	11.9600
16.00	0.878674	16.9631	25.00	1.11807	5.46711
17.00	0.951574	10.6293	30.00	1.28306	4.10010
18.00	1.01801	8.07165	35.00	1.41343	3.47350
19.00	1.07891	6.68263	40.00	1.52364	3.09679
20.00	1.13507	5.80606	45.00	1.62142	2.83572
25.00	1.36584	3.90646	50.00	1.71138	2.63809
30.00	1.54791	3.17900	55.00	1.79662	2.47903
35.00	1.70738	2.76152	60.00	1.87949	2.34496
40.00	1.85839	2.47092	65.00	1.96209	2.22754
45.00	2.01162	2.24221	70.00	2.04658	2.12111
50.00	2.17971	2.04310	71.00	2.06393	2.10074
50.25	2.18884	2.03348	72.00	2.08148	2.08061
50.50	2.19807	2.02388	72.25	2.08590	2.07560
50.75	2.20741	2.01428	72.50	2.09034	2.07062
UB(XII)	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**
52.00	−4.55316	0.974223	73.00	−4.41295	0.994085
60.00	−4.45332	0.820405	80.00	−4.36640	0.906302
80.00	−4.30949	0.628755	100.00	−4.27629	0.754437
100.00	−4.23335	0.536875	200.00	−4.12214	0.532845
200.00	−4.10249	0.384996	300.00	−4.07798	0.474732
300.00	−4.06506	0.340088	400.00	−4.05720	0.447751
400.00	−4.04752	0.318016	500.00	−4.04514	0.432137
500.00	−4.03739	0.304750	600.00	−4.03728	0.421948
600.00	−4.03080	0.295846	700.00	−4.03174	0.414773
689.00	−4.02662	0.290052	800.00	−4.02764	0.409445
689.50	−4.02660	0.290024	900.00	−4.02447	0.405331
689.75	−4.02659	0.290010	925.00	−4.02379	0.404445
UB(III)	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**

$\alpha_1 = 3.50$			$\alpha_1 = 3.75$		
α_2	c	k	α_2	c	k
LB(III)	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**
12.50	−23.7027	0.002713	14.50	−29.5681	0.001912
12.75	−14.1411	0.004720	14.75	−15.4794	0.003779
13.00	−11.3078	0.006108	15.00	−12.0814	0.004998
14.00	−7.81410	0.009903	16.00	−8.13445	0.008254
15.00	−6.70154	0.012617	17.00	−6.91542	0.010557
16.00	−6.12757	0.014837	18.00	−6.29140	0.012432
17.00	−5.77040	0.016749	19.00	−5.90446	0.014044
18.00	−5.52405	0.018442	20.00	−5.63812	0.015470
19.00	−5.34261	0.019969	21.00	−5.44223	0.016755
20.00	−5.20276	0.021363	22.00	−5.29137	0.017926
20.25	−5.17261	0.021694	23.00	−5.17119	0.019005
20.50	−5.14406	0.022018	23.50	−5.11973	0.019514
LB(XII)	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**
20.60	0.701878	808.654	24.00	0.674926	569.488
20.75	0.710226	161.710	25.00	0.718806	36.3506
21.00	0.723905	70.5851	50.00	1.26562	3.78336
25.00	0.908816	9.12267	75.00	1.48680	2.96696
30.00	1.07568	5.51844	100.00	1.62767	2.63127
40.00	1.29859	3.79613	125.00	1.73298	2.43347
50.00	1.45178	3.18423	150.00	1.81845	2.29700
60.00	1.57164	2.84653	160.00	1.84888	2.25268
70.00	1.67274	2.62143	170.00	1.87766	2.21259
80.00	1.76238	2.45456	175.00	1.89151	2.19390
90.00	1.84483	2.32191	176.00	1.89424	2.19026
100.00	1.92302	2.21094	177.00	1.89695	2.18665
105.00	1.96125	2.16114	178.00	1.89966	2.18307
108.00	1.98407	2.13267	179.00	1.90235	2.17952
108.25	1.98597	2.13034	179.25	1.90302	2.17864
108.50	1.98787	2.12802	179.50	1.90369	2.17776
108.75	1.98977	2.12570	179.75	1.90436	2.17688
UB(XII)	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**
110.00	−4.28474	0.991039	180.00	−4.18116	0.999867
150.00	−4.19647	0.808913	190.00	−4.17029	0.972037
200.00	−4.14124	0.709432	200.00	−4.16064	0.947985
300.00	−4.09020	0.624878	400.00	−4.07512	0.757307
400.00	−4.06619	0.587075	600.00	−4.04897	0.705631
500.00	−4.05226	0.565622	800.00	−4.03631	0.681527
600.00	−4.04317	0.551791	1000.00	−4.02885	0.667572
700.00	−4.03677	0.542131	1200.00	−4.02394	0.658469
800.00	−4.03202	0.535002	1400.00	−4.02045	0.652062
900.00	−4.02835	0.529524	1600.00	−4.01785	0.647309
1000.00	−4.02544	0.525183	1800.00	−4.01584	0.643641
1100.00	−4.02307	0.521657	2000.00	−4.01423	0.640725
1200.00	−4.02111	0.518738	2025.00	−4.01405	0.640403
1300.00	−4.01945	0.516280	2040.00	−4.01395	0.640213
1400.00	−4.01803	0.514183	2042.00	−4.01394	0.640188
1447.50	−4.01743	0.513291	2042.50	−4.01393	0.640181
UB(III)	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**

$\alpha_1 = 4.00$		
α_2	c	k
LB(III)	**LB(III)**	**LB(III)**
16.60	−101.529	0.000491
16.80	−20.2311	0.002526
17.00	−14.7786	0.003542
18.00	−8.78235	0.006606
19.00	−7.27466	0.008652
20.00	−6.54084	0.010293
21.00	−6.09648	0.011693
22.00	−5.79485	0.012926
23.00	−5.57502	0.014034
24.00	−5.40684	0.015042
25.00	−5.27351	0.015969
26.00	−5.16492	0.016828
27.00	−5.07457	0.017628
27.20	−5.05828	0.017782
27.40	−5.04251	0.017933
27.50	−5.03482	0.018009
LB(XII)	**LB(XII)**	**LB(XII)**
27.60	0.647667	32722.8
28.00	0.662552	109.723
30.00	0.731156	20.5905
40.00	0.971179	6.11723
50.00	1.11608	4.52013
75.00	1.32217	3.39810
100.00	1.44083	3.00308
125.00	1.52235	2.78914
150.00	1.58347	2.65106
175.00	1.63175	2.55293
200.00	1.67125	2.47883
225.00	1.70441	2.42045
250.00	1.73278	2.37303
275.00	1.75743	2.33358
300.00	1.77910	2.30014
325.00	1.79836	2.27137
350.00	1.81561	2.24631
375.00	1.83119	2.22424
UB(XII)	**UB(XII)**	**UB(XII)**
400.00	−4.08421	0.984940
800.00	−4.04064	0.867108
1200.00	−4.02678	0.832512
1600.00	−4.01997	0.815969
2000.00	−4.01592	0.806270
2400.00	−4.01323	0.799897
2800.00	−4.01132	0.795389
3200.00	−4.00989	0.792031
3250.00	−4.00974	0.791671
3264.00	−4.00970	0.791572
3264.50	−4.00970	0.791569
UB(III)	**UB(III)**	**UB(III)**

$\alpha_1 = -0.25$			$\alpha_1 = -0.50$		
α_2	c	k	α_2	c	k
LB(III)	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**
−1.00	−41.9663	0.033085	−0.700	−84.9932	0.021869
−0.900	−24.5954	0.060425	−0.600	−33.8589	0.058811
−0.800	−19.7405	0.080467	−0.500	−25.9341	0.082198
−0.700	−17.3270	0.097890	−0.400	−22.3910	0.101891
−0.600	−15.8768	0.114018	−0.300	−20.3785	0.119836
−0.500	−14.9222	0.129463	−0.200	−19.1124	0.136843
−0.400	−14.2642	0.144580	−0.100	−18.2815	0.153357
−0.300	−13.8028	0.159611	0.000	−17.7367	0.169664
−0.200	−13.4828	0.174741	0.100	−17.3988	0.185971
−0.150	−13.3651	0.182392	0.200	−17.2230	0.202444
−0.130	−13.3250	0.185475	0.250	−17.1876	0.210789
LB(XII)	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**
−0.100	5.12020	64.6724	0.260	7.51393	838.343
0.000	5.79408	12.9526	0.280	7.73410	72.8202
0.100	6.57310	7.35653	0.300	7.96268	38.2642
0.200	7.48958	5.19530	0.400	9.25185	11.5886
0.300	8.59123	4.03854	0.500	10.8544	6.94927
0.400	9.95123	3.31059	0.600	12.9112	5.00913
0.500	11.6877	2.80472	0.700	15.6635	3.93511
0.600	14.0039	2.42835	0.800	19.5617	3.24729
0.700	17.2830	2.13376	0.900	25.5530	2.76481
0.800	22.3432	1.89372	1.00	36.0265	2.40434
0.900	31.3057	1.69149	1.10	59.2480	2.12211
1.00	51.9240	1.51594	1.20	156.272	1.89289
1.10	153.528	1.35912	1.21	186.230	1.87223
UB(XII)	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**
1.15	−20.9537	0.498926	1.25	−26.1573	0.426947
1.20	−23.0118	0.527010	1.30	−27.9140	0.443012
1.25	−26.1089	0.560373	1.40	−32.9874	0.478514
1.30	−31.4994	0.602276	1.50	−42.3520	0.520053
1.35	−44.6031	0.661532	1.60	−65.9365	0.570853
1.39	−129.240	0.755391	1.70	−267.761	0.638272
UB(III)	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**

$\alpha_1 = -0.75$			$\alpha_1 = -1.00$		
α_2	c	k	α_2	c	k
LB(III)	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**	**LB(III)**
−0.190	−155.523	0.016587	0.550	−132.789	0.028322
−0.100	−48.2338	0.056901	0.600	−77.4817	0.050260
−0.050	−40.1647	0.070728	0.700	−53.0043	0.078833
0.000	−35.5305	0.082763	0.800	−44.4236	0.101057
0.100	−30.3361	0.103909	0.900	−40.0834	0.120529
0.200	−27.5226	0.122878	1.00	−37.6223	0.138473
0.300	−25.8290	0.140649	1.10	−36.2191	0.155486
0.400	−24.7800	0.157731	1.20	−35.5139	0.171916
0.500	−24.1561	0.174436	1.30	−35.3323	0.187991
0.600	−23.8445	0.190982	1.40	−35.5903	0.203873
0.700	−23.7853	0.207531	1.50	−36.2578	0.219688
0.800	−23.9491	0.224218	1.60	−37.3436	0.235535
0.850	−24.1111	0.232650	1.70	−38.8935	0.251502
0.875	−24.2123	0.236894	1.75	−39.8680	0.259554
LB(XII)	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**	**LB(XII)**
0.900	13.8614	94.7616	1.78	41.8975	288.685
1.00	17.0148	14.8843	1.80	45.7111	72.9600
1.10	21.5294	8.22657	1.82	50.2162	41.9021
1.20	28.5573	5.74174	1.85	58.7479	25.6708
1.30	41.0655	4.43550	1.90	81.0735	15.6916
1.40	69.7286	3.62522	1.92	95.2023	13.6028
1.50	204.303	3.07001	1.95	128.144	11.3547
UB(XII)	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**	**UB(XII)**
1.55	−34.7421	0.365361	2.00	−47.5399	0.300870
1.60	−36.7275	0.376559	2.10	−52.6239	0.318054
1.70	−41.9981	0.400035	2.20	−59.7567	0.335725
1.80	−50.1276	0.425191	2.25	−64.4719	0.344767
1.90	−64.1740	0.452392	2.30	−70.2945	0.353962
2.00	−94.1038	0.482128	2.40	−87.1901	0.372853
2.05	−126.722	0.498153	2.50	−118.295	0.392499
2.10	−201.663	0.515093	2.60	−193.668	0.413012
2.15	−554.434	0.533090	2.70	−633.180	0.434529
UB(III)	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**	**UB(III)**

c	k	α_1	α_2
−370.586	0.01	−1.00	0.528799
−434.960	0.01	−1.10	0.881051
−516.170	0.01	−1.20	1.27132
−621.380	0.01	−1.30	1.70128
−762.580	0.01	−1.40	2.17292
−961.320	0.01	−1.50	2.68823
−44.7270	0.10	−1.00	0.794885
−52.6910	0.10	−1.10	1.14702
−62.9910	0.10	−1.20	1.53922
−76.7670	0.10	−1.30	1.97308
−96.0540	0.10	−1.40	2.45047
−124.864	0.10	−1.50	2.97331
−172.395	0.10	−1.60	3.54395
−265.280	0.10	−1.70	4.16467
−526.600	0.10	−1.80	4.83829
−981.124	0.10	−1.85	5.19580
−35.48897	0.20	−1.00	1.37555
−43.0768	0.20	−1.10	1.73755
−53.8449	0.20	−1.20	2.14357
−70.2401	0.20	−1.30	2.59526
−98.0950	0.20	−1.40	3.09453
−155.590	0.20	−1.50	3.64354
−342.590	0.20	−1.60	4.24468
−795.800	0.20	−1.65	4.56560
−47.3200	0.30	−1.00	1.99488
−64.6020	0.30	−1.10	2.38207
−98.6650	0.30	−1.20	2.81772
−196.279	0.30	−1.30	3.30369
−137.629	0.40	−1.00	2.53710
−852.500	0.40	−1.10	2.95980

Appendix B. Approximate values of the lower boundary (LB) and upper boundary (UB) of kurtosis for the Type III and Type XII distributions in the tables of Appendix A.

$\alpha_1 = .00$

LB(III)	−1.1999975259810600	−1531.8534192235800	0.0006527558000171320
LB(XII)	−0.2831389179221800	3.6023494257189600	Infinity
UB(XII)	1.20	Infinity	1.00
UB(III)	1.2903020932270200	−27.658759663905600	0.7299814725364500

$\alpha_1 = .25$

LB(III)	−1.1879182994534800	−947.4623508491720	0.0007962750911985100
LB(XII)	−0.2304184621255770	2.7656299151425700	Infinity
UB(XII)	1.3547424358032000	35.02294611136490	1.00
UB(III)	1.5245793593464300	−13.965101416892600	0.6455984580349740

$\alpha_1 = .50$

LB(III)	−1.0177772742865700	−808.3772140762210	0.0007136504223431020
LB(XII)	0.028004457141406600	2.2155977834006700	Infinity
UB(XII)	1.8277601041089300	17.803614853388400	1.00
UB(III)	2.129137179211670	−9.641023962104130	0.5663797972947810

$\alpha_1 = .75$

LB(III)	−0.6950059039032010	−667.5648449484920	0.0006700971502532650
LB(XII)	0.49057487329392900	1.8358807672751200	Infinity
UB(XII)	2.6461715472751800	12.183618755201600	1.00
UB(III)	3.169694470780740	−7.579428249677900	0.4946967105670540

$\alpha_1 = 1.00$

LB(III)	−0.22333305667913900	−690.3619670115560	0.0005096167182371810
LB(XII)	1.1591374468706200	1.5639140222157900	Infinity
UB(XII)	3.8578691537503500	9.453794577377300	1.00
UB(III)	4.763927949783530	−6.405051359585	0.4318300397486560

$\alpha_1 = 1.25$

LB(III)	0.3947123929110210	−475.62575464988600	0.0005901795019437870
LB(XII)	2.037979246154570	1.3631779212065600	Infinity
UB(XII)	5.536310510524430	7.871273342311370	1.00
UB(III)	7.114148592982970	−5.664766013962820	0.37798203243016600

$\alpha_1 = 1.50$

LB(III)	1.1572353047855600	−827.9884050348280	0.00027429248849887800
LB(XII)	3.133024205697250	1.2111243388484900	Infinity
UB(XII)	7.788906706405670	6.85514445182749	1.00
UB(III)	10.575801330332400	−5.165722051604490	0.3325672598826120

$\alpha_1 = 1.75$

LB(III)	2.0632368940042100	−654.1336322729040	0.00028474067199035700
LB(XII)	4.451204300183300	1.093264316483310	Infinity
UB(XII)	10.771284908462000	6.156982123655090	1.00
UB(III)	15.813927862040900	−4.812339958056170	0.2945770585821110

	$\alpha_1 = 2.00$		
LB(III)	3.1117840905154200	−631.2751999800900	0.00024506902859725400
LB(XII)	6.00	1.00	Infinity
UB(XII)	14.711705533161400	5.65324308585641	1.00
UB(III)	24.210744519704400	−4.5522678822382500	0.26287302904354400

	$\alpha_1 = 2.25$		
LB(III)	4.302352538827770	−958.0222463622800	0.00013571709899864300
LB(XII)	7.787122675680750	0.9248122199484170	Infinity
UB(XII)	19.954142797604500	5.275920734424100	1.00
UB(III)	39.16019495539240	−4.354742125310470	0.23636729281034500

	$\alpha_1 = 2.50$		
LB(III)	5.634613629258550	−980.5369429410990	0.00011266067114394700
LB(XII)	9.820305552719350	0.8631745147305940	Infinity
UB(XII)	27.038114932743800	4.984734578248810	1.00
UB(III)	71.77261503501210	−4.200707419395350	0.21410637463456500

	$\alpha_1 = 2.75$		
LB(III)	7.108672950382420	−741.9032648514840	0.0001277659831352040
LB(XII)	12.10717250475590	0.8118803542680700	Infinity
UB(XII)	36.856976109595600	4.7544616520397700	1.00
UB(III)	191.50555613913700	−4.077854737873690	0.1952728383175950

	$\alpha_1 = 3.00$		
LB(III)	8.727749898778970	−764.871419098459	0.00010727293239711600
LB(XII)	14.655160175116700	0.7686155182291970	Infinity
UB(XII)	51.002147158594800	4.5685900000	1.00
UB(III)	689.7543310989990	−4.0200	0.11983600

	$\alpha_1 = 3.25$		
LB(III)	10.478914284126800	−822.7001385732750	0.0000871059778453576
LB(XII)	17.471475424275700	0.7316804290877360	Infinity
UB(XII)	72.600484570407	4.415950000155090	1.00
UB(III)	925.0837323167740	−4.0150	0.08577150

	$\alpha_1 = 3.50$		
LB(III)	12.375945811929000	−753.91130071296800	0.00008361063093333150
LB(XII)	20.563075547423600	0.6998063983865790	Infinity
UB(XII)	108.7746658007350	4.288688202415080	1.00
UB(III)	1447.8404258791200	−4.0100	0.06813000

	$\alpha_1 = 3.75$		
LB(III)	14.413795891067100	−758.9880659741690	0.00007355319868765170
LB(XII)	23.936662792761300	0.672031865058582	Infinity
UB(XII)	179.9557272424410	4.181207576606010	1.00
UB(III)	2042.6955026479100	−4.0075	0.05621950

	$\alpha_1 = 4.00$		
LB(III)	16.592364537808100	−828.0185325255550	0.00006008319629168
LB(XII)	27.59868759438910	0.6476176430783680	Infinity
UB(XII)	378.32930662991700	4.089391793019120	1.00
UB(III)	3264.859278230980	−4.0050	0.04764655

	$\alpha_1 = -.25$		
LB(III)	−1.04635820408785	−824.9152274072160	0.001629763950085830
LB(XII)	−0.12414849605625500	4.970536608883070	Infinity
UB(XII)	1.1152818251350400	220.00042557851500	1.336396505942700
UB(III)	1.396185175186670	−1120.436598051390	0.7998370746964720

	$\alpha_1 = -.50$		
LB(III)	−0.7171725247634950	−1293.457607799400	0.0014199814712539400
LB(XII)	0.25810926173229600	7.493531535546860	Infinity
UB(XII)	1.218028708942510	220.00005597875700	1.8558949823153600
UB(III)	1.7221365709068400	−1826.7756965590200	0.6569791225338680

	$\alpha_1 = -.75$		
LB(III)	−0.19857011961895000	−1025.7380032091900	0.002499999840538190
LB(XII)	0.8818665657334750	13.388171864910700	Infinity
UB(XII)	1.5038040546867900	220.000	3.0522772727272700
UB(III)	2.1758862296024700	−3460.086639163930	0.5416587051792580

	$\alpha_1 = -1.00$		
LB(III)	0.5259438684990860	−1479.4010245307800	0.0024999982616121600
LB(XII)	1.7732728048672000	40.74306711415020	Infinity
UB(XII)	1.9870253273082700	220.0000016645500	9.449545370622180
UB(III)	2.7399934037154	−3413.482202884210	0.4420805584421760

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