

11-2016

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## Recommended Citation

Headrick, Todd C. "A Note on the Relationship between the Pearson Product-Moment and the Spearman Rank-Based Coefficients of Correlation." *Open Journal of Statistics* 6 (Nov 2016): 1025-1027. doi:10.4236/ojs.2016.66082.

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# A Note on the Relationship between the Pearson Product-Moment and the Spearman Rank-Based Coefficients of Correlation

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**How to cite this paper:** Headrick, T.C. (2016) A Note on the Relationship between the Pearson Product-Moment and the Spearman Rank-Based Coefficients of Correlation. *Open Journal of Statistics*, 6, 1025-1027.

<http://dx.doi.org/10.4236/ojs.2016.66082>

**Received:** September 15, 2016

**Accepted:** November 14, 2016

**Published:** November 17, 2016

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## Abstract

This note derives the relationship between the Pearson product-moment coefficient of correlation and the Spearman rank-based coefficient of correlation for the bivariate normal distribution. This new derivation shows the relationship between the two correlation coefficients through an infinite cosine series. A computationally efficient algorithm is also provided to estimate the relationship between the Pearson product-moment coefficient of correlation and the Spearman rank-based coefficient of correlation. The algorithm can be implemented with relative ease using current modern mathematical or statistical software programming languages e.g. R, SAS, Mathematica, Fortran, *et al.* The algorithm is also available from the author of this article.

## Keywords

Bivariate Normal Distribution, Product-Moment Correlation, Rank-Based Correlation, Gibbs Phenomenon

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## 1. Introduction

The Pearson product-moment coefficient of correlation can be interpreted as the cosine of the angle between variable vectors in  $n$  dimensional space (e.g. [1] and [[2], p. 702]). Pearson [3] showed that the relationship of turning Spearman rank-based correlation coefficients ( $\rho_s$ ) for the bivariate normal distribution into Pearson product-moment correlations ( $\rho$ ), which was contrived based on the so-called correlation of grades, for large samples to be:

$$\rho_s = (6/\pi) \sin^{-1}((1/2)\rho). \quad (1)$$

For finite (small) samples, Moran [4] derived the relationship between the Pearson and Spearman coefficients of correlation for the bivariate normal distribution, which also appears in Headrick [5] p. 114, to be:

$$\rho_s = (6/\pi)\{((n-2)/(n+1))\sin^{-1}((1/2)\rho) + (1/(n+1))\sin^{-1}(\rho)\}. \tag{2}$$

Taking the limit as  $n \rightarrow \infty$  in Equation (2) will reduce Equation (2) to Equation (1). We would also note that Höfdding [6] demonstrated that the Spearman rank correlation tends to normality for any given parent population.

## 2. Mathematical Development

In view of the above, this note derives the relationship between the Pearson product-moment correlation coefficient and the Spearman rank-based correlation coefficient for the bivariate normal distribution, in a different manner from either the Pearson [3] or the Moran [4] derivations, through the following infinite cosine series:

$$\sum_{n=1}^{\infty} \cos(nx)/n. \tag{3}$$

Specifically, if we let  $z = \cos(x) + i \sin(x)$ , then

$$\sum_{n=1}^m y^{n-1} z^n = \left( z \{1 - (yz)^m\} \right) / (1 - yz) \tag{4}$$

where it follows that for  $|y| < 1$ , that

$$\begin{aligned} & \sum_{n=1}^{\infty} y^{n-1} (\cos(nx) + i \sin(nx)) \\ &= (\cos(x) + i \sin(x)) / (1 - y \cos(x) - yi \sin(x)) \\ &= ((\cos(x) - y) + i \sin(x)) / (1 - 2y \cos(x) + y^2). \end{aligned} \tag{5}$$

Thus, from Equation (5) we have:

$$\sum_{n=1}^{\infty} y^{n-1} \cos(nx) = (\cos(x) - y) / (1 - 2y \cos(x) + y^2). \tag{6}$$

The series associated with Equation (6) is uniformly convergent for all values of  $y$  and for  $|y| \leq p < 1$ . As such, integrating with respect to  $y$ , where  $0 < y < 1$  yields:

$$\begin{aligned} \sum_{n=1}^{\infty} y^n (\cos(nx)) / n &= \int_0^y ((\cos(x) - t) / (1 - 2t \cos(x) + t^2)) dt \\ &= (-1/2) (\ln(1 - 2y \cos(x) + y^2)). \end{aligned} \tag{7}$$

Let  $x$  neither be zero nor a multiple of  $2\pi$ . As such, it necessarily follows that the series in Equation (3) is convergent. Hence, for  $0 \leq y \leq 1$ ;  $y^n$  is positive, monotonic, decreasing, and bounded. Whence, the series

$$\sum_{n=1}^{\infty} y^n \cos(nx) / n \tag{8}$$

is, therefore, uniformly convergent for  $0 \leq x \leq 1$ . Subsequently letting  $y \rightarrow 1$ , noting again that  $x$  is neither zero nor a multiple of  $2\pi$ , it follows that Equation (3) can be expressed as

$$\sum_{n=1}^{\infty} \cos(nx) / n = (-1/2) \ln(2 - 2 \cos(x)) = -\ln(2 \sin((1/2)x)). \tag{9}$$

### 3. Main Result and Conclusions

Setting  $x = (\pi/3)\rho_s$  in Equation (9), and through subsequent inverse exponentiation ( $1/e$ ) of Equation (9), yields the relationship (for large samples) between the Pearson product-moment correlation and the Spearman rank-based correlation coefficients as

$$\rho = 2 \sin\left(\left(\frac{\pi}{6}\right)\rho_s\right) \quad (10)$$

for the bivariate normal distribution. In conclusion, the algorithm provided below in Equation (11), which has an oscillating effect of the Gibbs phenomenon [7], to demonstrate the analytical derivation above is given as:

$$\hat{\rho} = \left(1 - \frac{1}{k} \sum_{n=1}^k \cos\left(n\left(\frac{\pi}{3}\right)\rho_s\right)/n\right)^k \quad (11)$$

where  $0 \leq |\rho_s| \leq 1$ ,  $k$  is finite, and where Equation (11) converges to Equation (10) as  $k \rightarrow \infty$ . Finally, in terms of the error associated with Equation (11), it is straight-forward to see through real analysis, that  $\rho_s$  and  $\rho$  have a maximum absolute deviation when  $\rho_s = 0.566467\dots$  and hence Equation (10) would result in  $\rho = 0.584543\dots$ . As such, at this maximum point of deviation, given that  $k = 10000$  in Equation (11), that the absolute error is less than  $5.42 \times 10^{-5}$  when juxtaposed with Equation (10).

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