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Numerics of Stochastic Systems with Memory (Mittag-Leffler Institute Seminar)

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NUMERICS OF STOCHASTIC SYSTEMS WITH MEMORY

Salah Mohammed

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Institut Mittag-Leffler

Royal Swedish Academy of Sciences

Sweden: December 13, 2007
Acknowledgment

■ Joint work with E. Buckwar, R. Kuske and T. Shardlow.
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- Research supported by NSF Grants DMS-9703852, DMS-9975462, DMS-0203368, DMS-0705970 and Canadian PIMS.
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*No semimartingale properties* for processes of the form $(x(t - \tau), x(t))$. But need an Itô formula!

Get a “tame” Itô formula for $\psi(x(t - \tau), x(t))$. 
Proof of the Euler scheme: Uses tame Itô formula, tame character and Fréchet differentiability of the Euler approximation in the initial path, estimates on the Malliavin derivatives of the solution, Malliavin and Fréchet derivatives of the Euler approximation.
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Set-up is non-anticipating, but proof of convergence requires anticipating stochastic calculus techniques.
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Implementation of the scheme does not require knowledge of the Malliavin calculus.

Unlike (sode’s), sdde’s do not correspond to diffusions on Euclidean space. Thus techniques from deterministic pde’s do not apply.
Motivation

Sdde’s model noisy physical processes with memory:

- Laser dynamics with delayed feedback (Buldú, et al. (2001), and Masoller (2002, 2003)).
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- Models of delayed visual feedback systems (Beuter and Vasilakos (1993)), and (Longtin, Milton, Bos and Mackey (1990)).
- Option-pricing models with memory (Arriojas, Hu, Mohammed and Pap (2007)).
Motivation-Contd

Model equations are non-linear and do not allow for explicit solutions.
Motivation-Contd

Model equations are non-linear and do not allow for explicit solutions.
Hence need numerical approximation methods of solution:
Sfde approximations

Strong (or almost sure) Euler scheme (order $1^2$) and strong Milstein scheme (order $1$) for sdde's were developed by Ahmed, Elsanousi and Mohammed [A], Mohammed [Mo.1], Hu, Mohammed and Yan [H.M.Y] and Baker and Buckwar [B.B], Küchler and Platen [Kü.P].

Weak approximations for sode's (without memory) are well-developed (Bally and Talay [B.T], Kloeden and Platen [K.P], Kohatsu-Higa [K]).

Weak Euler scheme of order $1$ for semilinear sfde's: Buckwar and Shardlow (2005); linear smooth memory drift term; memoryless diffusion term; 1-dim'l noise:
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Weak Euler scheme of order 1 for semilinear sfde’s: Buckwar and Shardlow (2005); linear smooth memory drift term; memoryless diffusion term; 1-dim’l noise:
Approximations – cont’d

\[
x(t) = \begin{cases} 
\nu + \int_{-\tau}^{0} \int_{0}^{t} x(u + s) \mu(s) \, ds \, du \\
+ \int_{0}^{t} f(x(u)) \, du + \int_{0}^{t} g(x(u)) \, dW(u), & t \geq 0 \\
\eta(t), & -\tau < t < 0.
\end{cases}
\]

Initial condition \((\nu, \eta) \in M_2 := \mathbb{R}^d \times L^2([-\tau, 0], \mathbb{R}^d)\).
**Approximations – cont’d**

\[ x(t) = \begin{cases} 
  v + \int_0^t \int_{-\tau}^0 x(u + s) \mu(s) \, ds \, du \\
  + \int_0^t f(x(u)) \, du + \int_0^t g(x(u)) \, dW(u), & t \geq 0 \\
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Initial condition \((v, \eta) \in M_2 := \mathbb{R}^d \times L^2([-\tau, 0], \mathbb{R}^d)\). Embed sfde as a semilinear see (without memory) in Hilbert space \(M_2\). Weak approximation in \(M_2\).
Approximations – cont’d

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Embed sfde as a semilinear see (without memory) in Hilbert space \(M_2\).
Weak approximation in \(M_2\).
Duality methods for weak Euler scheme-independently by Clément, Kohatsu-Higa and Lamberton [CK-HL].
In this talk, we prove weak convergence of order 1 of the Euler scheme for fully non-linear sdde’s.
Approximations– cont’d

In this talk, we prove weak convergence of order 1 of the Euler scheme for *fully non-linear* sdde's.

Allow for:
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In this talk, we prove weak convergence of order 1 of the Euler scheme for *fully non-linear* sdde’s.

Allow for:

- multiple discrete delays
- smooth memory
Approximations– cont’d

In this talk, we prove weak convergence of order 1 of the Euler scheme for *fully non-linear* sdde’s.

Allow for:

- multiple discrete delays
- smooth memory
- multidimensional Brownian noise
The weak Euler scheme

\[
\begin{align*}
    x(t) &= (0) + \int_0^t f(x(u_1)) \, du + \int_0^t g(x(u_2)) \, dW(u_2) \\
    &\text{with } t < t_2.
\end{align*}
\]
The weak Euler scheme

The SDDE:

\[ x(t) = \begin{cases} 
(0) + \int_0^t f(x(u_1)) \, du + \int_0^t g(x(u_2)) \, dW(u), 
\end{cases} \]

Coefficients \( f;g \) are of class \( C^3 \); initial (random) path \( \mathbb{C}^1(0; \mathbb{R}) \).
The weak Euler scheme

The SDDE:

\[
x(t) = \begin{cases} 
\eta(0) + \int_{\tau}^{t} f(x(u - \tau_1), x(u)) \, du \\
+ \int_{\sigma}^{\tau} g(x(u - \tau_2), x(u)) \, dW(u), & t \geq \sigma, \\
\eta(t - \sigma), & \sigma - \tau \leq t < \sigma, \\
\end{cases}
\]

(I)
The weak Euler scheme

**The SDDE:**

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The weak Euler scheme

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\quad + \int_{\sigma}^{t} g(x(u - \tau_2), x(u)) \, dW(u), & t \geq \sigma, \\
\eta(t - \sigma), & \sigma - \tau \leq t < \sigma, \quad \tau := \tau_1 \lor \tau_2. 
\end{cases} \] (I)

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Initial instant \( \sigma \geq 0 \).
The weak Euler scheme – cont’d

**Noise:** One-dimensional Brownian motion $W$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. 
The weak Euler scheme – cont’d

**Noise:** One-dimensional Brownian motion $W$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$.

**Unique solution** $x := x(\cdot; \sigma, \eta) : [\sigma - \tau, a] \times \Omega \to \mathbb{R}$ of (I), fixed $a > \sigma$. 

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The weak Euler scheme – cont’d

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**Partition** \( \pi := \{-\tau = t_{-m} < t_{-m+1} < \cdots < t_{-1} < 0 \)

\( = t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n = a\} \) of \([-\tau, a]\), with mesh:

\[
|\pi| := \max\{(t_i - t_{i-1}) : -m + 1 \leq i \leq n\}.
\]
The weak Euler scheme – cont’d

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$$|\pi| := \max\{(t_i - t_{i-1}) : -m + 1 \leq i \leq n\}.$$ 

For any $u \in [\sigma, a]$, define $[u] := t_{i-1} \lor \sigma$ whenever $u \in [t_{i-1}, t_i] \cap [\sigma, a]$. 

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The weak Euler scheme – cont’d

\[ y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s)) \, ds + \int_{t_0}^{t} g(s, y(s)) \, dW(s) \]

where

\[ R(x) \]

is the solution of the SDE:

\[ y(t) = y(0) + \int_{0}^{t} f(s, x(s)) \, ds + \int_{0}^{t} g(s, x(s)) \, dW(s) \]

The parameters \( a, b, c, d, \) etc., are determined by the specific problem at hand.
The weak Euler scheme – cont’d

Euler approximation:

\[ y := y(\cdot; \sigma, \eta) : [\sigma - \tau, a] \times \Omega \to \mathbb{R} \]

of \( x(\cdot; \sigma, \eta) \) is the solution of the sdde:

\[
y(t) = \begin{cases} 
\eta(0) + \int_\sigma^t f(y([u] - \tau_1), y([u])) \, du \\
\quad + \int_\sigma^t g(y([u] - \tau_2), y([u])) \, dW(u), t \geq \sigma, \\
\eta(t - \sigma), \quad \sigma - \tau \leq t < \sigma, \quad \tau := \tau_1 \lor \tau_2.
\end{cases}
\]

(II)

\[
\eta_\pi(s) := \left(\frac{t_i - s}{t_i - t_{i-1}}\right) \eta(t_{i-1}) + \left(\frac{s - t_{i-1}}{t_i - t_{i-1}}\right) \eta(t_i),
\]

\( s \in [t_{i-1}, t_i), -m + 1 \leq i \leq 0. \)
The weak Euler scheme – cont’d

Main result:

Weak convergence of order 1 for the Euler scheme of the sdde (I).
Theorem 1-Weak Convergence

Let $\pi$ be a partition of $[-\tau, a]$ with mesh $|\pi|$, and $\phi : \mathbb{R} \to \mathbb{R}$ be of class $C^3_b$. In the sdde $(I)$, assume that the coefficients $f, g$ are $C^3_b$. Let $x(\cdot; \sigma, \eta)$ be the unique solution of $(I)$ starting at $\sigma \in (0, a]$ with initial path $\eta \in C^1([-\tau, 0], \mathbb{R})$. 
Theorem 1-Weak Convergence

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Theorem 1-Weak Convergence

Let \( \pi \) be a partition of \([−τ, a]\) with mesh \(|\pi|\), and \( \phi : \mathbb{R} \to \mathbb{R} \) be of class \( C_\beta^3 \). In the sdde (I), assume that the coefficients \( f, g \) are \( C_\beta^3 \). Let \( x(\cdot; \sigma, \eta) \) be the unique solution of (I) starting at \( \sigma \in (0, a] \) with initial path \( \eta \in C^1([−τ, 0], \mathbb{R}) \). Denote by \( y(\cdot; \sigma, \eta) \) the Euler approximation to \( x(\cdot; \sigma, \eta) \) defined by (II).

Then there is a positive constant \( C \) and a positive integer \( q \) such that
Theorem 1 – cont’d

\[ |E\phi(x(t; \sigma, \eta)) - E\phi(y(t; \sigma, \eta^\tau))| \leq C(1 + \|\eta\|_{C^1}^q)|\pi| \]

for all \( \eta \in C^1([-\tau, 0], \mathbb{R}) \) and all \( t \in [\sigma - \tau, a] \).
Theorem 1 – cont’d

\[ |E\phi(x(t; \sigma, \eta)) - E\phi(y(t; \sigma, \eta^\tau))| \leq C(1 + \|\eta\|_{C^1}^q)|\pi| \]

for all \( \eta \in C^1([-\tau, 0], \mathbb{R}) \) and all \( t \in [\sigma - \tau, a] \). The constant \( C \) may depend on \( a, q \) and the test function \( \phi \), but is independent of \( \pi, \eta, \sigma \in [0, a] \) and \( t \in [\sigma - \tau, a] \).
Markov Property

For the solution

\[ x : [-\tau, a] \times \Omega \rightarrow \mathbb{R} \]

of sdde (I), denote its \textit{segment} \( x_t \in C([-\tau, 0], \mathbb{R}) \), \( t \in [0, a] \), by

\[ x_t(s) := x(t + s), \quad s \in [-\tau, 0], \quad t \in [0, a]. \]
Markov Property

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of sdde (I), denote its segment \( x_t \in C([-\tau, 0], \mathbb{R}), \) \( t \in [0, a], \) by

\[ x_t(s) := x(t + s), \quad s \in [-\tau, 0], \quad t \in [0, a]. \]

\( x_t \in C([-\tau, 0], \mathbb{R}), \) \( t \geq 0, \) is Markov.
The segment process
Outline of Proof

Step 1:

Let \( t \in [\sigma, a] \) and \( \pi := \{t_0, t_1, t_2, \cdots, t_n\} \) be a partition of \([0, a]\). W.l.o.g, assume that \( \sigma = t_0 = 0, t = t_n \).
Outline of Proof

Step 1:

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Using telescoping, the Markov property for $x_t$ and $y_t$, and Fréchet differentiability of $y(t_n; t_i, \eta)$ in $\eta$: 
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Using telescoping, the Markov property for $x_t$ and $y_t$, and Fréchet differentiability of $y(t_n; t_i, \eta)$ in $\eta$:

$$E\phi(x(t_n; 0, \eta)) - E\phi(y(t_n; 0, \eta))$$

$$= E\phi(y(t_n; t_n, x_{t_n}(\cdot; 0, \eta))) - E\phi(y(t_n; 0, \eta))$$

$$= \sum_{i=1}^{n} \{ E\phi(y(t_n; t_i, x_{t_i}(\cdot; 0, \eta))) - E\phi(y(t_n; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)))) \}$$
Outline of Proof – cont’d

\[
= \sum_{i=1}^{n} \left\{ E\phi(y(t_n; t_i, x_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)))) \right. \\
- E\phi(y(t_n; t_i, y_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)))) \right\} \\
= \sum_{i=1}^{n} E \int_{0}^{1} D(\phi \circ y)(t_n; t_i, \lambda x_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta))) \\
\hphantom{=} \quad + (1 - \lambda)y_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta))) \, d\lambda \\
\hphantom{=} \quad \cdot \left[ x_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)) - y_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)) \right].
\]

(M.V. Theorem)
Outline of Proof—cont’d

**Step 2:**

Main task is to show that each of the terms in the above sum is $O((t_i - t_{i-1})^2)$: Use the **tame Itô formula**. Get multiple Skorohod integrals of the form...
Outline of Proof– cont’d

**Step 2:**

Main task is to show that each of the terms in the above sum is $O((t_i - t_{i-1})^2)$: Use the tame Itô formula. Get multiple Skorohod integrals of the form

\[ J_1^i := \int_{t_{i-1}-t_i}^0 Y(ds) \int_{t_{i-1}}^{t_i+s} \int_{t_{i-1}}^{u} \Sigma_1(v) \, dv \, dW(u), \]

\[ J_2^i := \int_{t_{i-1}-t_i}^0 Y(ds) \int_{t_{i-1}}^{t_i+s} \int_{t_{i-1}}^{u} \Sigma_2(v) \, dW(v - \tau_2) \, dW(u), \]

\[ J_3^i := \int_{t_{i-1}-t_i}^0 Y(ds) \int_{t_{i-1}}^{t_i+s} \int_{t_{i-1}}^{u} \Sigma_3(v) \, dW(v - \tau_1) \, du. \]
The discrete random measure $Y$ and the processes $\Sigma_j, j = 1, 2, 3$, are Malliavin smooth and possibly anticipate the lagged Brownian motions $W(\cdot - \tau_i)$, $i = 1, 2$. 
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**Step 3:**

To estimate the expectations $E J^i_j$ in Step 2, use the definition of the Skorohod integral as adjoint of the weak differentiation operator, coupled with estimates on higher-order moments of Malliavin derivatives of $\Sigma_j$’s, $j = 1, 2, 3$. 
The discrete random measure $Y$ and the processes $\Sigma_j, j = 1, 2, 3$, are Malliavin smooth and possibly anticipate the lagged Brownian motions $W(\cdot - \tau_i)$, $i = 1, 2$.

**Step 3:**

To estimate the expectations $E J_j^i$ in Step 2, use the definition of the Skorohod integral as adjoint of the weak differentiation operator, coupled with estimates on higher-order moments of Malliavin derivatives of $\Sigma_j$’s, $j = 1, 2, 3$. These estimates follow from higher moments of the solution $x$, its Euler approximations $y$ and Malliavin derivatives of linearizations of $y$. 
This gives

$$|EJ_j^i| = O((t_i - t_{i-1})^2), \quad j = 1, 2, 3.$$  

Summing over $i = 1, \ldots, n$, we get the required order of convergence 1 for the weak Euler scheme.

**Step 4:**

Replace $\eta$ in $y(t; \sigma, \eta)$ by its P-L approx $\eta^{\pi}$ via the estimates

$$|E\phi(x(t; \sigma, \eta)) - E\phi(x(t; \sigma, \eta^{\pi}))|$$

$$\leq C||\eta - \eta^{\pi}||_C \leq C||\eta'||_{\infty}|\pi|.$$
THE PROOF
Example
Example

One-dimensional sdde:

\[ dX(t) = g(X(t - 1), X(t)) \, dW(t), \quad t > 0, \]
\[ X(t) = W(t), \quad -1 \leq t \leq 0. \]

\( g : \mathbb{R}^2 \rightarrow \mathbb{R} \) smooth function. For Euler scheme of order 1, seek a stochastic differential of \( g(X(t - 1), X(t)) \) on RHS of sdde.
For $t \in (0, 1]$, formally expect something like:

\[
dg(X(t - 1), X(t)) = \frac{\partial g}{\partial x_2}(W(t - 1), X(t)) g(W(t - 1), X(t)) dW(t)
\]

\[
+ \frac{\partial g}{\partial x_1}(W(t - 1), X(t)) dW(t - 1) \quad (\text{anticipating!})
\]

+ second-order terms \cdots
Example – Cont’d

- LHS is *adapted* but *anticipating* integral(s) on RHS.
Example – Cont’d

- LHS is *adapted but anticipating* integral(s) on RHS.

- $(\mathcal{F}_t)_{0 \leq t \leq 1}$-adapted process $(X(t - 1), X(t)) \in \mathbb{R}^2$ is not a semimartingale with respect to any natural filtration.
Example – Cont’d

- **LHS is adapted but anticipating integral(s) on RHS.**

- \((\mathcal{F}_t)_{0 \leq t \leq 1}\)-adapted process \((X(t - 1), X(t)) \in \mathbb{R}^2\) is not a semimartingale with respect to any natural filtration.

- Still need an Itô formula for **tame functions**: 

  \[ g(X(t - 1), X(t)) = g(X_t(-1), X_t(0)). \]

  where \(X_t(s) := X(t + s), \ s \in [-1, 0], t \geq 0.\)
The tame Itô formula

First, some notation:

\[ W(t); t \geq 0, \ \text{one-dimensional standard Brownian motion} \]

on a filtered probability space \([\mathcal{F}; (\mathcal{F}_t); t \geq 0]; \mathbb{P} \).
The tame Itô formula

Objective is to obtain an Itô formula for tame functionals on the Banach $C'([-\tau, 0], \mathbb{R}^d)$, acting on segments of sample-continuous random processes $[-\tau, \infty] \times \Omega \to \mathbb{R}^d$: tame Itô formula ([H.M.Y])
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For simplicity, set $W(t) := 0$ if $t \leq 0$. 
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For simplicity, set $W(t) := 0$ if $t \leq 0$.

$\mathcal{D}$ := the weak (Malliavin) differentiation operator associated with $\{W(t) : t \geq 0\}$. 
The tame Itô formula – Cont’d

Let $p > 1$, $k$ a positive integer; $L^{k,p} := L^p([0, a], \mathbb{D}^{k,p})$, where $\mathbb{D}^{k,p}$ is the closure of all random variables $Y$ with $k$-th weak derivatives in $L^p(\Omega, H^{\otimes k})$ under the norm

$$
\|Y\|_{k,p} := (E|Y|^p)^{1/p} + \left( \sum_{j=1}^{k} E\|\mathcal{D}^j Y\|_{H^{\otimes j}}^p \right)^{1/p}.
$$
The tame Itô formula – Cont’d

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\[
\| Y \|_{k,p} := (E|Y|^p)^{1/p} + \left( \sum_{j=1}^{k} E\| \mathcal{D}^j Y \|_{H^\otimes j}^p \right)^{1/p}.
\]

In above formula, \( H := L^2([0, a], \mathbb{R}) \). The spaces \( \mathbb{L}^{k,p}_{loc}, p > 4 \), are defined to be the set of all processes \( X \) such that there is an increasing sequence of \( \mathcal{F} \)-measurable sets \( A_n, n \geq 1 \), and processes \( X_n \in \mathbb{L}^{k,p} \),
$n \geq 1$, such that $X = X_n$ a.s. on $A_n$ for each $n \geq 1$, and

$\bigcup_{n=1}^{\infty} A_n = \Omega$. Weak differentiation operator $\mathcal{D}$ is local and hence extends unambiguously to the spaces $L_{loc}^{k,p}, p > 4$. 

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Weak differentiation operator $\mathcal{D}$ is local and hence extends unambiguously to the spaces $\mathbb{L}^{k,p}_{loc}, p > 4$.

See ([Nu.1], pp. 61, 151, 161) for further properties of weak derivatives and the spaces $\mathbb{L}^{k,p}$. 
\( n \geq 1 \), such that \( X = X_n \) a.s. on \( A_n \) for each \( n \geq 1 \), and \( \bigcup_{n=1}^{\infty} A_n = \Omega \). Weak differentiation operator \( D \) is local and hence extends unambiguously to the spaces \( \mathbb{L}^{k,p}_{loc}, p > 4 \).

See ([Nu.1], pp. 61, 151, 161) for further properties of weak derivatives and the spaces \( \mathbb{L}^{k,p} \).

\( C^{1,2}([0, a] \times \mathbb{R}^k, \mathbb{R}) := \) space of all functions \( \phi : [0, a] \times \mathbb{R}^k \to \mathbb{R} \) which are \( C^1 \) in the time variable \([0, a]\) and \( C^2 \) in the space variables \( \mathbb{R}^k \).
Let $X : [-\tau, \infty) \times \Omega \to \mathbb{R}$ be a pathwise-continuous (not necessarily adapted) $\mathbb{R}$-valued process given by

$$X(t) = \begin{cases} 
\eta(0) + \int_0^t u(s) \, dW(s) + \int_0^t v(s) \, ds, & t > 0, \\
\eta(t), & -\tau \leq t \leq 0,
\end{cases}$$

(1)

where $\eta \in C := C([-\tau, 0], \mathbb{R})$ and is of bounded variation, $u \in \mathbb{L}_{loc}^{2,p}, p > 4$, and $v \in \mathbb{L}_{loc}^{1,4}$. 
The tame Itô formula – Cont’d

Let $X : [-\tau, \infty) \times \Omega \to \mathbb{R}$ be a pathwise-continuous (not necessarily adapted) $\mathbb{R}$-valued process given by

$$X(t) = \begin{cases} 
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\end{cases}$$

where $\eta \in C := C([-\tau, 0], \mathbb{R})$ and is of bounded variation, $u \in \mathbb{L}^{2, p}_{loc}, p > 4$, and $v \in \mathbb{L}^{1,4}_{loc}$. The stochastic integral is a Skorohod integral.
The tame Itô formula – Cont’d

Set $u(t) := 0$ for $t < 0$, and

$$v(t) := \eta'(t), \quad -\tau \leq t \leq 0,$$

where $\eta'$ is the (usual!) derivative of $\eta$. 
The tame Itô formula – Cont’d

Set \( u(t) := 0 \) for \( t < 0 \), and

\[ v(t) := \eta'(t), \quad -\tau \leq t \leq 0, \]

where \( \eta' \) is the (usual!) derivative of \( \eta \).

Let \( \Pi : C'([-\tau, 0], \mathbb{R}) \to \mathbb{R}^k \) be the tame projection
associated with \( s_1, \cdots, s_k \in [-\tau, 0] \); that is

\[ \Pi(\eta) := (\eta(s_1), \cdots, \eta(s_k)) \]

for all \( \eta \in C := C'([-\tau, 0], \mathbb{R}) \).
The tame Itô formula – Cont’d

For any sample-continuous process

\[ X : [-\tau, a] \times \Omega \to \mathbb{R} \]

recall its segment \( X_t \in C([-\tau, 0], \mathbb{R}), t \in [0, a] \):

\[ X_t(s) := X(t + s), \quad s \in [-\tau, 0], \quad t \in [0, a]. \]

Get the tame Itô formula:
Theorem 2 (Tame Itô Formula)

Assume that $X$ is a continuous process defined by (1), where $\eta : [-\tau, 0] \to \mathbb{R}$ is of bounded variation, $u \in \mathbb{L}^{2,4}_{loc}$, and $v \in \mathbb{L}^{1,4}_{loc}$. Suppose $\phi \in C^{1,2}([0, a] \times \mathbb{R}^k, \mathbb{R})$. Then for all $t \in [0, a]$ we have a.s.
Theorem 2 (Tame Itô Formula)

Assume that $X$ is a continuous process defined by (1), where $\eta : [-\tau, 0] \rightarrow \mathbb{R}$ is of bounded variation, $u \in L_{loc}^{2,4}$, and $v \in L_{loc}^{1,4}$. Suppose $\phi \in C^{1,2}([0, a] \times \mathbb{R}^k, \mathbb{R})$. Then for all $t \in [0, a]$ we have a.s.

$$
\phi(t, \Pi(X_t)) - \phi(0, \Pi(X_0)) = \int_0^t \frac{\partial \phi}{\partial s}(s, \Pi(X_s)) \, ds \\
+ \sum_{i=1}^k \int_0^t \frac{\partial \phi}{\partial x_i}(s, \Pi(X_s)) \, dX(s + s_i) \\
+ \frac{1}{2} \sum_{i,j=1}^k \int_0^t \frac{\partial^2 \phi}{\partial x_i \partial x_j}(s, \Pi(X_s)) u(s + s_i) \, \nabla_{s_i, s_j} X(s) \, ds
$$
Theorem 2 – Cont’d

where

\[ \nabla_{s_i,s_j} X(s) := D^+_{s+s_i} X(s + s_j) + D^-_{s+s_i} X(s + s_j) \]

and
Theorem 2 – Cont’d

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\[ \nabla_{s_i,s_j} X(s) := D^+_{s+s_i} X(s + s_j) + D^-_{s+s_i} X(s + s_j) \]

and

\[ D^+_{s+s_i} X(s + s_j) := \lim_{\epsilon \to 0^+} D_{s+s_i} X(s + s_j + \epsilon), \]
\[ D^-_{s+s_i} X(s + s_j) := \lim_{\epsilon \to 0^+} D_{s+s_i} X(s + s_j - \epsilon). \]
Theorem 2 – Cont’d

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\[ \nabla_{s_i, s_j} X(s) := D_{s+s_i}^+ X(s + s_j) + D_{s+s_i}^- X(s + s_j) \]

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\[ D_{s+s_i}^+ X(s + s_j) := \lim_{\epsilon \to 0+} D_{s+s_i} X(s + s_j + \epsilon), \]
\[ D_{s+s_i}^- X(s + s_j) := \lim_{\epsilon \to 0+} D_{s+s_i} X(s + s_j - \epsilon). \]

Proof. Hu, Mohammed and Yan [H.M.Y], Theorem 2.3.
Corollary 3

Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be of class $C^2$, and suppose $x$ solves the sdde

$$x(t) = \begin{cases} 
\eta(0) + \int_0^t f(x(u - \tau_1), x(u)) \, du \\
+ \int_0^t g(x(u - \tau_2), x(u)) \, dW(u), & t > 0 \\
\eta(t), & -\tau < t < 0, \tau := \tau_1 \lor \tau_2,
\end{cases}$$

(I)

where the coefficients $f, g : \mathbb{R}^2 \to \mathbb{R}$ are of class $C^2_b$, and $\eta \in C([-\tau, 0], \mathbb{R})$ is of bounded variation.
Corollary 3 – cont’d

Suppose $\delta > 0$. Then a.s.
Corollary 3 – cont’d

Suppose $\delta > 0$. Then a.s.

$$d\psi(x(t - \delta), x(t))$$

$$= \frac{\partial \psi}{\partial x_1}(x(t - \delta), x(t)) 1_{[0,\delta)}(t) \, d\eta(t - \delta)$$

$$+ \frac{\partial \psi}{\partial x_1}(x(t - \delta), x(t)) 1_{[\delta,\infty)}(t)$$

$$\times \left[ f(x(t - \delta - \tau_1), x(t - \delta)) \, dt + g(x(t - \delta - \tau_2), x(t - \delta)) \, dW(t - \delta) \right]$$

$$+ \frac{\partial \psi}{\partial x_2}(x(t - \delta), x(t)) f(x(t - \tau_1), x(t)) \, dt$$
Corollary 3 – cont’d

\[ + \frac{\partial \psi}{\partial x_2}(x(t - \delta), x(t)) g(x(t - \tau_2), x(t)) \, dW(t) \]

\[ + \frac{\partial^2 \psi}{\partial x_1 \partial x_2}(x(t - \delta), x(t)) g(x(t - \delta - \tau_2), x(t - \delta)) \times \]

\[ \times 1_{[\delta, \infty)}(t) \mathcal{D}_{t-\delta} x(t) \, dt \]

\[ + \frac{1}{2} \frac{\partial^2 \psi}{\partial x_1^2}(x(t - \delta), x(t)) g(x(t - \delta - \tau_2), x(t - \delta))^2 1_{[\delta, \infty)}(t) \, dt \]

\[ + \frac{1}{2} \frac{\partial^2 \psi}{\partial x_2^2}(x(t - \delta), x(t)) g(x(t - \tau_2), x(t))^2 \, dt, \quad t > 0. \]

(2)
Proof of Corollary 3

Suppose $t > \delta$. Apply Theorem 2 with $\phi := \psi(x_1, x_2)$, $X = x$, $s_1 = -\delta$, $s_2 = 0$, where $x$ solves the sdde (I).

For $0 < t < \delta$, result follows from classical Itô formula because $\eta$ is BV. □
Remark

In the second term on the right hand side of (2), the \((\mathcal{F}_t)_{t \geq 0}\)-adapted factor \(\frac{\partial \psi}{\partial x_1}(x(t - \delta), x(t))\) *anticipates* the differential \(dW(t - \delta)\).
Lemma 1

Fix a partition point $t_i$ in $\pi$. Then for a.a. $\omega \in \Omega$, the function

$$[t_i, a] \times C([-\tau, 0], \mathbb{R}) \ni (t, \eta) \mapsto y(t, \omega; t_i, \eta) \in \mathbb{R}$$

is tame: That is, there exists a deterministic function $F : \mathbb{R}^+ \times \mathbb{R}^k \times \mathbb{R}^h \times \mathbb{R}^l \rightarrow \mathbb{R}$ which is continuous in the time variable $\mathbb{R}^+$, of class $C^2_b$ in all space variables $\mathbb{R}^k, \mathbb{R}^h, \mathbb{R}^l$, and fixed numbers $t_1, t_2, \ldots, t_k \leq t$, $s_1, s_2, \ldots, s_h \leq t$, $\mu_1, \mu_2, \ldots, \mu_l \in [-\tau, 0]$ such that a.s.
Lemma 1 – cont’d

\[ y(t; t_i, \eta) = F(t, W(t), W(t_1), W(t_2), \ldots, W(t_k), s_1, s_2, \ldots, s_h, \eta(\mu_1), \eta(\mu_2), \ldots, \eta(\mu_l)) \]

for all \( \eta \in C([-\tau, 0], \mathbb{R}) \). In particular, for a.a. \( \omega \in \Omega \) and each \( t \in [t_i, a] \), the map

\[ C([-\tau, 0], \mathbb{R}) \ni \eta \mapsto y(t, \omega; t_i, \eta) \in \mathbb{R} \]

is \( C^1 \) (in the Fréchet sense), and
Lemma 1 – cont’d

\[ Dy(t, \omega; t_i, \eta)(\xi) = \sum_{m=1}^{l} \partial_m F(t, W(t, \omega), W(t_1, \omega), \ldots, \]
\[ W(t_k, \omega), s_1, \ldots, s_h, \eta(\mu_1), \]
\[ \ldots, \eta(\mu_m), \ldots, \eta(\mu_l)) \xi(\mu_m) \]

for all \( \eta, \xi \in C([-\tau, 0], \mathbb{R}) \). \( \partial_m F \) is the partial derivative of \( F \) with respect to the variable \( \eta(\mu_m) \).
Lemma 1 – cont’d

\[ Dy(t, \omega; t_i, \eta)(\xi) = \sum_{m=1}^{l} \partial_m F(t, W(t, \omega), W(t_1, \omega), \ldots, W(t_k, \omega), s_1, \ldots, s_h, \eta(\mu_1), \ldots, \eta(\mu_m)) \xi(\mu_m) \]

for all \( \eta, \xi \in C([-\tau, 0], \mathbb{R}) \). \( \partial_m F \) is the partial derivative of \( F \) with respect to the variable \( \eta(\mu_m) \).

**Proof of Lemma 1:** By induction, forward steps along partition intervals \([0, t_1], (t_1, t_2], \ldots\). \(\square\)
Warning!

Lemma 1 is false if the Euler approximation $y$ is replaced by the exact solution $x$ of the sDE (I):

$$x(t; \theta, 0)$$

is highly irregular in $\theta$. This dictates telescoping arguments with respect to Euler approximation $y$ and not the solution $x$ of (I).
Lemma 1 is \textit{false} if the Euler approximation $y$ is replaced by the exact solution $x$ of the sdde (I):
Warning!

Lemma 1 is *false* if the Euler approximation $y$ is replaced by the exact solution $x$ of the sdde (I):

$$x(t, \omega; 0, \eta) \text{ is highly irregular in } \eta!$$
Lemma 1 is *false* if the Euler approximation $y$ is replaced by the exact solution $x$ of the sdde (I):

$$x(t, \omega; 0, \eta) \text{ is highly irregular in } \eta!$$

*This dictates telescoping argument is wrt Euler approximation $y$ and not the solution $x$ of (I)*
Lemma 2

Assume that $f, g$ are $C^2_b$ and let $\pi := \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, a]$. For each $1 \leq i \leq n$, define the process $\Lambda^i : [-\tau, 0] \times \Omega \to \mathbb{R}$ by

$$
\Lambda^i := x_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)) \nonumber
$$

$$
- y_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)) \nonumber.
$$

Denote

$$
x(u) := x(u; 0, \eta), \quad y(u) := y(u; 0, \eta) \nonumber
$$

for $u \in [-\tau, a]$. Then
Lemma 2 – cont’d

\[ \Lambda^i(s) = \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \left[ f \left( x(u - \tau_1), x(u) \right) \right. \\
- f \left( x \left( \lfloor u \rfloor - \tau_1 \right), x(\lfloor u \rfloor) \right) \, du \\
+ \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \left[ g \left( x(u - \tau_2), x(u) \right) \right. \\
- g \left( x \left( \lfloor u \rfloor - \tau_2 \right), x(\lfloor u \rfloor) \right) \right] \, dW(u) \]

:= \sum_{j=1}^{10} \Lambda^i_j(s), \quad s \in [-\tau, 0],

where
Lemma 2 – cont’d

\[ \Lambda^i_1(s) := \int_{t_{i-1}}^{(t_i+s)\wedge t_{i-1}} \int_{[u]}^u \frac{\partial f}{\partial x_1}(x(v - \tau_1), x(v)) \]
\[ \times f(x(v - 2\tau_1), x(v - \tau_1)) 1_{[\tau_1, \infty)}(v) \, dv \, du \]
\[ + \int_{t_{i-1}}^{(t_i+s)\wedge t_{i-1}} \int_{[u]}^u \frac{\partial f}{\partial x_1}(x(v - \tau_1), x(v)) \]
\[ \times g(x(v - \tau_1 - \tau_2), x(v - \tau_1)) \]
\[ \times 1_{[\tau_1, \infty)}(v) \, dW(v - \tau_1) \, du \]
\[ + \int_{t_{i-1}}^{(t_i+s)\wedge t_{i-1}} \int_{[u]}^u \frac{\partial f}{\partial x_1}(x(v - \tau_1), x(v)) \]
\[ \times 1_{[0, \tau_1)}(v) \, d\eta(v - \tau_1) \, du \]
Lemma 2 – cont’d

\[ \Lambda_2^i(s) : = \int_{t_{i-1}}^{(t_i + s) \land t_{i-1}} \int_{[u]} \partial f \frac{\partial f}{\partial x_2} (x(v - \tau_1), x(v)) \]
\[ \times f(x(v - \tau_1), x(v)) \, dv \, du \]
\[ + \int_{t_{i-1}}^{(t_i + s) \land t_{i-1}} \int_{[u]} \partial f \frac{\partial f}{\partial x_2} (x(v - \tau_1), x(v)) \]
\[ \times g(x(v - \tau_2), x(v)) \, dW(v) \, du \]
Lemma 2 – cont’d

\[ \Lambda_i^3(s) := \int_{t_{i-1}}^{(t_i+s) \vee t_{i-1}} \int_u^u \frac{\partial^2 f}{\partial x_1 \partial x_2} (x(v - \tau_1), x(v)) \]
\[ \times g(x(v - \tau_1 - \tau_2), x(v - \tau_1)) \]
\[ \times 1_{[\tau_1, \infty)}(v) \mathcal{D}_{v-\tau_1} x(v) \, dv \, du \]
\[ \Lambda^i_4(s) := \frac{1}{2} \int_{t_{i-1}}^{(t_i+s) \vee t_{i-1}} \int_u^u \frac{\partial^2 f}{\partial x_1^2} (x(u - \tau_1), x(v)) \]
\[ \times g(x(u - \tau_1 - \tau_2), x(v - \tau_1))^2 \]
\[ \times 1_{[\tau_1, \infty)}(v) \, dv \, du \]
Lemma 2 – cont’d

\[ \Lambda^i_4(s) : = \frac{1}{2} \int_{t_{i-1}}^{(t_i+s)\wedge t_{i-1}} \int_{[u]}^{u} \frac{\partial^2 f}{\partial x_1^2} (x(v - \tau_1), x(v)) \]
\[ \times g(x(v - \tau_1 - \tau_2), x(v - \tau_1))^2 \]
\[ \times 1_{[\tau_1, \infty)}(v) \, dv \, du \]

\[ \Lambda^i_5(s) : = \frac{1}{2} \int_{t_{i-1}}^{(t_i+s)\wedge t_{i-1}} \int_{[u]}^{u} \frac{\partial^2 f}{\partial x_2^2} (x(v - \tau_1), x(v)) \]
\[ \times g(x(v - \tau_2), x(v))^2 \, dv \, du \]
Lemma 2 – cont’d

\[ \Lambda_6^i(s) : = \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{[u]}^{u} \frac{\partial g}{\partial x_1}(x(v - \tau_2), x(v)) \]
\[ \times f(x(v - \tau_1 - \tau_2), x(v - \tau_2)) \]
\[ \times 1_{[\tau_2, \infty)}(v) dv \, dW(u) \]

\[ + \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{[u]}^{u} \frac{\partial g}{\partial x_1}(x(v - \tau_2), x(v)) \]
\[ \times 1_{[0, \tau_2)}(v) d\eta(v - \tau_2) \, dW(u) \]
Lemma 2 – cont’d

\[ \Lambda_7^i(s) : = \int_{t_{i-1}}^{(t_i+s) \wedge t_{i-1}} \int_u^u \frac{\partial g}{\partial x_1} \left( x(v - \tau_2), x(v) \right) \times g \left( x(v - 2\tau_2), x(v - \tau_2) \right) \times 1_{[\tau_2, \infty)}(v) \, dW(v - \tau_2) \, dW(u) \]
Lemma 2 – cont’d

\[ \Lambda^i_7(s) : = \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{[u]}^u \frac{\partial g}{\partial x_1} (x(v - \tau_2), x(v)) \times g(x(v - 2\tau_2), x(v - \tau_2)) \times 1_{[\tau_2, \infty)}(v) \, dW(v - \tau_2) \, dW(u) \]

\[ \Lambda^i_8(s) : = \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{[u]}^u \frac{\partial^2 g}{\partial x_1 \partial x_2} (x(v - \tau_2), x(v)) \times g(x(v - 2\tau_2), x(v - \tau_2)) \times 1_{[\tau_2, \infty)}(v) D_{v-\tau_2} x(v) \, dv \, dW(u) \]
Lemma 2 – cont’d

\[ \Lambda^i_9(s) := \frac{1}{2} \int_{t_{i-1}}^{(t_i+s)\vee t_{i-1}} \int_{[u]\vee \tau_1}^u \frac{\partial^2 g}{\partial x_1^2}(x(v - \tau_2), x(v)) \]
\[ \times g(x(v - 2\tau_2), x(v - \tau_2))^2 \times \]
\[ \times 1_{[\tau_2, \infty)}(v) \, dv \, dW(u) \]
\[ \Lambda_9^i(s) := \frac{1}{2} \int_{t_{i-1}}^{(t_i + s) \lor t_{i-1}} \int_{[u] \lor \tau_1}^u \frac{\partial^2 g}{\partial x_1^2}(x(v - \tau_2), x(v)) \]
\[ \times g(x(v - 2\tau_2), x(v - \tau_2))^2 \times \]
\[ \times 1_{[\tau_2, \infty)}(v) \, dv \, dW(u) \]
\[ \Lambda_{10}^i(s) := \frac{1}{2} \int_{t_{i-1}}^{(t_i + s) \lor t_{i-1}} \int_{[u]}^u \frac{\partial^2 g}{\partial x_2^2}(x(v - \tau_2), x(v)) \]
\[ \times g(x(v - \tau_2), x(v))^2 \, dv \, dW(u) \]
for all \( s \in [-\tau, 0] \).
From now on all positive constants will be denoted by the same letter $C$

e.g.

$$C = 2C = \frac{1}{2}C = \cdots$$

etc..
Lemma 3

Suppose $f, g \in C^2_b$. Then for any $p \geq 1$ there is a positive constant $C := C(p, a, f, g)$ such that

$$\sup_{\sigma - \tau \leq u, t \leq a} E|D_{u} y(t; \sigma, \eta)|^{2p}$$

$$< C \left(1 + E\|\eta\|_{C}^{2p} + \sup_{\sigma - \tau \leq s \leq \sigma} E\|D_{s} \eta\|_{\infty}^{2p}\right);$$
Lemma 3

Suppose \( f, g \in C^2_b \). Then for any \( p \geq 1 \) there is a positive constant \( C := C(p, a, f, g) \) such that

\[
\sup_{\sigma - \tau \leq u, t \leq a} E |D_u y(t; \sigma, \eta)|^{2p} < C \left(1 + E \|\eta\|_{C}^{2p} + \sup_{\sigma - \tau \leq s \leq \sigma} E \|D_s \eta\|_{\infty}^{2p}\right);
\]

\[
\sup_{\sigma - \tau \leq u, t \leq a \atop \|\xi\|_{\infty} \leq 1} E |D_u D_y(t; \sigma, \eta)(\xi)|^{2p} < C \left(1 + E \|\eta\|_{C}^{4p} + \sup_{\sigma - \tau \leq s \leq \sigma} E \|D_s \eta\|_{\infty}^{4p}\right)^{1/2}
\]

for all \( \eta \in L^{4p}(\Omega, C([-\tau, 0], \mathbb{R}); \mathcal{F}_\sigma) \) with finite RHS.
Lemma 4

Suppose \( f, g \in \mathcal{C}^3_b \). Then for any \( p \geq 1, \sigma \in [0, a] \),

\[
\sup_{\sigma - \tau \leq u, w, t \leq a} E|D_w D_u y(t; \sigma, \eta)|^{2p} < C \left( 1 + E\|\eta\|^{4p}_{\mathcal{C}} + \sup_{\sigma - \tau \leq s \leq \sigma} E\|D_s \eta\|^{4p}_{\infty} \right. \\
+ \left. \sup_{\sigma - \tau \leq s_1, s_2 \leq \sigma} E\|D_{s_1} D_{s_2} \eta\|^{4p}_{\infty} \right)
\]

for all \( \eta \in L^{4p}(\Omega, \mathcal{C}([-\tau, 0], \mathbb{R}); \mathcal{F}_\sigma) \) with RHS finite.

\( C := C(p, a, f, g) > 0 \) independent of \( t \in [\sigma - \tau, a], \sigma, \eta \).
Lemma 4

Suppose $f, g \in C^3_b$. Then for any $p \geq 1$, $\sigma \in [0, a]$, 

$$
\sup_{\sigma - \tau \leq u, w, t \leq a} E|\mathcal{D}_w \mathcal{D}_u y(t; \sigma, \eta)|^{2p} < C (1 + E\|\eta\|_C^{4p} + \sup_{\sigma - \tau \leq s \leq \sigma} E\|\mathcal{D}_s \eta\|_\infty^{4p}) 
+ \sup_{\sigma - \tau \leq s_1, s_2 \leq \sigma} E\|\mathcal{D}_{s_1} \mathcal{D}_{s_2} \eta\|_\infty^{4p})
$$

for all $\eta \in L^{4p}(\Omega, C([-\tau, 0], \mathbb{R}); \mathcal{F}_\sigma)$ with RHS finite. 

$C := C(p, a, f, g) > 0$ independent of $t \in [\sigma - \tau, a], \sigma, \eta$. 

Similar estimate for $E|\mathcal{D}_w \mathcal{D}_u \mathcal{D} y(t; \sigma, \eta)(\xi)|^{2p}$.
Proof of Theorem 1

Fix $t \in [\sigma, a]$. Let $\pi := \{0 = t_0, t_1, t_2, \cdots, t_n = a\}$ be a partition of $[0, a]$. W.l.o.g, assume that $\sigma = 0$, $t = t_n$. 

Proof of Theorem 1

Fix $t \in [\sigma, a]$. Let $\pi := \{0 = t_0, t_1, t_2, \cdots, t_n = a\}$ be a partition of $[0, a]$. W.l.o.g, assume that $\sigma = 0$, $t = t_n$.

By telescoping and the Markov property for the segments $x_t$ and $y_t$ ([Mo.1], [Mo.2]), write:
Proof of Theorem 1

Fix \( t \in [\sigma, a] \). Let \( \pi := \{0 = t_0, t_1, t_2, \cdots, t_n = a\} \) be a partition of \([0, a]\). W.l.o.g, assume that \( \sigma = 0, t = t_n \).

By telescoping and the Markov property for the segments \( x_t \) and \( y_t \) ([Mo.1], [Mo.2]), write:

\[
E\phi(x(t_n; 0, \eta)) - E\phi(y(t_n; 0, \eta)) = E\phi(y(t_n; t_n, x_{t_n}(\cdot; 0, \eta))) - E\phi(y(t_n; 0, \eta)) \\
= \sum_{i=1}^{n} \left\{ E\phi(y(t_n; t_i, x_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)))) - E\phi(y(t_n; t_i, y_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)))) \right\}
\]
Proof of Theorem 1 – cont’d

\[ = \sum_{i=1}^{n} E \int_{0}^{1} D(\phi \circ y)(t_n; t_i, \lambda x_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)) \]

\[ + (1 - \lambda) y_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta))) \, d\lambda \]

\[ \cdot \left[ x_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)) \right. \]

\[ \left. - y_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta)) \right] . \]
Proof of Theorem 1 – cont’d

For simplicity, denote each random measure

\[
\{ D(\phi \circ y)(t_n; t_i, \lambda x_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta))) \\
+ (1 - \lambda)y_{t_i}(\cdot; t_{i-1}, x_{t_{i-1}}(\cdot; 0, \eta))) \} (ds)
\]

by

\[
D(\phi \circ y)_i(\lambda, ds)
\]

for each \( \lambda \in [0, 1] \).
Thus, by Fubini’s theorem,

\[
E(\phi(x(t_n; 0, \eta))) - E\phi(y(t_n; 0, \eta)) = \sum_{j=1}^{10} \sum_{i=1}^{n} \int_{0}^{1} \int_{-\tau}^{0} E(D(\phi \circ y)_i(\lambda, ds)) \Lambda^i_j(s) d\lambda.
\]  

(3)
Proof of Theorem 1 – cont’d

Thus, by Fubini’s theorem,

\[
E \left( \phi \left( x(t_n; 0, \eta) \right) \right) - E \phi \left( y(t_n; 0, \eta) \right) = \sum_{j=1}^{10} \sum_{i=1}^{n} \int_{0}^{1} \int_{-\tau}^{0} E \left[ D(\phi \circ y)_i(\lambda, ds) \Lambda_j^i(s) \right] d\lambda.
\]

Estimate each of the 10 terms

\[
\sum_{i=1}^{n} \int_{0}^{1} \int_{-\tau}^{0} E \left\{ D(\phi \circ y)_i(\lambda, ds) \Lambda_j^i(s) \right\} d\lambda, j = 1, 2, \ldots, 10
\]

on RHS of (3), for any fixed \( \lambda \in [0, 1] \).
Proof of Theorem 1 – cont’d

Let $j = 10$. Fix $\lambda \in [0, 1]$. Since the Skorohod integral is the adjoint of the Malliavin derivative, a computation via Lemma 2 ($Dy$ tame) gives:

$$I_{10}^i := \int_{-\tau}^{0} E D(\phi \circ y)_i(\lambda, ds) \Lambda_{10}^i(s)$$

$$= \int_{-\tau}^{0} E D\phi(y(t_n; t_i, \lambda x_{t_i} + (1 - \lambda)y_{t_i}))(ds) \Lambda_{10}^i(s)$$

$$\times Dy(t_n; t_i, \lambda x_{t_i} + (1 - \lambda)y_{t_i})(ds) \Lambda_{10}^i(s)$$

$$= Y_1^i + Y_2^i,$$

(4)
Proof of Theorem 1 – cont’d

\[ Y^i_1 := \int_{t_{i-1}}^{t_i} EX(u) Dy(t_n; t_i, \lambda x_t + (1 - \lambda)y_t)(\xi_u) \, du \]  

(5)
Proof of Theorem 1 – cont’d

\[ Y_i^i := \int_{t_{i-1}}^{t_i} EX(u) \frac{\partial^2 g}{\partial x_2^2}(x(v - \tau_2), x(v)) g(x(v - \tau_2), x(v))^2 dv \]

where

\[ X(u) := \frac{1}{2} D_u D\phi(y(t_n; t_i, \lambda x_{t_i} + (1 - \lambda)y_{t_i})) \]

(5)
Proof of Theorem 1 – cont’d

\[ Y^i_1 := \int_{t_{i-1}}^{t_i} E X(u) D y(t_n; t_i, \lambda x_{t_i} + (1 - \lambda) y_{t_i})(\xi^u) \, du \]

where

\[ X(u) := \frac{1}{2} D_u D \phi (y(t_n; t_i, \lambda x_{t_i} + (1 - \lambda) y_{t_i}) \]

\[ \times \int_{[u]} \frac{\partial^2 g}{\partial x_2^2} (x(u - \tau_2), x(u)) g(x(u - \tau_2), x(u))^2 \, dv, \]

and \( \xi^u \in L^\infty([-\tau, 0], \mathbb{R}) \) is given by

\[ \xi^u(s) := 1_{[t_{i-1}, (t_i + s) \vee t_{i-1})} (u), \quad s \in [-\tau, 0], \quad u \in [0, a]; \]
Proof of Theorem 1 – cont’d
Proof of Theorem 1 – cont’d

\[ Y_2^i := \int_{t_{i-1}}^{t_i} E Z(u) D_u D_y(t_n; t_i, \lambda x_{t_i} + (1 - \lambda)y_{t_i})(\xi^u) \, du \] (5)

where

\[ Z(u) := \frac{1}{2} D\phi(y(t_n; t_i, \lambda x_{t_i} + (1 - \lambda)y_{t_i})) \]

\[ \times \int_u \frac{\partial^2 g}{\partial x_2^2}(x(u - \tau_2), x(u)) g(x(u - \tau_2), x(u))^2 \, dv. \]
By the linearization of (II) and Gronwall’s lemma, we get

$$\sup_{\xi \in L^\infty([-\tau,0],[R]) \atop \|\xi\|_\infty \leq 1} \sup_{\sigma - \tau \leq t \leq a} \mathbb{E}|D_y(t; \sigma, \eta)(\xi)|^{2p} \leq C$$  \hspace{1cm} (6)

for every $p \geq 1$. 
Proof of Theorem 1 – cont’d

Using (similar) moment estimates on the solution, the Euler approximation and its Fréchet and Malliavin derivatives:

\[ |Y^i_1| \leq C (1 + E \| \eta \|_C^3) (t_i - t_{i-1})^2 \]  \hspace{1cm} (7)

\[ |Y^i_2| \leq C (1 + E \| \eta \|_C^4) (t_i - t_{i-1})^2 \]  \hspace{1cm} (8)

Positive constants \( C \) are independent of \( \eta \) and the partition \( \{ t_1, t_2, \cdots, t_n \} \).
Proof of Theorem 1 – cont’d

Putting things together:

\[
\left| \sum_{i=1}^{n} I_{10}^i \right| = \left| \sum_{i=1}^{n} \int_{0}^{1} \int_{-\tau}^{0} E\{ D(\phi \circ y)_i(\lambda, ds) \Lambda_{10}^i(s) \} \, d\lambda \right|
\]

\[
\leq C(1 + E\|\eta\|_C^3) \sum_{i=1}^{n} (t_i - t_{i-1})^2
\]

\[
+ C(1 + E\|\eta\|_C^4) \sum_{i=1}^{n} (t_i - t_{i-1})^2
\]

\[
\leq C(1 + E\|\eta\|_C^4) |\pi|.
\]

(9)
Proof of Theorem 1 – cont’d

Develop estimates similar to (9) for the 9 cases $j = 1, 2, 3, 4, 5, 6, 7, 8, 9$. □
Extensions

Extensions cover:

- Nonlinear $\mathbb{R}^d$-valued sdde’s with several delays.
Extensions

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THE END
THANK YOU!
Extensions

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Extensions—Notation

Let $W(t) := (W_1(t), W_2(t), \cdots, W_m(t))$ for $t \geq 0$, be $m$-dimensional standard Brownian motion on a filtered probability space $(\Omega, F, (F_t)_{t \geq 0}, P)$. 
Extensions—Notation

Let \( W(t) := (W_1(t), W_2(t), \ldots, W_m(t)) \) \( t \geq 0 \), be \( m \)-dimensional standard Brownian motion on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq0}, P)\).

Consider a finite number of delays \( \{\tau_i^j : 1 \leq i \leq k_1\} \), \( \{\tau_j^i, l : 1 \leq j \leq k_{2,l}, 1 \leq l \leq m\} \), with maximum delay \( \tau := \max\{\tau_i^j, \tau_j^i, l : 1 \leq i \leq k_1, 1 \leq j \leq k_{2,l}, 1 \leq l \leq m\} \). We will designate the memory in our sfde by a collection of *tame projections*
Extensions—cont’d

\[ \Pi^1 : C := C([−\tau, \mathbb{R}^d]) \to \mathbb{R}^{d_1}, \quad \Pi^{2, l} : C \to \mathbb{R}^{d_{2, l}} \]

\[ \Pi^1(\eta) := (\eta(\tau^1_{1, l}), \eta(\tau^2_{1, l}), \cdots, \eta(\tau^{k_1}_{1, l})), \]
\[ \Pi^{2, l}(\eta) := (\eta(\tau^1_{2, l}), \eta(\tau^2_{2, l}), \cdots, \eta(\tau^{k_2}_{2, l, l})), \]

for all \( \eta \in C \), and *quasitame projections*

\[ \Pi^1_q : C \to \mathbb{R}^{d^q_1}, \quad \Pi^{2, l}_q : C \to \mathbb{R}^{d^q_{2, l}} \]

where \( d_1 = k_1 d, d^q_1 = k_2 d, d_{2, l} = k_{2, l} d, d^q_{2, l} = k_{2, l} d \) are integer multiples of \( d \), for \( 1 \leq l \leq m \).
The quasitame projections are of the form:

\[ \Pi^1_q(\eta) := \left( \int_{-\tau}^{0} \sigma^1_1(\eta(s))\mu^1_1(s) \, ds, \int_{-\tau}^{0} \sigma^1_2(\eta(s))\mu^2_2(s) \, ds, \right. \]

\[ \left. \cdots, \int_{-\tau}^{0} \sigma^1_{k_2}(\eta(s))\mu^1_{k_2}(s) \, ds \right) \]
Extensions—cont’d

\[
\Pi_{q,l}^{2}(\eta) := \left( \int_{-\tau}^{0} \sigma_{1}^{2}(\eta(s)) \mu_{1}^{2}(s) \, ds, \int_{-\tau}^{0} \sigma_{2}^{2}(\eta(s)) \mu_{2}^{2}(s) \, ds, \ldots, \int_{-\tau}^{0} \sigma_{k_{2},l}^{2}(\eta(s)) \mu_{k_{2},l}^{2}(s) \, ds \right)
\]

for all \( \eta \in C \). The functions \( \sigma_{i}^{1}, \sigma_{j}^{2}, \mu_{i}^{1}, \mu_{j}^{2} \) are smooth.
Let
\[ f : \mathbb{R}^+ \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_1^q} \to \mathbb{R}^d, \quad g_l : \mathbb{R}^+ \times \mathbb{R}^{d_2,l} \times \mathbb{R}^{d_2^q,l} \to \mathbb{R}^d \]
be functions of class $C^1$ in the first variable and $C^3_b$ in all space variables.
Consider the sfde

\[ dx(t) = f(t, \Pi^1_{x_t}, \Pi^1_{q_{x_t}}) \, dt \]

\[ + \sum_{l=1}^{m} g_l(t, \Pi^{2,l}_{x_t}, \Pi^{2,l}_{q_{x_t}}) \, dW_l(t), \quad \sigma < t < a, \]

(III)

with initial path

\[ x_\sigma = \eta \in H_1^1,\infty([-\tau, 0], \mathbb{R}^d). \]
Let \( \pi := \{ t_{-m}, \ldots, t_0, t_1, t_2, \ldots, t_n \} \) be a partition of \([-\tau, a]\) with mesh \(|\pi|\). The Euler approximations \( y \) of \( x \) are given by

\[
dy(t) = f([t], \Pi^1(y_{[t]}), \Pi^1_q(y_{[t]})) \, dt \\
+ \sum_{l=1}^{m} g_l([t], \Pi^{2,l}(y_{[t]}), \Pi^{2,l}_q(y_{[t]})) \, dW_l(t),
\]

\( \sigma < t < a, \)  

(IV)

with initial path

\[
y_\sigma = \eta \in H^{1,\infty}([-\tau, 0], \mathbb{R}^d). \]
Under sufficient regularity hypotheses on the coefficients of (III), one gets weak convergence of order 1 for the Euler approximations $y$ in (IV) to the exact solution $x$. 
Theorem 4

Let $\phi : \mathbb{R} \to \mathbb{R}$ be of class $C_b^3$. In the sfde (III), let $f$, $g_1$, $1 \leq l \leq m$, be $C^1$ in the time variable and $C_b^3$ in all space variables. Let $x(\cdot; \sigma, \eta)$ be the unique solution of (III) with initial process $\eta \in H^{1,\infty}([-\tau, 0], \mathbb{R}^d)$. 
Theorem 4

Let $\phi : \mathbb{R} \to \mathbb{R}$ be of class $C^3_b$. In the sfde (III), let $f$, $g$ be $C_1$ in the time variable and $C^3_b$ in all space variables. Let $x(\cdot; \sigma, \eta)$ be the unique solution of (III) with initial process $\eta \in H^{1,\infty}([-\tau, 0], \mathbb{R}^d)$. Denote by $y(\cdot; \sigma, \eta)$ the Euler approximation to $x(\cdot; \sigma, \eta)$ defined by (IV). Let $\eta^\pi$ be the piecewise-linear approximation of $\eta$. 
Theorem 4

Let \( \phi : \mathbb{R} \to \mathbb{R} \) be of class \( C^3_b \). In the sfde (III), let \( f, g \), \( 1 \leq l \leq m \), be \( C^1 \) in the time variable and \( C^3_b \) in all space variables. Let \( x(\cdot; \sigma, \eta) \) be the unique solution of (III) with initial process \( \eta \in H^{1,\infty}([-\tau, 0], \mathbb{R}^d) \). Denote by \( y(\cdot; \sigma, \eta) \) the Euler approximation to \( x(\cdot; \sigma, \eta) \) defined by (IV). Let \( \eta^\pi \) be the piecewise-linear approximation of \( \eta \). Then there is a positive constant \( C \) and a positive integer \( q \) such that

\[
|E\phi(x(t; \sigma, \eta)) - E\phi(y(t; \sigma, \eta^\pi))| \leq C(1 + E\|\eta\|_{1,\infty}^q)|\pi|
\]

for all \( \eta \in H^{1,\infty}([-\tau, 0], \mathbb{R}^d) \), all \( t \in [\sigma - \tau, a] \), and all \( \sigma \in [0, a] \).
The constant $C$ may depend on $a, q$ and the test function $\phi$, but is independent of $\pi, \eta$, $t \in [\sigma - \tau, a]$ and $\sigma \in [0, a]$. 

Proof: Very similar to that of Theorem 1: Main difference is a straightforward application of the classical Itô formula combined with the tame Itô formula.
The constant $C$ may depend on $a, q$ and the test function $\phi$, but is independent of $\pi, \eta$, $t \in [\sigma - \tau, a]$ and $\sigma \in [0, a]$.

Proof:

Very similar to that of Theorem 1: Main difference is a straightforward application of the classical Itô formula combined with the tame Itô formula. □
Lemma 5

Let \( \psi : \mathbb{R}^+ \times \mathbb{R}^3 \rightarrow \mathbb{R} \) be of class \( C^1 \) in the time-variable and \( C^2 \) in the three space variables \( x_1, x_2, x_3 \). Suppose \( x \) solves the sfde (III) (for \( d = 1 \)) with coefficients satisfying the hypotheses of Theorem 4. Assume that \( h, \mu \) are smooth functions. Let \( \delta > 0 \). Then:
Lemma 5

Let \( \psi : \mathbb{R}^+ \times \mathbb{R}^3 \rightarrow \mathbb{R} \) be of class \( C^1 \) in the time-variable and \( C^2 \) in the three space variables \( x_1, x_2, x_3 \). Suppose \( x \) solves the sfde (III) (for \( d = 1 \)) with coefficients satisfying the hypotheses of Theorem 4. Assume that \( h, \mu \) are smooth functions. Let \( \delta > 0 \). Then:

\[
\begin{align*}
    d\psi(t, x(t - \delta), x(t), \int_{-\delta}^{0} h(x(t + s))\mu(s) \, ds) &= \frac{\partial \psi}{\partial t}(t, x(t - \delta), x(t), \int_{-\delta}^{0} h(x(t + s))\mu(s) \, ds) \, dt
\end{align*}
\]
Lemma 5—contd

\[ + \frac{\partial \psi}{\partial x_1}(t, x(t - \delta), x(t), \int_{-\delta}^{0} h(x(t + s))\mu(s) \, ds) \, 1_{[0,\delta)}(t) \cdot d\eta(t - \delta) \]

\[ + \frac{\partial \psi}{\partial x_1}(t, x(t - \delta), x(t), \int_{-\delta}^{0} h(x(t + s))\mu(s) \, ds) \, 1_{[\delta,\infty)}(t) \times \]

\[ \times [f(t - \delta, \Pi^1(x_{t-\delta}), \Pi^1_q(x_{t-\delta})) \, dt \]

\[ + \sum_{l=1}^{m} g_l(t - \delta, \Pi^{2,l}(x_{t-\delta}), \Pi^{2,l}_q(x_{t-\delta})) \, dW_l(t - \delta)] \]

\[ + \frac{\partial \psi}{\partial x_2}(t, x(t - \delta), x(t), \int_{-\delta}^{0} h(x(t + s))\mu(s) \, ds) \times \]
Lemma 5 – cont’d

\[
\times \left[ f(t, \Pi^1(x_t), \Pi^1_q(x_t)) \right] dt + \sum_{l=1}^{m} g_l(t, \Pi^{2,l}(x_t), \Pi^{2,l}_q(x_t)) dW_l(t) \\
+ \frac{\partial \psi}{\partial x_3}(t, x(t - \delta), x(t), \int_{-\delta}^{0} h(x(t + s))\mu(s)\, ds) \times \\
\times \left[ h(x(t)\mu(0) - h(x(t - \delta)\mu(-\delta) - \int_{t-\delta}^{t} h(x(u))\mu'(u - t)\, du \right] dt \\
+ \frac{\partial^2 \psi}{\partial x_1 \partial x_2}(t, x(t - \delta), x(t), \int_{-\delta}^{0} h(x(t + s))\mu(s)\, ds) \times \\
\times \sum_{l=1}^{m} g_l(t - \delta, \Pi^{2,l}(x_{t-\delta}), \Pi^{2,l}_q(x_{t-\delta})) 1_{[\delta, \infty)}(t) D_{t-\delta} x(t) \, dt
\]
Lemma 5 – cont’d

\[
+ \frac{1}{2} \frac{\partial^2 \psi}{\partial x_1^2} (t, x(t - \delta), x(t), \int_{-\delta}^{0} h(x(t + s)) \mu(s) \, ds)
\]

\[
\times \sum_{l=1}^{m} g_l(t - \delta, \Pi^{2,l}_1(x_{t-\delta}), \Pi^{2,l}_q(x_{t-\delta}))^2 1_{[\delta, \infty)}(t) \, dt
\]

\[
+ \frac{1}{2} \frac{\partial^2 \psi}{\partial x_2^2} (t, x(t - \delta), x(t), \int_{-\delta}^{0} h(x(t + s)) \mu(s) \, ds)
\]

\[
\times \sum_{l=1}^{m} g_l(t, \Pi^{2,l}_1(x_t), \Pi^{2,l}_q(x_t))^2 \, dt,
\]

for all \( t > 0 \).
Appropriate generalizations of Lemma 5 hold for higher dimensional versions of the sfde (III) \((d > 1)\).
Duality Methods

Weak convergence of the Euler scheme for a class of SFDE’s with smooth coefficients:

\[ b \left( \int_{-r}^{0} x(u + s) \, d\nu(s) \right) \]

and

\[ \sigma \left( \int_{-r}^{0} x(u + s) \, d\nu(s) \right) \]

\( \nu \) a finite measure on \([-r, 0]\) and \(b, \sigma : \mathbb{R} \rightarrow \mathbb{R}\) sufficiently smooth real-valued functions- due independently to Clément, Kohatsu-Higa and Lamberton [CK-HL]. Uses duality techniques.
THE VERY END