MARKOV BEHAVIOR
AND THE WEAK GENERATOR

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MARKOV BEHAVIOR AND THE GENERATOR

Consider the sfde

$$\left.\begin{aligned} dx(t) &= H(t, x_t)\, dt + G(t, x_t)\, dW(t), \qquad t > 0 \\ x_0 &= \eta \in C := C([-r, 0], \mathbf{R}^d) \end{aligned}\right\} \qquad (XIII)$$

with coefficients $H : [0,T] \times C \to \mathbf{R}^d$, $\quad G : [0,T] \times C \to \mathbf{R}^{d \times m}$, m-dimensional Brownian motion W and trajectory field $\{{}^{\eta}x_t : t \geq 0, \eta \in C\}$.

1. Questions

(i) For the sfde (XIII) does the trajectory field x_t give a diffusion in C (or M_2)?

(ii) How does the trajectory x_t transform under smooth non-linear functionals $\phi : C \to \mathbf{R}$?

(iii) What "diffusions" on C (or M_2) correspond to sfde's on $\mathbf{R}^d$?

We will only answer the first two questions. More details in [Mo], Pitman Books, 1984, Chapter III, pp. 46-112. Third question is OPEN.

Difficulties

(i) Although the current state $x(t)$ is a semimartingale, the trajectory x_t does *not* seem to possess any martingale properties when viewed as C-(or M_2)-valued process: e.g. for Brownian motion W $\big(H \equiv 0, G \equiv 1\big)$:

$$[E(W_t|\mathcal{F}_{t_1})](s) = W(t_1) = W_{t_1}(0), \qquad s \in [-r, 0]$$

whenever $t_1 \leq t - r$.

(ii) Lack of strong continuity leads to the use of weak limits in C which tend to live outside C.

(iii) We will show that x_t is a Markov process in C. However almost all tame functions lie *outside* the domain of the (weak) generator.

(iv) Lack of an Itô formula makes the computation of the generator hard.

Hypotheses (M)

(i) $\mathcal{F}_t :=$ completion of $\sigma\{W(u) : 0 \leq u \leq t\}, \quad t \geq 0.$

(ii) H, G are jointly continuous and globally Lipschitz in second variable uniformly wrt the first:

$$|H(t,\eta_1) - H(t,\eta_2)| + \|G(t,\eta_1) - G(t,\eta_2)\| \leq L\|\eta_1 - \eta_2\|_C$$

for all $t \in [0, T]$ and $\eta_1, \eta_2 \in C$.

2. The Markov Property

${}^{\eta}x^{t_1} :=$ solution starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_{t_1})$ at $t = t_1$ for the sfde:

$${}^{\eta}x^{t_1}(t) = \begin{cases} \eta(0) + \int_{t_1}^t H(u, x_u^{t_1})\, du + \int_{t_1}^t G(u, x_u^{t_1})\, dW(u), & t > t_1 \\ \eta(t - t_1), & t_1 - r \leq t \leq t_1. \end{cases}$$

This gives a two-parameter family of mappings

$$T_{t_2}^{t_1} : L^2(\Omega, C; \mathcal{F}_{t_1}) \to L^2(\Omega, C; \mathcal{F}_{t_2}),\ t_1 \le t_2,$$

$$T_{t_2}^{t_1}(\theta) := {}^{\theta}x_{t_2}^{t_1}, \qquad \theta \in L^2(\Omega, C; \mathcal{F}_{t_1}). \tag{1}$$

Uniqueness of solutions gives the *two-parameter* semigroup property:

$$T_{t_2}^{t_1} \circ T_{t_1}^{0} = T_{t_2}^{0}, \quad t_1 \le t_2. \tag{2}$$

([Mo], Pitman Books, 1984, Theorem II (2.2), p. 40.)

Theorem II.1 (Markov Property)([Mo], 1984).

In (XIII) suppose Hypotheses (M) hold. Then the trajectory field $\{{}^{\eta}x_t : t \ge 0, \eta \in C\}$ is a Feller process on C with transition probabilities

$$p(t_1, \eta, t_2, B) := P\big({}^{\eta}x_{t_2}^{t_1} \in B\big) \quad t_1 \le t_2, \quad B \in \text{ Borel } C, \quad \eta \in C.$$

i.e.

$$P\big(x_{t_2} \in B \big| \mathcal{F}_{t_1}\big) = p(t_1, x_{t_1}(\cdot), t_2, B) = P\big(x_{t_2} \in B \big| x_{t_1}\big) \text{ a.s.}$$

Further, if H and G do not depend on t, then the trajectory is time-homogeneous:

$$p(t_1, \eta, t_2, \cdot) = p(0, \eta, t_2 - t_1, \cdot), \quad 0 \le t_1 \le t_2, \quad \eta \in C.$$

Proof.

[Mo], 1984, Theorem III.1.1, pp. 51-58. [Mo], 1984, Theorem III.2.1, pp. 64-65. □

3. The Semigroup

In the autonomous sfde

$$\left.\begin{aligned} dx(t) &= H(x_t)\,dt + G(x_t)\,dW(t) \quad t>0 \\ x_0 &= \eta \in C \end{aligned}\right\} \tag{XIV}$$

suppose the coefficients $H : C \to \mathbf{R}^d$, $G : C \to \mathbf{R}^{d\times m}$ are *globally bounded* and globally Lipschitz.

$C_b :=$ Banach space of all bounded uniformly continuous functions $\phi : C \to \mathbf{R}$, with the sup norm

$$\|\phi\|_{C_b} := \sup_{\eta\in C} |\phi(\eta)|, \quad \phi \in C_b.$$

Define the operators $P_t : C_b \hookrightarrow C_b, t \geq 0$, on C_b by

$$P_t(\phi)(\eta) := E\phi\big({}^\eta x_t\big) \quad t \geq 0,\ \eta \in C.$$

A family $\phi_t, t > 0$, *converges weakly* to $\phi \in C_b$ as $t \to 0+$ if $\lim_{t\to 0+} < \phi_t, \mu > = < \phi, \mu >$ for all finite regular Borel measures μ on C. Write $\phi := w - \lim_{t\to 0+} \phi_t$. This is equivalent to

$$\begin{cases} \phi_t(\eta) \to \phi(\eta) \text{ as } t \to 0+, \text{ for all } \eta \in C \\ \\ \{\|\phi_t\|_{C_b} : t \geq 0\} \text{ is bounded .} \end{cases}$$

(Dynkin, [Dy], Vol. 1, p. 50). Proof uses uniform boundedness principle and dominated convergence theorem.

Theorem II.2([Mo], Pitman Books, 1984)

(i) $\{P_t\}_{t\geq 0}$ *is a one-parameter contraction semigroup on* C_b.

(ii) $\{P_t\}_{t\geq 0}$ *is weakly continuous at* $t=0$*:*

$$\begin{cases} P_t(\phi)(\eta) \to \phi(\eta) \text{ as } t \to 0+ \\ \\ \{|P_t(\phi)(\eta)| : t \geq 0, \eta \in C\} \text{is bounded by } \|\phi\|_{C_b}. \end{cases}$$

(iii) If $r > 0$, $\{P_t\}_{t\geq 0}$ *is never strongly continuous on* C_b *under the sup norm.*

Proof.

(i) One parameter semigroup property

$$P_{t_2} \circ P_{t_1} = P_{t_1+t_2}, \quad t_1, t_2 \geq 0$$

follows from the continuation property (2) and time-homogeneity of the Feller process x_t (Theorem II.1).

(ii) Definition of P_t, continuity and boundedness of ϕ and sample-continuity of trajectory ${}^\eta x_t$ give weak continuity of $\{P_t(\phi) : t > 0\}$ at $t = 0$ in C_b.

(iii) Lack of strong continuity of semigroup:

Define the canonical shift (static) semigroup

$$S_t : C_b \to C_b, \ t \geq 0,$$

by

$$S_t(\phi)(\eta) := \phi(\tilde{\eta}_t), \quad \phi \in C_b, \quad \eta \in C,$$

where $\tilde{\eta} : [-r, \infty) \to \mathbf{R}^d$ is defined by

$$\tilde{\eta}(t) = \begin{cases} \eta(0) & t \geq 0 \\ \eta(t) & t \in [-r, 0). \end{cases}$$

Then P_t is strongly continuous iff S_t is strongly continuous. P_t and S_t have the same "domain of strong continuity" independently of H, G, and W. This follows from the global boundedness of H and G. ([Mo], Theorem IV.2.1, pp. 72-73). Key relation is

$$\lim_{t\to 0+} E\|{}^\eta x_t - \tilde{\eta}_t\|_C^2 = 0$$

uniformly in $\eta \in C$. But $\{S_t\}$ is strongly continuous on C_b iff C is locally compact iff $r=0$ (no memory) ! ([Mo], Theorems IV.2.1 and IV.2.2, pp.72-73). Main idea is to pick any $s_0 \in [-r, 0)$ and consider the function $\phi_0 : C \to \mathbf{R}$ defined by

$$\phi_0(\eta) := \begin{cases} \eta(s_0) & \|\eta\|_C \le 1 \\ \dfrac{\eta(s_0)}{\|\eta\|_C} & \|\eta\|_C > 1 \end{cases}$$

Let C_b^0 be the domain of strong continuity of P_t, viz.

$$C_b^0 := \{\phi \in C_b : P_t(\phi) \to \phi \text{ as } t \to 0+ \text{ in } C_b\}.$$

Then $\phi_0 \in C_b$, but $\phi_0 \notin C_b^0$ because $r > 0$. □

4. The Generator

Define the *weak generator* $A : D(A) \subset C_b \to C_b$ by the weak limit

$$A(\phi)(\eta) := w - \lim_{t \to 0+} \frac{P_t(\phi)(\eta) - \phi(\eta)}{t}$$

where $\phi \in D(A)$ iff the above weak limit exists. Hence $D(A) \subset C_0^b$ (Dynkin [Dy], Vol. 1, Chapter I, pp. 36-43). Also $D(A)$ is weakly dense in C_b and A is weakly closed. Further

$$\frac{d}{dt} P_t(\phi) = A(P_t(\phi)) = P_t(A(\phi)), \quad t > 0$$

for all $\phi \in D(A)$ ([Dy], pp. 36-43).

Next objective is to derive a formula for the weak generator A. *We need to augment C by adjoining a canonical d-dimensional direction. The generator A will be equal to the weak generator of the shift semigroup $\{S_t\}$ plus a second order linear partial differential operator along this new direction.* Computation requires the following lemmas.

Let

$$F_d = \{v\chi_{\{0\}} : v \in \mathbf{R}^d\}$$
$$C \oplus F_d = \{\eta + v\chi_{\{0\}} : \eta \in C, v \in \mathbf{R}^d\}, \quad \|\eta + v\chi_{\{0\}}\| = \|\eta\|_C + |v|$$

Lemma II.1.([Mo], Pitman Books, 1984)

Suppose $\phi : C \to \mathbf{R}$ is C^2 and $\eta \in C$. Then $D\phi(\eta)$ and $D^2\phi(\eta)$ have unique weakly continuous linear and bilinear extensions

$$\overline{D\phi(\eta)} : C \oplus F_d \to \mathbf{R}, \quad \overline{D^2\phi(\eta)} : (C \oplus F_d) \times (C \oplus F_d) \to \mathbf{R}$$

respectively.

Proof.

First reduce to the one-dimensional case $d = 1$ by using coordinates.

Let $\alpha \in C^* = [C([-r,0],\mathbf{R})]^*$. We will show that there is a weakly continuous linear extension $\overline{\alpha} : C \oplus F_1 \to \mathbf{R}$ of α; viz. If $\{\xi^k\}$ is a bounded sequence in C such that $\xi^k(s) \to \xi(s)$ as $k \to \infty$ for all $s \in [-r,0]$, where $\xi \in C \oplus F_1$, then $\alpha(\xi^k) \to \overline{\alpha}(\xi)$ as $k \to \infty$. By the Riesz representation theorem there is a unique finite regular Borel measure μ on $[-r,0]$ such that

$$\alpha(\eta) = \int_{-r}^{0} \eta(s)\, d\mu(s)$$

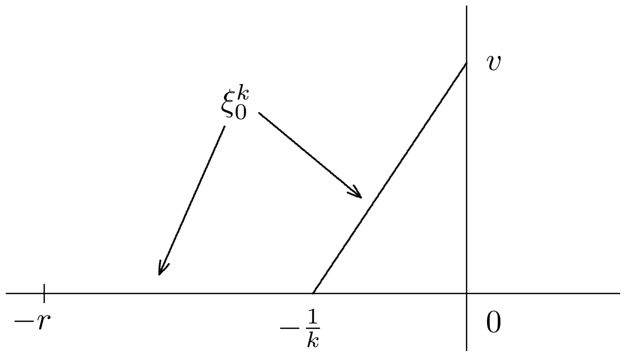
for all $\eta \in C$. Define $\overline{\alpha} \in [C \oplus F_1]^*$ by

$$\overline{\alpha}(\eta + v\chi_{\{0\}}) = \alpha(\eta) + v\mu(\{0\}), \quad \eta \in C, \quad v \in \mathbf{R}.$$

Easy to check that $\overline{\alpha}$ is weakly continuous. (*Exercise:* Use Lebesgue dominated convergence theorem.)

Weak extension $\overline{\alpha}$ is unique because each function $v\chi_{\{0\}}$ can be approximated weakly by a sequence of continuous functions $\{\xi_0^k\}$:

$$\xi_0^k(s) := \begin{cases} (ks+1)v, & -\frac{1}{k} \le s \le 0 \\ 0 & -r \le s < -\frac{1}{k}. \end{cases}$$



Put $\alpha = D\phi(\eta)$ to get first assertion of lemma.

To construct a weakly continuous bilinear extension $\overline{\beta} : (C \oplus F_1) \times (C \oplus F_1) \to \mathbf{R}$ for any continuous bilinear form $\beta : C \times C \to \mathbf{R}$, use classical theory of vector measures (Dunford and Schwartz, [D-S], Vol. I, Section 6.3). Think of β as a continuos *linear* map $C \to C^*$. Since C^* is weakly complete ([D-S], I.13.22, p. 341), then β is a weakly compact linear operator ([D-S], Theorem I.7.6, p. 494): i.e. it maps norm-bounded sets in C into weakly sequentially compact sets in C^*. By the Riesz representation theorem (for vector measures), there is a unique C^*-valued Borel measure λ on $[-r, 0]$ (of finite semi-variation) such that

$$\beta(\xi) = \int_{-r}^{0} \xi(s)\, d\lambda(s)$$

for all $\xi \in C$. ([D-S], Vol. I, Theorem VI.7.3, p. 493). By the dominated convergence theorem for vector measures ([D-S], Theorem IV.10.10, p. 328), one could reach elements in F_1 using weakly convergent sequences of type $\{\xi_0^k\}$. This gives a unique weakly continuous extension $\hat{\beta} : C \oplus F_1 \to C^*$. Next for each $\eta \in C$, $v \in \mathbf{R}$, extend $\hat{\beta}(\eta + v\chi_{\{0\}}) : C \to \mathbf{R}$ to a weakly continuous linear map $\overline{\hat{\beta}(\eta + v\chi_{\{0\}})} : C \oplus F_1 \to \mathbf{R}$. Thus $\overline{\beta}$ corresponds to the weakly continuous bilinear extension $\overline{\hat{\beta}(\cdot)}(\cdot) : [C \oplus F_1] \times [C \oplus F_1] \to \mathbf{R}$ of β. (Check this as exercise).

Finally use $\beta = D^2\phi(\eta)$ for each fixed $\eta \in C$ to get the required bilinear extension $\overline{D^2\phi(\eta)}$. □

Lemma II.2. ([Mo], Pitman Books, 1984)

For $t > 0$ define $W_t^* \in C$ by

$$W_t^*(s) := \begin{cases} \frac{1}{\sqrt{t}}[W(t+s) - W(0)], & -t \le s < 0, \\ 0 & -r \le s \le -t. \end{cases}$$

Let β be a continuous bilinear form on C. Then

$$\lim_{t \to 0+} \left[\frac{1}{t} E\beta({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t) - E\beta(G(\eta) \circ W_t^*, G(\eta) \circ W_t^*) \right] = 0$$

Proof.

Use

$$\lim_{t \to 0+} E\| \frac{1}{\sqrt{t}}({}^\eta x_t - \tilde{\eta}_t - G(\eta) \circ W_t^* \|_C^2 = 0.$$

The above limit follows from the Lipschitz continuity of H and G and the martingale properties of the Itô integral. Conclusion of lemma is obtained by a computation using the bilinearity of β, Hölder's inequality and the above limit.([Mo], Pitman Books, 1984, pp. 86-87.)

□

Lemma II.3. ([Mo], Pitman Books, 1984)

Let β be a continuous bilinear form on C and $\{e_i\}_{i=1}^m$ be any basis for $\mathbf{R}^m$. Then

$$\lim_{t \to 0+} \frac{1}{t} E\beta({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t) = \sum_{i=1}^m \overline{\beta}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big)$$

for each $\eta \in C$.

Proof.

By taking coordinates reduce to the one-dimensional case $d = m = 1$:

$$\lim_{t\to 0+} E\beta(W_t^*, W_t^*) = \overline{\beta}(\chi_{\{0\}}, \chi_{\{0\}})$$

with W one-dimensional Brownian motion. The proof of the above relation is lengthy and difficult. A key idea is the use of the projective tensor product $C\otimes_\pi C$ in order to view the continuous *bilinear* form β as a continuous *linear* functional on $C\otimes_\pi C$. At this level β commutes with the (Bochner) expectation. Rest of computation is effected using Mercer's theorem and some Fourier analysis. See [Mo], 1984, pp. 88-94. □

Theorem II.3.([Mo], Pitman Books, 1984)

In (XIV) suppose H and G are globally bounded and Lipschitz. Let $S : D(S) \subset C_b \to C_b$ be the weak generator of $\{S_t\}$. Suppose $\phi \in D(S)$ is sufficiently smooth (e.g. ϕ is C^2, $D\phi$, $D^2\phi$ globally bounded and Lipschitz). Then $\phi \in D(A)$ and

$$A(\phi)(\eta) = S(\phi)(\eta) + \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big) + \frac{1}{2}\sum_{i=1}^{m} \overline{D^2\phi(\eta)}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big).$$

where $\{e_i\}_{i=1}^m$ is any basis for $\mathbf{R}^m$.

Proof.

Step 1.

For fixed $\eta \in C$, use Taylor's theorem:

$$\phi({}^\eta x_t) - \phi(\eta) = \phi(\tilde{\eta}_t) - \phi(\eta) + D\phi(\tilde{\eta}_t)({}^\eta x_t - \tilde{\eta}_t) + R(t)$$

a.s. for $t > 0$; where

$$R(t) := \int_0^1 (1-u)D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)]({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t)\, du.$$

Take expectations and divide by $t > 0$:

$$\left.\begin{aligned} \frac{1}{t}E[\phi({}^\eta x_t) - \phi(\eta)] = \frac{1}{t}[S_t(\phi(\eta) - \phi(\eta)] + D\phi(\tilde{\eta}_t)\left\{E[\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)]\right\} \\ + \frac{1}{t}ER(t) \end{aligned}\right\} \tag{3}$$

for $t > 0$.

As $t \to 0+$, the first term on the RHS converges to $S(\phi)(\eta)$, because $\phi \in D(S)$.

Step 2.

Consider second term on the RHS of (3). Then

$$\begin{aligned} \lim_{t\to 0+}\left[E\left\{\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right\}\right](s) &= \begin{cases} \lim\limits_{t\to 0+}\dfrac{1}{t}\displaystyle\int_0^t E[H({}^\eta x_u)]\,du, & s = 0 \\ 0 & -r \le s < 0. \end{cases} \\ &= [H(\eta)\chi_{\{0\}}](s), \qquad\qquad -r \le s \le 0. \end{aligned}$$

Since H is bounded, then $\|E\{\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\}\|_C$ is bounded in $t > 0$ and $\eta \in C$ (*Exercise*). Hence

$$w - \lim_{t\to 0+}\left[E\left\{\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right\}\right] = H(\eta)\chi_{\{0\}} \quad (\notin C).$$

Therefore by Lemma II.1 and the continuity of $D\phi$ at η:

$$\begin{aligned} \lim_{t\to 0+} D\phi(\tilde{\eta}_t)\left\{E\left[\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right]\right\} &= \lim_{t\to 0+} D\phi(\eta)\left\{E\left[\frac{1}{t}({}^\eta x_t - \tilde{\eta}_t)\right]\right\} \\ &= \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big) \end{aligned}$$

Step 3.

To compute limit of third term in RHS of (3), consider

$$
\begin{aligned}
&\Big| \frac{1}{t} E D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)]({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t) \\
&\quad - \frac{1}{t} E D^2\phi(\eta)({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t) \Big| \\
&\quad \le (E\|D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)] - D^2\phi(\eta)\|^2)^{1/2} \left[\frac{1}{t^2} E\|{}^\eta x_t - \tilde{\eta}_t\|^4 \right]^{1/2} \\
&\quad \le K(t^2+1)^{1/2} [E\|D^2\phi[\tilde{\eta}_t + u({}^\eta x_t - \tilde{\eta}_t)] - D^2\phi(\eta)\|^2]^{1/2} \\
&\quad \to 0
\end{aligned}
$$

as $t \to 0+$, uniformly for $u \in [0,1]$, by martingale properties of the Itô integral and the Lipschitz continuity of $D^2\phi$. Therefore by Lemma II.3

$$
\begin{aligned}
\lim_{t\to 0+} \frac{1}{t} ER(t) &= \int_0^1 (1-u) \lim_{t\to 0+} \frac{1}{t} E D^2\phi(\eta)({}^\eta x_t - \tilde{\eta}_t, {}^\eta x_t - \tilde{\eta}_t)\, du \\
&= \frac{1}{2} \sum_{i=1}^m \overline{D^2\phi(\eta)}\big(G(\eta)(e_i)\chi_{\{0\}}, G(\eta)(e_i)\chi_{\{0\}}\big).
\end{aligned}
$$

The above is a weak limit since $\phi \in D(S)$ and has first and second derivatives globally bounded on C. □

5. Quasitame Functions

Recall that a function $\phi : C \to \mathbf{R}$ is *tame* (or a *cylinder function*) if there is a finite set $\{s_1 < s_2 < \cdots < s_k\}$ in $[-r, 0]$ and a C^∞-bounded function $f : (\mathbf{R}^d)^k \to \mathbf{R}$ such that

$$
\phi(\eta) = f(\eta(s_1), \cdots, \eta(s_k)), \qquad \eta \in C.
$$

The set of all tame functions is a weakly dense subalgebra of C_b, invariant under the static shift S_t and generates $Borel\, C$. For $k \ge 2$ the tame function ϕ *lies outside* the domain of strong continuity C_b^0 of P_t, and hence *outside* $D(A)$ ([Mo], Pitman Books, 1984, pp.98-103; see also proof of Theorem IV .2.2, pp. 73-76). To overcome this difficulty we introduce

Definition.

Say $\phi : C \to \mathbf{R}$ is *quasitame* if there are C^∞-bounded maps $h : (\mathbf{R}^d)^k \to \mathbf{R}$, $f_j : \mathbf{R}^d \to \mathbf{R}^d$, and piecewise C^1 functions $g_j : [-r,0] \to \mathbf{R}, 1 \leq j \leq k-1$, such that

$$\phi(\eta) = h\Big(\int_{-r}^0 f_1(\eta(s))g_1(s)\,ds, \cdots, \int_{-r}^0 f_{k-1}(\eta(s))g_{k-1}(s)\,ds, \eta(0)\Big) \qquad (4)$$

for all $\eta \in C$.

Theorem II.4. ([Mo], Pitman Books, 1984)

The set of all quasitame functions is a weakly dense subalgebra of C_b^0, invariant under S_t, generates Borel C and belongs to $D(A)$. In particular, if ϕ is the quasitame function given by (4), then

$$\begin{aligned} A(\phi)(\eta) = \sum_{j=1}^{k-1} D_j h(m(\eta))\{&f_j(\eta(0))g_j(0) - f_j(\eta(-r))g_j(-r) \\ &- \int_{-r}^0 f_j(\eta(s))g_j'(s)\,ds\} \\ &+ D_k h(m(\eta))(H(\eta)) + \frac{1}{2}\text{trace}[D_k^2 h(m(\eta)) \circ (G(\eta) \times G(\eta))]. \end{aligned}$$

for all $\eta \in C$, where

$$m(\eta) := \Big(\int_{-r}^0 f_1(\eta(s))g_1(s)\,ds, \cdots, \int_{-r}^0 f_{k-1}(\eta(s))g_{k-1}(s)\,ds, \eta(0)\Big).$$

Remarks.

(i) Replace C by the Hilbert space M_2. No need for the weak extensions because M_2 is weakly complete. Extensions of $D\phi(v,\eta)$ and $D^2\phi(v,\eta)$ correspond to partial derivatives in the $\mathbf{R}^d$-variable. *Tame functions do not exist on M_2* but quasitame functions do! (with $\eta(0)$ replaced by $v \in \mathbf{R}^d$).

Analysis of supermartingale behavior and stability of $\phi({}^{\eta}x_t)$ given in Kushner ([Ku], JDE, 1968). Infinite fading memory setting by Mizel and Trützer ([M-T], JIE, 1984) in the weighted state space $\mathbf{R}^d \times L^2((-\infty, 0], \mathbf{R}^d; \rho)$.

(ii) For each quasitame ϕ on C, $\phi({}^{\eta}x_t)$ *is* a semimartingale, and the Itô formula holds:

$$d[\phi({}^{\eta}x_t)] = A(\phi)({}^{\eta}x_t)\, dt + \overline{D\phi(\eta)}\big(H(\eta)\chi_{\{0\}}\big)\, dW(t).$$