STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH CONSTRAINTS

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Salah-Eldin A. Mohammed

Southern Illinois University

Carbondale, IL 62901–4408 USA

Web site: http://sfde.math.siu.edu

Outline

- Theory of stochastic functional differential equations (SFDE's) in flat space: Itô and Nisio ([IN], Kushner ([Ku]), Mohammed ([Mo_2], [Mo_3]) and Mohammed-Scheutzow ([MoS_1], [MoS_2]).

- **Objective:** to constrain the solution to live on a smooth submanifold of Euclidean space.

- **Main difficulty:** Tangent space along a solution path is random (cf. unlike flat case).

- Difficulty resolved by pulling back the calculus on the tangent space at the starting point of the initial semimartingale using stochastic parallel transport. Get SFDE on a linear space of semimartingales with values in the tangent space at a given point on the manifold.

- Solve SFDE on flat space by Picard's iteration method. (cf. Driver [Dr]). But two levels of randomness:
  (1) stochastic parallel transport over initial semimartingale path;
  (2) driving Brownian motion.

Law of solution at a given time may not be absolutely continuous with respect to law of initial semimartingale.

- Example of SDDE on the manifold with a type of Markov property in space of semimartingales.

- Regularity of solution of SDDE in initial semimartingale: stochastic Chen-Souriau calculus (Léandre [Le_2], [Le_3]). Requires Fréchet topology on semimartingales.

The Existence Theorem

Notation:

M smooth compact Riemannian manifold, dimension d.

Delay $\delta > 0$, $T > 0$.

$(\Omega, \mathcal{F}_t, t \geq -\delta, P)$ filtered probability space-usual conditions.

$W : [-\delta, \infty) \times \Omega \to \mathbf{R}^p$ Brownian motion on $(\Omega, \mathcal{F}_t, t \geq -\delta, P)$, $W(-\delta) = 0$.
($p = 1$ for simplicity.)

N any smooth finite-dimensional Riemannian manifold; $x \in N$.

$\mathcal{S}([-\delta, T], N; -\delta, x) :=$ space of all N-valued $(\mathcal{F}_t)_{t \geq -\delta}$-adapted continuous semimartingales

$$\gamma : [-\delta, T] \times \Omega \to N$$

with $\gamma(-\delta) = x$.

The Itô Map:

Fix $x \in M$.

$T(M) :=$ tangent bundle over M.

Define the *Itô map* by

$$\mathcal{S}([-\delta, T], M; -\delta, x) \ni \gamma \to \tilde{\gamma} \in \mathcal{S}([-\delta, T], T_x(M); -\delta, 0)$$

$$\left.\begin{aligned} d\tilde{\gamma}(t) &= \tau_{t,-\delta}^{-1}(\gamma) \circ d\gamma(t) \\ \tilde{\gamma}(-\delta) &= 0 \end{aligned}\right\} \qquad (1)$$

(Stratonovich).

$\tau_{t,-\delta}(\gamma) :=$ (stochastic) parallel transport from $x = \gamma(-\delta)$ to $\gamma(t)$ along semimartingale γ.([E.E], [Em])

Itô map is a bijection.

$\tilde{\mathcal{S}}_2^T :=$ Hilbert space of all semimartingales $\tilde{\gamma} \in \mathcal{S}([-\delta, T], T_x(M); -\delta, 0)$ such that

$$\tilde{\gamma}(t) = \int_{-\delta}^{t} A(s)\, dW(s) + \int_{-\delta}^{t} B(s)\, ds, \qquad -\delta \leq t \leq T \tag{2}$$

and

$$\|\tilde{\gamma}\|_2^2 := E[\int_{-\delta}^{T} |A(s)|^2\, ds] + E[\int_{-\delta}^{T} |B(s)|^2\, ds] < \infty \tag{3}$$

$A(s), \ B(s) \in T_x(M)$ adapted previsible processes-*characteristics* of $\tilde{\gamma}$ (or γ). $\|\cdot\|_2$ gives slightly different topology than traditional semi-martingale topologies ([D.M]).

$\mathcal{S}_2^T :=$ inverse image of $\tilde{\mathcal{S}}_2^T$ under the Itô map with induced topology.

Let $\gamma \in \mathcal{S}_2^T$, $t \in [-\delta, T]$. Set

$$\gamma^t(s) := \gamma(s \wedge t), \qquad s \in [-\delta, T].$$

Then $\widetilde{(\gamma^t)} = (\tilde{\gamma})^t$.

Evaluation map

$$e : [0, T] \times \mathcal{S}_2^T \to L^0(\Omega, M)$$

$$e(t, \gamma) := \gamma(t)$$

Vector bundle $L^0(\Omega, T(M))$ over $L^0(\Omega, M)$ with fiber over $Z \in L^0(\Omega, M)$ given by

$$L^0(\Omega, T(M))_Z := \{Y : Y(\omega) \in T_{Z(\omega)}M \text{ a.a. } \omega \in \Omega\}$$

$e^* L^0(\Omega, T(M) :=$ pull-back bundle of $L^0(\Omega, T(M))$ over $[0, T] \times \mathcal{S}_2^T$ by e.

A SFDE on M is a map

$$F : [0,T] \times \mathcal{S}_2^T \to L^0(\Omega, T(M))$$

such that $F(t,\gamma^t) \in T_{\gamma(t)}(M)$ a.s. for all $\gamma \in \mathcal{S}_2^T$, $0 \leq t \leq T$. I.e. F is a section of $e^* L^0(\Omega, T(M))$.

Consider SFDE

$$\left.\begin{aligned} dx(t) =& F(t,x^t) \circ dW(t), \qquad t \geq 0 \\ x^0 =& \gamma^0 \end{aligned}\right\} \tag{4}$$

- Pullback SFDE (4) over $T_x(M)$.
  Then:

$$\left.\begin{aligned} d\tilde{x}(t) &= \tau_{t,-\delta}^{-1}(x^t) F(t,x^t) \circ dW(t) \\ &= \tilde{F}(t,\tilde{x}^t) \circ dW(t), \qquad t \geq 0 \\ \tilde{x}^0 &= \tilde{\gamma}^0 \end{aligned}\right\} \tag{5}$$

$(t,\tilde{\gamma}) \mapsto \tilde{F}(t,\tilde{\gamma}) := \tau_{t,-\delta}^{-1}(\gamma)F(t,\gamma)$ can be viewed as a functional

$$[0,T] \times \tilde{\mathcal{S}}_2^T \to L^0(\Omega, T_x(M))$$

on the flat space $\tilde{\mathcal{S}}_2^T$,

- Use Stratonovich correction $\Delta\tilde{F}(t,\tilde{\gamma}^t)$ and impose "boundedness" and "Lipschitz condition" on F in terms of $\tilde{F}$ to get existence and uniqueness:

Hypothesis (H):

(i) "**Boundedeness**". There exists a deterministic constant C_1 such that

$$|\tilde{F}(t, \tilde{\gamma}^t)| + |\Delta\tilde{F}(t, \tilde{\gamma}^t)| < C_1 < \infty, \quad \text{a.s.} \qquad (6)$$

for all $(t, \tilde{\gamma}) \in [0, T] \times \tilde{\mathcal{S}}_2^T$.

(ii) "**Local Lipschitz property**". Suppose $\tilde{\gamma}, \tilde{\gamma}' \in \mathcal{S}_2^T$ have characteristics $(A(.), B(.))$ and $(A'(.), B'(.))$ respectively which are a.s. bounded by a deterministic constant R. Then

$$E[|\tilde{F}(t, \tilde{\gamma}^t) - \tilde{F}(t, (\tilde{\gamma}')^t)|^2 + |\Delta\tilde{F}(t, \tilde{\gamma}^t) - \Delta\tilde{F}(t, (\tilde{\gamma}')^t)|^2] \leq K(R)\|\tilde{\gamma}^t - (\tilde{\gamma}')^t\|_2^2 \qquad (7)$$

Examples:

1. $X :=$ a smooth section of k-frame bundle $L(\underline{\mathbf{R}^k}, T(M)) \to M$.

SDDE:

$$dx(t) = \tau_{t,t-\delta}(x)X(x(t-\delta)), \qquad t > 0 \qquad (8)$$

with

$$F(t,\gamma) := \tau_{t,t-\delta}(\gamma)X(\gamma(t-\delta));$$

and

$$\tilde{F}(t,\tilde{\gamma}) = \tau^{-1}_{t-\delta,-\delta}(\gamma)X(\gamma(t-\delta)). \qquad (8')$$

$\tilde{F}$ satisfies (H)(i) because parallel transport is a rotation and M is compact.

2. $X_1, X_2 :=$ smooth sections of k-frame bundle $L(\underline{\mathbf{R}^k}, T(M)) \to M$.

SFDE:

$$dx_{c,t} = \left\{ \int_{t-\delta}^{t} \tau_{t,s}(x_{c,.}) X_1(x_{c,s}) ds + X_2(x_{c,t}) \right\} \circ dw_t, \tag{9}$$

for $0 < t < T$.

For (H)(ii) embed M (isometrically) into $R^{d'}$ and extend the Riemannian structure over $R^{d'}$: the Riemannian metric has bounded derivatives of all orders and is uniformly non-degenerate. Extend the Levi-Civita connection over M to a connection which preserves the metric over $R^{d'}$ on the trivial tangent bundle of $R^{d'}$ with

Christoffel symbols having bounded derivatives of all order. The pair $(\gamma(t), \tau_{t,-\delta})$ corresponds to a process $\hat{x}(t) \in R^{d'} \times R^{d' \times d'}$ which solves the Stratonovitch SDE:

$$\left.\begin{aligned} d\hat{x}(t) &= \hat{Z}(\hat{x}(t)) \circ A(t)\, dW(t) + \hat{Z}(\hat{x}(t))B(t)\, dt \\ \hat{x}(-\delta) &= (x, Id_{T_x(M)}) \end{aligned}\right\} \tag{10}$$

on $R^{d'} \times R^{d' \times d'}$

$\hat{Z}$ is smooth (and hence has derivatives of all orders bounded over the range of existence of $\hat{x}$).

(10) in Itô form:

$$\left.\begin{aligned} d\hat{x}(t) = \hat{Z}(\hat{x}(t))A(t)\, dW(t) + \hat{Y}(\hat{x}(t))A(t)^2\, dt \\ + \hat{Z}(\hat{x}(t))B(t)\, dt \end{aligned}\right\} \tag{11}$$

In (11), $A(t) \in T_x(M)$, but we consider the one-dimensional case $d = 1$ for simplicity.

$\hat{Y}$ satisfies same hypotheses as the vector field $\hat{Z}$.

$\hat{x}(A, B)$ denotes dependence of $\hat{x}$ on A and B.

Lemma 1.

Suppose

$$|A(t)| + |B(t)| + |A'(t)| + |B'(t)| \leq R,$$

a.s. for all $t \in [-\delta, T]$ *and some deterministic* $R > 0$.

Then there exists a constant $K(R) > 0$ such that:

$$E[\sup_{-\delta \le s \le t} |\hat{x}(A,B)(s) - \hat{x}(A',B')(s)|^2] \le K(R) E[\int_{-\delta}^{t} (|A(s) - A'(s)|^2 + |B(s) - B'(s)|^2)\, ds] \quad (12)$$

Proof.

Follows from (11) by Burkholder's inequality and Gronwall's lemma. □

Put $t = 0$ in Lemma to show that SFDE's (8) and (9) satisfy (H)(ii).

Theorem 1.

Assume hypotheses (H).

Suppose that $\gamma^0 \in \mathcal{S}_2^0$ has characteristics $(A(t), B(t))$, $t \in [-\delta, 0]$, a.s. bounded by a deterministic constant $C > 0$.

Then the SFDE (4) has a unique global solution x such that $x|[-\delta, T] \in \mathcal{S}_2^T$ for every $T > 0$.

Proof.

Sufficient to prove theorem for the SFDE (5) in flat space.

Define $\tilde{x}^n$ inductively:

$$\left.\begin{aligned} d\tilde{x}^{n+1}(t) &= \tilde{F}(t, \tilde{x}^{n,t})\, dW(t) + \Delta\tilde{F}(t, \tilde{x}^{n,t})\, dt, \qquad t \geq 0 \\ \tilde{x}^{n+1,0} &= \tilde{\gamma}^0 \end{aligned}\right\} \tag{13}$$

By (H)(i),(ii),

$$\|\tilde{x}^{n+1,t} - \tilde{x}^{n,t}\|_2^2 \leq C \int_0^t \|\tilde{x}^{n,s} - \tilde{x}^{n-1,s}\|_2^2 ds \tag{14}$$

By induction:

$$\|\tilde{x}^{n+1,t} - \tilde{x}^{n,t}\|_2^2 \leq \frac{C^n t^n}{n!} \tag{15}$$

This gives existence.

For uniqueness, take two solutions $\tilde{x}^1, \tilde{x}^2$ of (5). By (H)(i), their characteristics are a.s. bounded. Then

$$\left.\begin{aligned} d\tilde{x}^1(t) =& \tilde{F}(t, \tilde{x}^{1,t})\, dW(t) + \Delta\tilde{F}(t, \tilde{x}^{1,t})\, dt \\ d\tilde{x}^2(t) =& \tilde{F}(t, \tilde{x}^{2,t})\, dW(t) + \Delta\tilde{F}(t, \tilde{x}^{2,t})\, dt \\ \tilde{x}^{1,0} =& \tilde{x}^{2,0} = \tilde{\gamma}^0 \end{aligned}\right\} \tag{16}$$

imply

$$\|\tilde{x}^{1,t} - \tilde{x}^{2,t}\|_2^2 \le C \int_0^t \|\tilde{x}^{1,s} - \tilde{x}^{2,s}\|_2^2 ds \tag{17}$$

Hence $\|\tilde{x}^{1,t} - \tilde{x}^{2,t}\|_2^2 = 0$. $\square$

Under the

Delay Condition:

$$\tilde{F}(t, \tilde{\gamma}^t) = \tilde{F}(t, \tilde{\gamma}^{t-\delta})$$

the *Stratonovich* equation (5) now becomes also the *Itô* equation:

$$\left.\begin{aligned} d\tilde{x}(t) =& \tilde{F}(t, \tilde{x}^{(t-\delta)})\, dW(t) \\ \tilde{x}^0 =& \tilde{\gamma}^0 \end{aligned}\right\}$$

Existence and uniqueness hold by forward steps of length δ.

Continuous dependence on initial process:

Theorem 2.

Assume hypotheses (H). Let $\mathcal{B}^T \subset \mathcal{S}_2^T$ *be the family of all* $\gamma \in \mathcal{S}_2^T$ *with characteristics* (A, B) *a.s. uniformly bounded on* $[-\delta, 0]$ *by a deterministic constant. Denote by* $x(\gamma^0)$ *the unique solution of SFDE (4) with initial semimartingale* $\gamma^0 \in \mathcal{B}^0$. *Then the mapping*

$$\mathcal{B}^0 \ni \gamma^0 \mapsto x(\gamma^0) \in \mathcal{B}^T$$

is continuous.

Proof.

Let $\tilde{\gamma}^0, (\tilde{\gamma}')^0$ have characteristics (A, B), (A', B') uniformly bounded on $[-\delta, 0]$ by a deterministic constant. Let $\tilde{x}(A, B)$ and $\tilde{x}(A', B')$ be corresponding solutions of (5). By Burkholder's inequality and (H)(ii):

$$\begin{aligned} &\|\tilde{x}^t(A, B) - \tilde{x}^t(A', B')\|_2^2 \\ &\quad \leq \|\tilde{\gamma}^0 - (\tilde{\gamma}')^0\|_2^2 + K \int_0^t \|\tilde{x}^s(A, B) - \tilde{x}^s(A', B')\|_2^2 ds \end{aligned} \tag{18}$$

By Gronwall's lemma:

$$\|\tilde{x}(A, B) - \tilde{x}(A', B')\|_2^2 \leq C\|\tilde{\gamma}^0 - (\tilde{\gamma}')^0\|_2^2 \tag{19}$$

□

Example-Markov Behavior.

Consider the SDDE:

$$\left.\begin{aligned} dx(t) &= \tau_{t,t-\delta}(x)X(x(t-\delta))\,dW(t) \\ x^0 &= \gamma^0, \end{aligned}\right\} \tag{20}$$

with $\gamma^0(-\delta) = x \in M$.

Replace x by a random variable $Z \in L^0(\Omega, M)$ independent of of $W(t), t \geq -\delta$.

Fix $t_0 > 0$. The process $x(t), t \geq t_0$ solves the SDDE:

$$\left.\begin{aligned} dx'(t) &= \tau_{t,t-\delta}(x')X(x'(t-\delta))\,dW(t), t \geq t_0 \\ x'(s) &= x(s),\ \ s \in [t_0-\delta, t_0] \end{aligned}\right\} \tag{21}$$

$x(t_0 - \delta)$ is independent of $dW(t), t \geq t_0 - \delta$, and parallel transport in (20) depends only on the path between $t - \delta$ and t.

Uniqueness implies

$$x'(t) = x(t),\ t \geq t_0.$$

For any semi-martingale $\gamma(t),\ t \geq -\delta$ in M, let $\gamma_t := \gamma|[t - \delta, t]$.
$x(\cdot)(\gamma^0)(W) :=$ solution of (20) with initial condition γ^0.

Then

$$x(t)(\gamma^0)(W) = x(t - t')(x_{t'}(\gamma^0))(W(t' + \cdot)), t \geq t' \quad (22)$$

$W(t' + \cdot) :=$ Brownian shift

$$s \mapsto W(t' + s) - W(t').$$

Differentiability in Chen-Souriau Sense:

Consider family of SDDE's:

$$\left.\begin{aligned} dx(t)(u) &= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta)(u)) \circ dW(t), t \geq 0 \\ x^0(u) &= \gamma^0(u) \end{aligned}\right\} \tag{23}$$

parametrized by $u \in U$, open subset of $\mathbf{R}^n$.

Embed M into $R^{d'}$.

Seek differentiability of $x(t)(u)$ in u. Can use Kolmogorov's lemma, Sobolev's imbedding theorem because u is finite-dimensional.

Flat version of (23) given by SDDE ($8'$) with an added parameter u.

For a parametrized semimartingale $\gamma(u)$ on M, the couple

$$(\gamma(u), \tau_{t,-\delta}(\gamma(u))) = \hat{x}_t$$

satisfies an Itô SDE depending on the parameter u:

$$d\hat{x}(t) = \hat{Z}(\hat{x}(t))A(u)(t)\,dW(t) + \hat{Y}(\hat{x}(t))A(u)(t)^2\,dt + \hat{Z}(\hat{x}(t))B(u)(t)\,dt \tag{24}$$

$\hat{Z}$ and $\hat{Y}$ are smooth.

Introduce family of norms:

$$\|\tilde{\gamma}\|_p^p := E[\int_{-\delta}^{T} |A(s)|^p\,ds + \int_{-\delta}^{T} |B(s)|^p\,ds] \tag{25}$$

on the space $\tilde{\mathcal{S}}_\infty^T$ of all semimartingales

$$\tilde{\gamma} \in \mathcal{S}([-\delta, T], T_x(M); -\delta, 0)$$

where $\tilde{\gamma}(t) = \int_{-\delta}^{t} A(s)\, dW(s) + \int_{-\delta}^{t} B(s)\, ds,\ 0 \leq t \leq T$ and $\|\tilde{\gamma}\|_p$ is finite for every $p \geq 1$.

Suppose $A(u)(\cdot)$ and $B(u)(\cdot)$ are bounded by a deterministic constant C independent of u, and

$$u \mapsto (A(u)(\cdot), B(u)(\cdot))$$

is Fréchet smooth in the Fréchet space $\tilde{\mathcal{S}}_{\infty}^{T}$ defined by the family of norms $\|\cdot\|_p$.

Theorem 3.

Consider the parametrized SDDE's:

$$\left.\begin{aligned} dx(t)(u) &= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta)(u)) \circ dW(t), t \geq 0, \\ x^0(u) &= \gamma^0(u) \end{aligned}\right\} \tag{26}$$

where X is smooth and $\gamma^0(u)$ is smooth in u as above. Then $x(t)(u)$ has a version which is a.s. smooth in u.

Theorem also holds if noise has a smooth parameter u:

$$dx(t)(u) = \tau_{t,t-\delta}(x^t(u))X(x(t-\delta))(\circ A(u)(t)\, dW(t) + B(u)(t)\, dt) \tag{27}$$

with initial conditions $x^0(u) = \gamma^0(u)$.

Proof of Theorem 3-Outline.

$\alpha := (\alpha_1, \cdots \alpha_k)$ multi-index.

$D^\alpha :=$ partial derivatives of order

$$|\alpha| := \sum_{i=1}^{k} \alpha_i.$$

- For a parametrized semimartingale $\gamma(u)$ on M, the couple

$$(\gamma(u), \tau_{t,-\delta}^{-1}(\gamma(u))) := \hat{x}(t)(u)$$

satisfies an Itô SDE depending on the parameter u:

$$d\hat{x}(t)(u) = \hat{Z}(\hat{x}(t)(u))A(u)(t)\,dW(t) + \hat{Y}(\hat{x}(t)(u))A(u)(t)^2\,dt + \hat{Z}(\hat{x}(t)(u))B(u)(t)\,dt$$

Since the inverse of the parallel transport is bounded, then $\hat{Z}$ and $\hat{Y}$ have bounded derivatives of all orders. If $\gamma(u) \in \mathcal{S}_\infty^T$ has a.s. bounded characteristics $(A(u), B(u))$ which are smooth in u into the Fréchet space $\mathcal{S}_\infty^T$, then the pair $\hat{x}(t)(u) := (\gamma(u), \tau_{t,-\delta}^{-1}(\gamma(u)))$ has characetristics Fréchet smooth in u. Follows by differentiating above SDE and applying Burkholder's inequality and Gronwall's lemma.

- Write the SDDE

$$\left.\begin{aligned} dx(t)(u) &= \tau_{t,t-\delta}(x^t(u))X(x(t-\delta)(u)) \circ dW(t), t \geq 0, \\ x^0(u) &= \gamma^0(u) \end{aligned}\right\} \tag{26}$$

in the form:

$$\left.\begin{aligned} d\tilde{x}(t)(u) &= g(\hat{x}((t-\delta))(u))dW(t) \\ \tilde{x}^0(u) &= \tilde{\gamma}^0(u) \end{aligned}\right\} \qquad (*)$$

where $\hat{x}(t) := (x(t), \tau_{t,-\delta}^{-1}(x))$, $g(y,z) := zX(y)$, and z represents parallel transport (orthogonal matrix), $y \in M$. Then g is bounded and has bounded derivatives of all orders.

$\tilde{\gamma}(t)^0(u) := \int_{-\delta}^t A_s^0(u)dw_s + \int_{-\delta}^t B_s^0 ds$ for $t < 0$ where $A^0(u)(\cdot)$ and $B^0(u)(\cdot)$ are bounded independently of u and differentiable in u in all the L^p semi-martingale norms $\|.\|_p$.

Hence $\tilde{\gamma}(t)^0(u)$ has u-derivatives of all orders in all L^p semi-martingale norms.

Follows from Kolmogorov's lemma and Burkholder's inequality.

- For $t \in [0, \delta]$, $\tilde{x}(t)(u)$ is a.s. differentiable in u and

$$dD^\alpha \tilde{x}(t)(u)$$
$$= Dg(\hat{x}(t-\delta)(u))D^\alpha \hat{x}(t-\delta)(u)\, dW(t) + l.o.$$

where $l.o.$ are terms containing lower-order derivatives of $\tilde{x}(t)(u)$.

- Get estimate:

$$\sup_{u \in U} \|D^\alpha \tilde{x}(\cdot)(u)\|_p \leq C(p, \alpha)$$

- Use forward steps of length δ to prove that $\tilde{x}(t)(u)$ has a smooth version in u for all $t \in [0, T]$.

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