II. MARKOV BEHAVIOR AND THE WEAK GENERATOR

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II. MARKOV BEHAVIOR AND THE GENERATOR

Consider the sfde

$$dx(t) = H(t, x_t) dt + G(t, x_t) dW(t), t > 0$$

$$x_0 = \eta \in C := C([-r, 0], \mathbf{R}^d)$$
(XIII)

with coefficients $H:[0,T]\times C\to \mathbf{R}^d$, $G:[0,T]\times C\to \mathbf{R}^{d\times m}$, m-dimensional Brownian motion W and trajectory field $\{^{\eta}x_t:t\geq 0,\eta\in C\}$.

1. Questions

- (i) For the sfde (XIII) does the trajectory field x_t give a diffusion in C (or M_2)?
- (ii) How does the trajectory x_t transform under smooth non-linear functionals $\phi: C \to \mathbf{R}$?
- (iii) What "diffusions" on C (or M_2) correspond to sfde's on \mathbf{R}^d ?

We will only answer the first two questions. More details in [Mo], Pitman Books, 1984, Chapter III, pp. 46-112. Third question is OPEN.

Difficulties

(i) Although the current state x(t) is a semimartingale, the trajectory x_t does not seem to possess any martingale properties when viewed as C-(or M_2)-valued process: e.g. for Brownian motion W ($H \equiv 0, G \equiv 1$):

$$[E(W_t|\mathcal{F}_{t_1})](s) = W(t_1) = W_{t_1}(0), \qquad s \in [-r, 0]$$

whenever $t_1 \leq t - r$.

- (ii) Lack of strong continuity leads to the use of weak limits in C which tend to live outside C.
- (iii) We will show that x_t is a Markov process in C. However almost all tame functions lie *outside* the domain of the (weak) generator.
- (iv) Lack of an Itô formula makes the computation of the generator hard.

Hypotheses (M)

- (i) $\mathcal{F}_t := \text{completion of } \sigma\{W(u) : 0 \le u \le t\}, \quad t \ge 0.$
- (ii) H, G are jointly continuous and globally Lipschitz in second variable uniformly wrt the first:

$$|H(t,\eta_1) - H(t,\eta_2)| + ||G(t,\eta_1) - G(t,\eta_2)|| \le L||\eta_1 - \eta_2||_C$$

for all $t \in [0, T]$ and $\eta_1, \eta_2 \in C$.

2. The Markov Property

 $^{\eta}x^{t_1}:=$ solution starting off at $\theta\in L^2(\Omega,C;\mathcal{F}_{t_1})$ at $t=t_1$ for the sfde:

$$^{\eta}x^{t_1}(t) = \begin{cases} \eta(0) + \int_{t_1}^t H(u, x_u^{t_1}) du + \int_{t_1}^t G(u, x_u^{t_1}) dW(u), & t > t_1 \\ \eta(t - t_1), & t_1 - r \le t \le t_1. \end{cases}$$

This gives a two-parameter family of mappings

$$T_{t_2}^{t_1}: L^2(\Omega, C; \mathcal{F}_{t_1}) \to L^2(\Omega, C; \mathcal{F}_{t_2}), \ t_1 \le t_2,$$

$$T_{t_2}^{t_1}(\theta) := {}^{\theta} x_{t_2}^{t_1}, \qquad \theta \in L^2(\Omega, C; \mathcal{F}_{t_1}). \tag{1}$$

Uniqueness of solutions gives the two-parameter semigroup property:

$$T_{t_2}^{t_1} \circ T_{t_1}^0 = T_{t_2}^0, \quad t_1 \le t_2.$$
 (2)

([Mo], Pitman Books, 1984, Theorem II (2.2), p. 40.)

Theorem II.1 (Markov Property)([Mo], 1984).

In (XIII) suppose Hypotheses (M) hold. Then the trajectory field $\{^{\eta}x_t:t\geq 0,\eta\in C\}$ is a Feller process on C with transition probabilities

$$p(t_1, \eta, t_2, B) := P(^{\eta} x_{t_2}^{t_1} \in B) \quad t_1 \le t_2, \quad B \in Borel C, \quad \eta \in C.$$

i.e.

$$P(x_{t_2} \in B | \mathcal{F}_{t_1}) = p(t_1, x_{t_1}(\cdot), t_2, B) = P(x_{t_2} \in B | x_{t_1}) \text{ a.s.}$$

Further, if H and G do not depend on t, then the trajectory is time-homogeneous:

$$p(t_1, \eta, t_2, \cdot) = p(0, \eta, t_2 - t_1, \cdot), \quad 0 \le t_1 \le t_2, \quad \eta \in C.$$

Proof.

[Mo], 1984, Theorem III.1.1, pp. 51-58. [Mo], 1984, Theorem III.2.1, pp. 64-65. $\hfill\Box$

3. The Semigroup

In the autonomous sfde

$$dx(t) = H(x_t) dt + G(x_t) dW(t) \quad t > 0$$

$$x_0 = \eta \in C$$
(XIV)

suppose the coefficients $H: C \to \mathbf{R}^d$, $G: C \to \mathbf{R}^{d \times m}$ are globally bounded and globally Lipschitz.

 $C_b := \text{Banach space of all bounded uniformly continuous functions}$ $\phi: C \to \mathbf{R}$, with the sup norm

$$\|\phi\|_{C_b} := \sup_{\eta \in C} |\phi(\eta)|, \quad \phi \in C_b.$$

Define the operators $P_t: C_b \hookrightarrow C_b, t \geq 0$, on C_b by

$$P_t(\phi)(\eta) := E\phi(^{\eta}x_t) \quad t \ge 0, \ \eta \in C.$$

A family ϕ_t , t > 0, converges weakly to $\phi \in C_b$ as $t \to 0+$ if $\lim_{t \to 0+} < \phi_t$, $\mu > = < \phi$, $\mu >$ for all finite regular Borel measures μ on C. Write $\phi := w - \lim_{t \to 0+} \phi_t$. This is equivalent to

$$\begin{cases} \phi_t(\eta) \to \phi(\eta) \text{ as } t \to 0+, \text{ for all } \eta \in C \\ \{ \|\phi_t\|_{C_b} : t \ge 0 \} \text{ is bounded } . \end{cases}$$

(Dynkin, [Dy], Vol. 1, p. 50). Proof uses uniform boundedness principle and dominated convergence theorem.

Theorem II.2([Mo], Pitman Books, 1984)

(i) $\{P_t\}_{t\geq 0}$ is a one-parameter contraction semigroup on C_b .

(ii) $\{P_t\}_{t>0}$ is weakly continuous at t=0:

$$\begin{cases} P_t(\phi)(\eta) \to \phi(\eta) \text{ as } t \to 0+\\ \{|P_t(\phi)(\eta)| : t \ge 0, \eta \in C\} \text{ is bounded by } \|\phi\|_{C_b}. \end{cases}$$

(iii) If r > 0, $\{P_t\}_{t \ge 0}$ is never strongly continuous on C_b under the sup norm.

Proof.

(i) One parameter semigroup property

$$P_{t_2} \circ P_{t_1} = P_{t_1 + t_2}, \quad t_1, t_2 \ge 0$$

follows from the continuation property (2) and time-homogeneity of the Feller process x_t (Theorem II.1).

- (ii) Definition of P_t , continuity and boundedness of ϕ and sample-continuity of trajectory ${}^{\eta}x_t$ give weak continuity of $\{P_t(\phi): t>0\}$ at t=0 in C_b .
- (iii) Lack of strong continuity of semigroup:

Define the canonical shift (static) semigroup

$$S_t: C_b \to C_b, \ t \ge 0,$$

by

$$S_t(\phi)(\eta) := \phi(\tilde{\eta}_t), \quad \phi \in C_b, \quad \eta \in C,$$

where $\tilde{\eta}: [-r, \infty) \to \mathbf{R}^d$ is defined by

$$\tilde{\eta}(t) = \begin{cases} \eta(0) & t \ge 0 \\ \eta(t) & t \in [-r, 0). \end{cases}$$

Then P_t is strongly continuous iff S_t is strongly continuous. P_t and S_t have the same "domain of strong continuity" independently of H, G, and W. This follows from the global boundedness of H and G. ([Mo], Theorem IV.2.1, pp. 72-73). Key relation is

$$\lim_{t \to 0+} E \|^{\eta} x_t - \tilde{\eta}_t \|_C^2 = 0$$

uniformly in $\eta \in C$. But $\{S_t\}$ is strongly continuous on C_b iff C is locally compact iff r = 0 (no memory)! ([Mo], Theorems IV.2.1 and IV.2.2, pp.72-73). Main idea is to pick any $s_0 \in [-r, 0)$ and consider the function $\phi_0 : C \to \mathbf{R}$ defined by

$$\phi_0(\eta) := \begin{cases} \eta(s_0) & \|\eta\|_C \le 1\\ \frac{\eta(s_0)}{\|\eta\|_C} & \|\eta\|_C > 1 \end{cases}$$

Let C_h^0 be the domain of strong continuity of P_t , viz.

$$C_b^0 := \{ \phi \in C_b : P_t(\phi) \to \phi \text{ as } t \to 0+ \text{ in } C_b \}.$$

Then $\phi_0 \in C_b$, but $\phi_0 \notin C_b^0$ because r > 0.

4. The Generator

Define the weak generator $A: D(A) \subset C_b \to C_b$ by the weak limit

$$A(\phi)(\eta) := w - \lim_{t \to 0+} \frac{P_t(\phi)(\eta) - \phi(\eta)}{t}$$

where $\phi \in D(A)$ iff the above weak limit exists. Hence $D(A) \subset C_0^b$ (Dynkin [Dy], Vol. 1, Chapter I, pp. 36-43). Also D(A) is weakly dense in C_b and A is weakly closed. Further

$$\frac{d}{dt}P_t(\phi) = A(P_t(\phi)) = P_t(A(\phi)), \quad t > 0$$

for all $\phi \in D(A)$ ([Dy], pp. 36-43).

Next objective is to derive a formula for the weak generator A. We need to augment C by adjoining a canonical d-dimensional direction. The generator A will be equal to the weak generator of the shift semigroup $\{S_t\}$ plus a second order linear partial differential operator along this new direction. Computation requires the following lemmas.

Let

$$F_d = \{v\chi_{\{0\}} : v \in \mathbf{R}^d\}$$

$$C \oplus F_d = \{\eta + v\chi_{\{0\}} : \eta \in C, v \in \mathbf{R}^d\}, \quad \|\eta + v\chi_{\{0\}}\| = \|\eta\|_C + |v|$$

Lemma II.1.([Mo], Pitman Books, 1984)

Suppose $\phi: C \to \mathbf{R}$ is C^2 and $\eta \in C$. Then $D\phi(\eta)$ and $D^2\phi(\eta)$ have unique weakly continuous linear and bilinear extensions

$$\overline{D\phi(\eta)}: C \oplus F_d \to \mathbf{R}, \quad \overline{D^2\phi(\eta)}: (C \oplus F_d) \times (C \oplus F_d) \to \mathbf{R}$$

respectively.

Proof.

First reduce to the one-dimensional case d=1 by using coordinates.

Let $\alpha \in C^* = [C([-r,0],\mathbf{R})]^*$. We will show that there is a weakly continuous linear extension $\overline{\alpha}: C \oplus F_1 \to \mathbf{R}$ of α ; viz. If $\{\xi^k\}$ is a bounded sequence in C such that $\xi^k(s) \to \xi(s)$ as $k \to \infty$ for all $s \in [-r,0]$, where $\xi \in C \oplus F_1$, then $\alpha(\xi^k) \to \overline{\alpha}(\xi)$ as $k \to \infty$. By the Riesz representation theorem there is a unique finite regular Borel measure μ on [-r,0] such that

$$\alpha(\eta) = \int_{-r}^{0} \eta(s) \, d\mu(s)$$

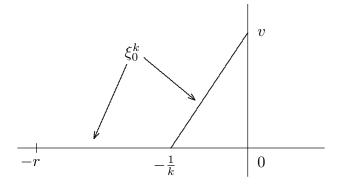
for all $\eta \in C$. Define $\overline{\alpha} \in [C \oplus F_1]^*$ by

$$\overline{\alpha}(\eta+v\chi_{\{0\}})=\alpha(\eta)+v\mu(\{0\}), \quad \eta\in C, \quad v\in \mathbf{R}.$$

Easy to check that $\overline{\alpha}$ is weakly continuous. (*Exercise*: Use Lebesgue dominated convergence theorem.)

Weak extension $\overline{\alpha}$ is unique because each function $v\chi_{\{0\}}$ can be approximated weakly by a sequence of continuous functions $\{\xi_0^k\}$:

$$\xi_0^k(s) := \begin{cases} (ks+1)v, & -\frac{1}{k} \le s \le 0\\ 0 & -r \le s < -\frac{1}{k}. \end{cases}$$



Put $\alpha = D\phi(\eta)$ to get first assertion of lemma.

To construct a weakly continuous bilinear extension $\overline{\beta}: (C \oplus F_1) \times (C \oplus F_1) \to \mathbf{R}$ for any continuous bilinear form

 $\beta: C \times C \to \mathbb{R}$, use classical theory of vector measures (Dunford and Schwartz, [D-S], Vol. I, Section 6.3). Think of β as a continuos linear map $C \to C^*$. Since C^* is weakly complete ([D-S], I.13.22, p. 341), then β is a weakly compact linear operator ([D-S], Theorem I.7.6, p. 494): i.e. it maps norm-bounded sets in C into weakly sequentially compact sets in C^* . By the Riesz representation theorem (for vector measures), there is a unique C^* -valued Borel measure λ on [-r,0] (of finite semi-variation) such that

$$\beta(\xi) = \int_{-r}^{0} \xi(s) \, d\lambda(s)$$

for all $\xi \in C$. ([D-S], Vol. I, Theorem VI.7.3, p. 493). By the dominated convergence theorem for vector measures ([D-S], Theorem IV.10.10, p. 328), one could reach elements in F_1 using weakly convergent sequences of type $\{\xi_0^k\}$. This gives a unique weakly continuous extension $\hat{\beta}: C \oplus F_1 \to C^*$. Next for each $\eta \in C$, $v \in \mathbf{R}$, extend $\hat{\beta}(\eta + v\chi_{\{0\}}): C \to \mathbf{R}$ to a weakly continuous linear map $\hat{\beta}(\eta + v\chi_{\{0\}}): C \oplus F_1 \to \mathbf{R}$. Thus $\bar{\beta}$ corresponds to the weakly continuous bilinear extension $\hat{\beta}(\cdot)(\cdot): [C \oplus F_1] \times [C \oplus F_1] \to \mathbf{R}$ of β . (Check this as exercise).

Finally use $\beta = D^2 \phi(\eta)$ for each fixed $\eta \in C$ to get the required bilinear extension $\overline{D^2 \phi(\eta)}$.

Lemma II.2. ([Mo], Pitman Books, 1984)

For t > 0 define $W_t^* \in C$ by

$$W_t^*(s) := \begin{cases} \frac{1}{\sqrt{t}} [W(t+s) - W(0)], & -t \le s < 0, \\ 0 & -r \le s \le -t. \end{cases}$$

Let β be a continuous bilinear form on C. Then

$$\lim_{t \to 0+} \left[\frac{1}{t} E\beta(^{\eta}x_t - \tilde{\eta}_t, ^{\eta}x_t - \tilde{\eta}_t) - E\beta(G(\eta) \circ W_t^*, G(\eta) \circ W_t^*) \right] = 0$$

Proof.

Use

$$\lim_{t \to 0+} E \| \frac{1}{\sqrt{t}} (^{\eta} x_t - \tilde{\eta}_t - G(\eta) \circ W_t^* \|_C^2 = 0.$$

The above limit follows from the Lipschitz continuity of H and G and the martingale properties of the Itô integral. Conclusion of lemma is obtained by a computation using the bilinearity of β , Hölder's inequality and the above limit. ([Mo], Pitman Books, 1984, pp. 86-87.)

Lemma II.3. ([Mo], Pitman Books, 1984)

Let β be a continuous bilinear form on C and $\{e_i\}_{i=1}^m$ be any basis for \mathbf{R}^m . Then

$$\lim_{t \to 0+} \frac{1}{t} E \beta(^{\eta} x_t - \tilde{\eta}_t, ^{\eta} x_t - \tilde{\eta}_t) = \sum_{i=1}^m \overline{\beta} \big(G(\eta)(e_i) \chi_{\{0\}}, G(\eta)(e_i) \chi_{\{0\}} \big)$$

for each $\eta \in C$.

Proof.

By taking coordinates reduce to the one-dimensional case d = m = 1:

$$\lim_{t \to 0+} E\beta(W_t^*, W_t^*) = \overline{\beta}(\chi_{\{0\}}, \chi_{\{0\}})$$

with W one-dimensional Brownian motion. The proof of the above relation is lengthy and difficult. A key idea is the use of the projective tensor product $C \otimes_{\pi} C$ in order to view the continuous bilinear form β as a continuous linear functional on $C \otimes_{\pi} C$. At this level β commutes with the (Bochner) expectation. Rest of computation is effected using Mercer's theorem and some Fourier analysis. See [Mo], 1984, pp. 88-94.

Theorem II.3.([Mo], Pitman Books, 1984)

In (XIV) suppose H and G are globally bounded and Lipschitz. Let S: $D(S) \subset C_b \to C_b$ be the weak generator of $\{S_t\}$. Suppose $\phi \in D(S)$ is sufficiently smooth (e.g. ϕ is C^2 , $D\phi$, $D^2\phi$ globally bounded and Lipschitz). Then $\phi \in D(A)$ and

$$A(\phi)(\eta) = S(\phi)(\eta) + \overline{D\phi(\eta)} \big(H(\eta) \chi_{\{0\}} \big)$$

$$+ \frac{1}{2} \sum_{i=1}^{m} \overline{D^2 \phi(\eta)} \big(G(\eta)(e_i) \chi_{\{0\}}, G(\eta)(e_i) \chi_{\{0\}} \big).$$

where $\{e_i\}_{i=1}^m$ is any basis for \mathbf{R}^m .

Proof.

Step 1.

For fixed $\eta \in C$, use Taylor's theorem:

$$\phi(^{\eta}x_t) - \phi(\eta) = \phi(\tilde{\eta}_t) - \phi(\eta) + D\phi(\tilde{\eta}_t)(^{\eta}x_t - \tilde{\eta}_t) + R(t)$$

a.s. for t > 0; where

$$R(t) := \int_0^1 (1 - u) D^2 \phi [\tilde{\eta}_t + u(^{\eta} x_t - \tilde{\eta}_t)] (^{\eta} x_t - \tilde{\eta}_t, ^{\eta} x_t - \tilde{\eta}_t) du.$$

Take expectations and divide by t > 0:

$$\frac{1}{t}E[\phi(^{\eta}x_{t}) - \phi(\eta)] = \frac{1}{t}[S_{t}(\phi(\eta) - \phi(\eta))] + D\phi(\tilde{\eta}_{t}) \left\{ E[\frac{1}{t}(^{\eta}x_{t} - \tilde{\eta}_{t})] \right\} + \frac{1}{t}ER(t)$$
(3)

for t > 0.

As $t \to 0+$, the first term on the RHS converges to $S(\phi)(\eta)$, because $\phi \in D(S)$.

Step 2.

Consider second term on the RHS of (3). Then

$$\lim_{t \to 0+} \left[E \left\{ \frac{1}{t} (^{\eta} x_t - \tilde{\eta}_t) \right\} \right] (s) = \begin{cases} \lim_{t \to 0+} \frac{1}{t} \int_0^t E[H(^{\eta} x_u)] du, & s = 0 \\ 0 & -r \le s < 0. \end{cases}$$
$$= [H(\eta) \chi_{\{0\}}](s), \qquad -r \le s \le 0.$$

Since H is bounded, then $||E\{\frac{1}{t}(^{\eta}x_t - \tilde{\eta}_t)\}||_C$ is bounded in t > 0 and $\eta \in C$ (Exercise). Hence

$$w - \lim_{t \to 0+} \left[E \left\{ \frac{1}{t} (^{\eta} x_t - \tilde{\eta}_t) \right\} \right] = H(\eta) \chi_{\{0\}} \quad (\notin C).$$

Therefore by Lemma II.1 and the continuity of $D\phi$ at η :

$$\lim_{t \to 0+} D\phi(\tilde{\eta}_t) \left\{ E\left[\frac{1}{t}(^{\eta}x_t - \tilde{\eta}_t)\right] \right\} = \lim_{t \to 0+} D\phi(\eta) \left\{ E\left[\frac{1}{t}(^{\eta}x_t - \tilde{\eta}_t)\right] \right\}$$
$$= \overline{D\phi(\eta)} \left(H(\eta)\chi_{\{0\}}\right)$$

Step 3.

To compute limit of third term in RHS of (3), consider

$$\left| \frac{1}{t} E D^{2} \phi [\tilde{\eta}_{t} + u(^{\eta} x_{t} - \tilde{\eta}_{t})] (^{\eta} x_{t} - \tilde{\eta}_{t}, ^{\eta} x_{t} - \tilde{\eta}_{t}) \right|
- \frac{1}{t} E D^{2} \phi (\eta) (^{\eta} x_{t} - \tilde{\eta}_{t}, ^{\eta} x_{t} - \tilde{\eta}_{t}) \Big|
\leq (E \| D^{2} \phi [\tilde{\eta}_{t} + u(^{\eta} x_{t} - \tilde{\eta}_{t})] - D^{2} \phi (\eta) \|^{2})^{1/2} \left[\frac{1}{t^{2}} E \|^{\eta} x_{t} - \tilde{\eta}_{t} \|^{4} \right]^{1/2}
\leq K (t^{2} + 1)^{1/2} [E \| D^{2} \phi [\tilde{\eta}_{t} + u(^{\eta} x_{t} - \tilde{\eta}_{t})] - D^{2} \phi (\eta) \|^{2}]^{1/2}
\rightarrow 0$$

as $t \to 0+$, uniformly for $u \in [0,1]$, by martingale properties of the Itô integral and the Lipschitz continuity of $D^2\phi$. Therefore by Lemma II.3

$$\lim_{t \to 0+} \frac{1}{t} ER(t) = \int_0^1 (1 - u) \lim_{t \to 0+} \frac{1}{t} ED^2 \phi(\eta) (\eta x_t - \tilde{\eta}_t, \eta x_t - \tilde{\eta}_t) du$$
$$= \frac{1}{2} \sum_{i=1}^m \overline{D^2 \phi(\eta)} (G(\eta)(e_i) \chi_{\{0\}}, G(\eta)(e_i) \chi_{\{0\}}).$$

The above is a weak limit since $\phi \in D(S)$ and has first and second derivatives globally bounded on C.

5. Quasitame Functions

Recall that a function $\phi: C \to \mathbf{R}$ is tame (or a cylinder function) if there is a finite set $\{s_1 < s_2 < \dots < s_k\}$ in [-r, 0] and a C^{∞} -bounded function $f: (\mathbf{R}^d)^k \to \mathbf{R}$ such that

$$\phi(\eta) = f(\eta(s_1), \cdots, \eta(s_k)), \qquad \eta \in C.$$

The set of all tame functions is a weakly dense subalgebra of C_b , invariant under the static shift S_t and generates $Borel\,C$. For $k \geq 2$ the tame function ϕ lies outside the domain of strong continuity C_b^0 of P_t , and hence outside D(A) ([Mo], Pitman Books, 1984, pp.98-103; see also proof of Theorem IV .2.2, pp. 73-76). To overcome this difficulty we introduce

Definition.

Say $\phi: C \to \mathbf{R}$ is *quasitame* if there are C^{∞} -bounded maps $h: (\mathbf{R}^d)^k \to \mathbf{R}, f_j: \mathbf{R}^d \to \mathbf{R}^d$, and piecewise C^1 functions $g_j: [-r, 0] \to \mathbf{R}, 1 \le j \le k-1$, such that

$$\phi(\eta) = h\left(\int_{-r}^{0} f_1(\eta(s))g_1(s) \, ds, \cdots, \int_{-r}^{0} f_{k-1}(\eta(s))g_{k-1}(s) \, ds, \eta(0)\right) \tag{4}$$

for all $\eta \in C$.

Theorem II.4. ([Mo], Pitman Books, 1984)

The set of all quasitame functions is a weakly dense subalgebra of C_b^0 , invariant under S_t , generates Borel C and belongs to D(A). In particular, if ϕ is the quasitame function given by (4), then

$$A(\phi)(\eta) = \sum_{j=1}^{k-1} D_j h(m(\eta)) \{ f_j(\eta(0)) g_j(0) - f_j(\eta(-r)) g_j(-r)$$

$$- \int_{-r}^0 f_j(\eta(s)) g_j'(s) \, ds \}$$

$$+ D_k h(m(\eta)) (H(\eta)) + \frac{1}{2} trace[D_k^2 h(m(\eta)) \circ (G(\eta) \times G(\eta))].$$

for all $\eta \in C$, where

$$m(\eta) := \left(\int_{-r}^{0} f_1(\eta(s)) g_1(s) \, ds, \cdots, \int_{-r}^{0} f_{k-1}(\eta(s)) g_{k-1}(s) \, ds, \eta(0) \right).$$

Remarks.

(i) Replace C by the Hilbert space M_2 . No need for the weak extensions because M_2 is weakly complete. Extensions of $D\phi(v,\eta)$ and $D^2\phi(v,\eta)$ correspond to partial derivatives in the \mathbf{R}^d -variable. Tame functions do not exist on M_2 but quasitame functions do! (with $\eta(0)$ replaced by $v \in \mathbf{R}^d$). Analysis of supermartingale behavior and stability of $\phi(^{\eta}x_t)$ given in Kushner ([Ku], JDE, 1968). Infinite fading memory setting by Mizel and Trützer ([M-T], JIE, 1984) in the weighted state space $\mathbf{R}^d \times L^2((-\infty, 0], \mathbf{R}^d; \rho)$.

(ii) For each quasitame ϕ on C, $\phi(^{\eta}x_t)$ is a semimartingale, and the Itô formula holds:

$$d[\phi(^{\eta}x_t)] = A(\phi)(^{\eta}x_t) dt + \overline{D\phi(\eta)} (H(\eta)\chi_{\{0\}}) dW(t).$$