STOCHASTIC DIFFERENTIAL SYSTEMS WITH MEMORY

THEORY, EXAMPLES AND APPLICATIONS

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Outline of Lectures

Lecture I. Existence.

1. Simple examples: The noisy feedback loop: $dx(t) = x(t-r)\, dW(t)$. Solution process $x(t)$ is not a Markov process in $\mathbf{R}$. No closed form solution when $r > 0$. Compare with the case $r = 0$. The logistic time-lag model with Gaussian noise:

   $$dx(t) = [\alpha - \beta x(t-r)]x(t)\, dt + \sigma x(t)\, dW(t).$$

   The classical "heat-bath" model of R. Kubo: Motion of a large molecule in a viscous fluid.

2. General Formulation. Choice of state space. Pathwise existence and uniquenss of solutions to sfde's under local Lipschitz and linear growth hypotheses on the coefficients. Existence theorem allows for stochastic white-noise perturbations of the memory, e.g.

   $$dx(t) = \left\{ \int_{[-r,0]} x(t+s)\, dW(s) \right\} dW(t) \quad t > 0$$

   Above sfde is *not* covered by classical results of Protter, Metivier and Pellaumail, Doleans-Dade.

3. Mean Lipschitz, smooth and sublinear dependence of the trajectory random field.

Lecture II. Markov Behavior and the Generator.

1. Markov (Feller) property holds for the trajectory random field. Time homogeneity.
2. Construction of the semigroup. Semigroup is not strongly continuous for positive delay. Domain of strong continuity does not contain tame (or cylinder) functions with evaluations away from 0, but contains "quasitame" functions. These are weakly dense in the underlying space of continuous functions and generate the Borel σ-algebra of the state space.
3. Derivation of a formula for the weak infinitesimal generator of the semigroup for sufficiently regular functions, and for a large class of quasitame functions.

Lecture III. Regularity. Classification of SFDE's.

1. Pathwise regularity of the trajectory random field in the time variable. α-Hölder continuity.
2. Almost sure (pathwise) dependence on the initial state. Non-existence of the stochastic flow for the singular sdde $dx(t) = x(t-r)\,dW(t)$. Breakdown of linearity and local boundedness. Classification of sfde's into regular and singular types.
3. Results on sufficient conditions for regularity of linear systems driven by white noise or semimartingales.
4. Sussman-Doss type nonlinear sfde's. Existence and compactness of semiflow.
5. Path regularity of general non-linear sfde's with "smooth memory".

Lecture IV. Ergodic Theory of Linear SFDE's.

1. Existence of stochastic semiflows for certain classes of linear sfde's with smooth memory terms. The cocycle and its perfection.
2. Compactness of the semiflow in the finite memory case.
3. Ruelle-Oseledec multiplicative ergodic theorem in Hilbert space. Existence of a discrete Lyapunov spectrum. The Stable Manifold Theorem (viz. *random saddles*) for hyperbolic linear sfde's driven by white noise. The case of helix noise.

Lecture V. Stability. Examples and Case Studies.

1. Estimates on the maximal exponential growth rate for the singular noisy feedback loop: $dx(t) = \sigma x(t-r)\, dW(t)$. Stability and instability for small σ (or large r) using a Lyapunov functional argument. Comparison with the non-delay case for large σ.
2. Derivation of estimates on the top Lyapunov exponent λ_1 for various examples of one-dimensional regular sfde's driven by white noise or a martingale with stationary ergodic increments.
3. Lyapunov spectrum for sdde $dx(t) = x((t-1)-)\, dN(t)$ driven by a Poisson process N. Characterization of the Lyapunov spectrum.

Lecture VI. Miscellany

1. Malliavin Calculus of SFDE's. Regularity of the solution $x(t, \omega)$ in ω. Malliavin smoothness and existence of smooth densities. Classical solution of a degenerate parabolic pde as an application.
2. Small delays. Applications to sode's. A proof of the classical existence theorem for solutions to sode's.
3. Affine systems of sfde's. Lyapunov spectrum. The hyperbolic splitting. Existence of stationary solutions in the hyperbolic case. Application to simple population model.
4. Random delays. Induced measure-valued process. Random families of Markov fields and random generators.
5. Infinite memory and stationary solutions.

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