SOME NUMERICS OF STOCHASTIC SYSTEMS WITH MEMORY

Berlin: March 2003

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Outline

- Strong Euler scheme for (general) SFDE's. Order of convergence 0.5.
- Strong Milstein scheme for SDDE's. Order of convergence 1.
- For Milstein scheme, use infinite dimensional Itô formula for "tame" functions acting on segment process of solution of SDDE. Presence of memory in SDDE requires use of Malliavin calculus + anticipating stochastic analysis of Nualart and Pardoux.
- *Conjecture*: Milstein scheme works for mixed discrete and continuous memory. *Open*: for general SFDE's?

Types of SFDE's

Suppose rate of change of physical system depends on *present state* and some noisy input. Model by SODE.

Rate of change depends on *present* and *past* states of the system: Model by SDDE or SFDE.

$\mathbf{R}^m :=$ m-dimensional Euclidean space.
Euclidean norm:

$$|x| := \sqrt{x_1^2 + \cdots + x_m^2}, \qquad x = (x_1, \cdots, x_m) \in \mathbf{R}^m.$$

$T := [0, a],\ J := [-r, 0],\ r, a > 0.$

$C := C(J; \mathbf{R}^m)$; sup norm:

$$\|\eta\|_C := \sup_{-r \leq s \leq 0} |\eta(s)|, \qquad \eta \in C := C([-r, 0], \mathbf{R}^m).$$

$W :=$ d-dimensional Brownian motion.

SDDE:

$$X(t) = \begin{cases} \eta(0) + \int_0^t g(s, \Pi_1(X_s))\, dW(s) \\ \qquad\qquad + \int_0^t h(s, \Pi_2(X_s))\, ds, \quad t \in [0, a] \\ \eta(t), \quad -r \leq t < 0. \end{cases}$$

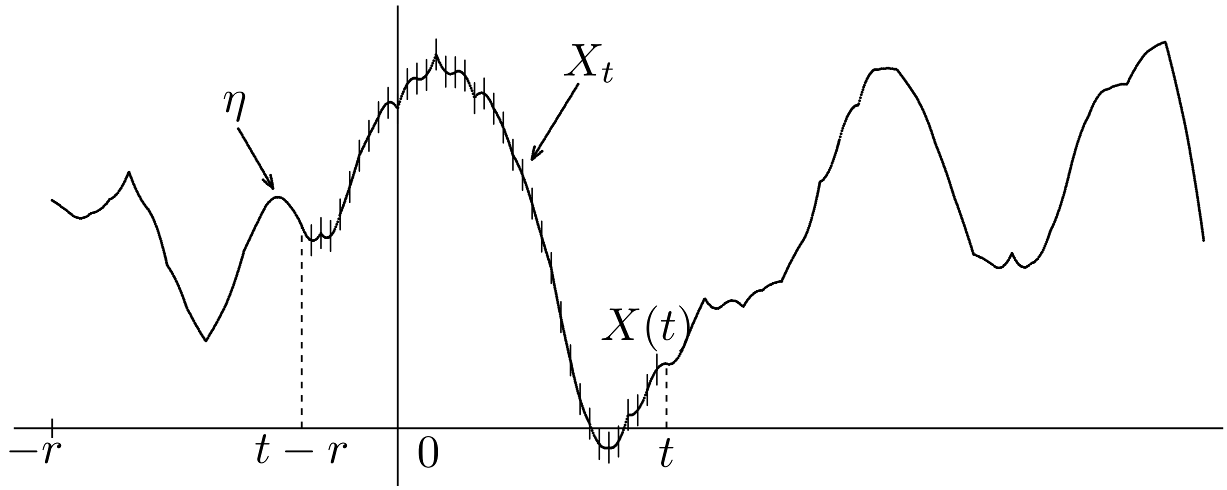
$\Pi_i : C \to \mathbf{R}^{mk_i}$, $i = 1, 2$, two projections of discrete type based on $s_{1,1}, \cdots, s_{1,k_1} \in [-r, 0]$ and $s_{2,1}, \cdots, s_{2,k_2} \in [-r, 0]$:

$$\Pi_i(\eta) := (\eta(s_{i,1}), \cdots, \eta(s_{i,k_i})) \in \mathbf{R}^{mk_i}, \quad \eta \in C, \ i = 1, 2.$$

Segment process X_t, $t \in [0, a]$:

$$X_t(s) = X(t + s), \quad t \in [0, a], \quad s \in [-r, 0].$$

$$g : T \times \mathbf{R}^{mk_1} \to L(\mathbf{R}^d, \mathbf{R}^m), \quad h : T \times \mathbf{R}^{mk_2} \to \mathbf{R}^m.$$



SFDE with mixed discrete and continuous memory:

$$X(t) = \eta(0) + \int_0^t g(s, \Pi_1(X_s), Q_1(X_s))\, dW(s) + \int_0^t h(s, \Pi_2(X_s), Q_2(X_s))\, ds, \quad t \in [0, a],$$

$$X_0 = \eta \in C = C(J; \mathbf{R}^m), J := [-r, 0].$$

$g : T \times \mathbf{R}^{mk_1} \times \mathbf{R}^{m_1} \to L(\mathbf{R}^d, \mathbf{R}^m), \quad h : T \times \mathbf{R}^{mk_2} \times \mathbf{R}^{m_2} \to \mathbf{R}^m.$

Π_1, Π_2 two projections of discrete type;

Q_1, Q_2 two projections of continuous type:

$$
\begin{aligned}
Q_i(\eta) &:= (Q_{i,1}(\eta), \cdots, Q_{i,m_i}(\eta)), \quad i = 1, 2, \\
Q_{ij}(\eta) &:= \int_{-r}^{0} \phi_{ij}(\eta(s)) a_{ij}(s)\, ds\,, \quad j = 1, \cdots, m_i.
\end{aligned}
$$

$a_{ij} : J \to \mathbf{R}$ and $\phi_{ij} : \mathbf{R}^m \to \mathbf{R}$ sufficiently regular,

$i = 1, 2$, $j = 1, \cdots, m_i$.

General SFDE:

$$
X(t) = \begin{cases} \eta(0) + \displaystyle\int_0^t G(s, X_s)\, dW(s) \\ \qquad\qquad + \displaystyle\int_0^t H(s, X_s)\, ds, \quad t \in [0, a] \\ \eta(t), \quad -r \le t < 0. \end{cases}
$$

$G : T \times C \to L(\mathbf{R}^d, \mathbf{R}^m)$, $\quad H : T \times C \to \mathbf{R}^m$.

Numerical Schemes

SDDE's and SFDE's cannot be solved explicitly: Need effective numerical techniques.

Numerical methods for SODE's : well developed; Kloeden and Platen, Kloeden, Platen and Schurz, McShane, Chapters 5 and 6), Hu, Talay, Protter, etc..

Cauchy-Maruyama scheme for SFDE's with continuous memory: On Delfour-Mitter state space $\mathbf{R}^m \times L^2([-r,0],\mathbf{R}^m)$ developed by Ahmed, Elsanousi and Mohammed (Ahmed, M.Sc. thesis, Khartoum 1983), Baker and Buckwar, 2000. See also [M], 1984, p. 227, and Hu-Mohammed, 1997.

Aims.

- *Strong Euler schemes* for general SFDE's. Allows for multiple delays and continuous memory. Estimates in supremum norm on $C([-r,0],\mathbf{R}^m)$ (cf. [A]).
- *Strong Milstein scheme* for SDDE's. Solution of SDDE is non-anticipating. But need methods from *anticipating* stochastic analysis and Malliavin calculus to derive Itô's formula for segment process. Itô's formula needed for convergence of Milstein scheme.

Preliminaries

Recall *segment process* X_t, $t \in [0,a]$:

$$X_t(s) = X(t+s), \quad t \in [0,a], \quad s \in [-r,0].$$

for continuous m-dimensional process $\{X(t)\}_{t\in[-r,a]}$.

$\{X_t\}$ is a C-valued or $L^2(J;\mathbf{R}^m)$-valued process.

Distinguish between finite-dimensional current state $X(t)$ *and infinite-dimensional segment* X_t, $t \in [0,a]$.

Itô SFDE:

$$X(t) = \begin{cases} \eta(0) + \int_0^t G(s,X_s)\,dW(s) \\ \qquad + \int_0^t H(s,X_s)\,ds, \quad t \in [0,a] \\ \eta(t), \quad -r \le t < 0. \end{cases}$$

Coefficients: $G : T \times C([-r,0],\mathbf{R}^m) \to L(\mathbf{R}^d;\mathbf{R}^m)$ and $H : T \times C([-r,0],\mathbf{R}^m) \to \mathbf{R}^m$.

$\{W(t) := (W^1(t),\cdots,W^d(t)) : t \geq 0\}$, d-dimensional standard Brownian motion on $(\Omega,\mathcal{F},P)$.

$(\mathcal{F}_t)_{t\geq 0}$ = Brownian filtration.

$\eta \in C([-r,0];\mathbf{R}^m)$ = random initial path independent of $\{W(t) : t \geq 0\}$.

Lipschitz Condition:

$$\|G(t,\eta) - G(t,\xi)\| + |H(t,\eta) - H(t,\xi)| \leq L\|\eta - \xi\|_C$$

for all $t \in T, \eta, \xi \in C$; $L > 0$ constant.

Boundedness Condition:

$$\sup_{0\leq t\leq a} \Big[\|G(t,0)\| + |H(t,0)|\Big] < \infty.$$

Lipschitz + bounded conditions imply SFDE has unique strong solution such that for each $q \geq 1$, there exists a constant $C = C(q, L, a) > 0$ with

$$E||X_t||_C^{2q} \leq C(1 + E||\eta||_C^{2q})$$

for all $\eta \in C, t \in [0, a]$ ([M], 1984).

Segment X_t, $t \geq 0$, is a C-valued Markov prcess.

Qualitative theory of SFDE's: [M], 1984, 1996, + references therein.

Strong versus Weak:

SFDE's do not lead to diffusions on Euclidean space. (*Highly degenerate infinite-dimensional diffusions on* C.) Hence no natural link to deterministic PDE's. Strong schemes give information on sample paths dynamics, a.s. financial option-pricing formulas with delays (Arriojas and Mohammed, 2001).

Strong Euler Scheme

Develop Euler scheme for general SFDE's (include discrete and/or continuous memory).

n, l positive integers, $T := [0, a]$, $a > 0$, $J := [-r, 0]$.

$\pi_n : t_{-l} < t_{-l+1} < \cdots < 0 = t_0 < t_1 < t_2 < \cdots < t_n = a$, partition of $[-r, a]$.

$|\pi_n| := \max_{-l \leq i \leq n-1} (t_{i+1} - t_i)$, mesh of π_n.

$X^n := X^{\pi_n}$.

SFDE:

$$X(t) = \begin{cases} \eta(0) + \displaystyle\int_0^t G(s, X_s))\, dW(s) \\ \qquad\qquad + \displaystyle\int_0^t H(s, X_s)\, ds, \quad t \in [0, a] \\ \eta(t), \quad -r \leq t < 0. \end{cases}$$

Euler scheme for SFDE:

$$X^n(t) = \begin{cases} X^n(t_i) + G(t_i, X^n_{t_i})(W(t) - W(t_i)) \\ +H(t_i, X^n_{t_i})(t - t_i),\, t \in (t_i, t_{i+1}], \quad t_i \in (0, a] \\ \eta^n(t), \qquad -r \le t \le 0 \end{cases}$$

Approx. initial path $\eta^n \in C(J, \mathbf{R}^m)$ is prescribed (e.g. a piece-wise linear approximation of η using partition points $\{t_{-l}, \cdots, t_0\}$).

Error function Z^n:

$$\begin{cases} Z^n(t) := X^n(t) - X(t), \quad 0 \le t \le a, \\ Z^n_0 := \eta^n - \eta. \end{cases}$$

Euler scheme for SFDE's has strong order of convergence 0.5 (as in SODE).

Theorem 1.

Assume that the coefficients $G : T \times C([-r,0], \mathbf{R}^m) \to L(\mathbf{R}^d; \mathbf{R}^m)$ *and* $H : T \times C([-r,0], \mathbf{R}^m) \to \mathbf{R}^m$ *in SFDE satisfy the following Lipschitz and regularity conditions:*

$$\|G(t,\eta) - G(t,\xi)\| + |H(t,\eta) - H(t,\xi)| \leq L\|\eta - \xi\|_C, \; t \in T$$

$$\sup_{0 \leq t \leq a} \left[\|G(t,0)\| + |H(t,0)|\right] < \infty$$

$$\|G(s,\eta) - G(t,\eta)\| \leq L_1(1 + \|\eta\|_C)|s-t|^\gamma, \; s,t \in T$$

$$|H(s,\eta) - H(t,\eta)| \leq L_1(1 + \|\eta\|_C)|s-t|^\gamma, \; s,t \in T$$

for all $\eta, \xi \in C([-r,0], \mathbf{R}^m)$, *where* L *and* L_1 *are positive constants. Fix any integer* $q \geq 2$. *Suppose that* $\eta : [-r,0] \to L^q(\Omega, \mathbf{R}^m)$ *is independent of*

W and Hölder continuous with exponent $\gamma \in (0,1]$, i.e., there is a positive constant K such that

$$E|\eta(s) - \eta(t)|^q \leq K|s-t|^{\gamma q}$$

for all $s, t \in [-r, 0]$. Suppose also that there is a positive constant $C' := C'(q)$ such that

$$E||\eta^n - \eta||_C^q \leq C'|\pi_n|^{\gamma q}.$$

Then there is a constant $C'' := C''(q, a) > 0$, depending on a and q, such that

$$E \sup_{0 \leq t \leq a} ||Z_t^n||_C^q \leq C''|\pi_n|^{\tilde{\gamma} q}$$

where $\tilde{\gamma} := \gamma \wedge (1/2)$.

Proof of Theorem 1.

Based on moment estimates:

$$E||X_t||_C^{2q} \leq C(1 + E||\eta||_C^{2q}), \ q \geq 1$$

for all $\eta \in C, t \in [0, a]$ ([M], 1984), and Burkholder's inequality. $\square$

Theorem 1 applies to SDDE's under Lipschitz and boundedness conditions. Also to SFDE's with mixed discrete and continuous memory:

$$X(t) = \eta(0) + \int_0^t g(s, \Pi_1(X_s), Q_1(X_s))\, dW(s)$$
$$+ \int_0^t h(s, \Pi_2(X_s), Q_2(X_s))\, ds, \qquad t \in [0, a],$$
$$X_0 = \eta \in C = C(J; \mathbf{R}^m)$$

Π_1, Π_2 two projections of discrete type;

Q_1, Q_2 two projections of continuous type:

$$Q_i(\eta) := (Q_{i,1}(\eta), \cdots, Q_{i,m_i}(\eta)), \qquad i = 1, 2,$$
$$Q_{ij}(\eta) := \int_{-1}^0 \phi_{ij}(\eta(s)) a_{ij}(s)\, ds\,, \qquad j = 1, \cdots, m_i.$$

$a_{ij} \in C^{\frac{1}{2}}(J)$, and $\phi_{ij} : \mathbf{R}^m \to \mathbf{R}$, $i = 1, 2$, $j = 1, \cdots, m_i$, satisfy Lipschitz and linear growth conditions.

Euler scheme for SFDE with mixed discrete and continuous memory:

$$
\begin{aligned}
X^n(t) = X^n(t_i) + g(t_i, \Pi_1(X^n_{t_i}), Q^n_1(X^n_{t_i}))(W(t) - W(t_i)) \\
+ h(t_i, \Pi_2(X^n_{t_i}), Q^n_2(X^n_{t_i}))(t - t_i), \quad t \in (t_i, t_{i+1}], \\
X^n(t) = \eta^n(t), \qquad -r \le t \le 0,
\end{aligned}
$$

where $Q^n_i(\eta), i = 1, 2$, are approximations of $Q_i(\eta)$ using partial sums of Riemann integral. Strong order of convergence 0.5 under Lipschitz and regularity conditions as in Theorem 1.

Example: Exact convergence rate.

One-dimensional SDDE:

$$\begin{cases} dX(t) = b(t)X(t-1)\,dW(t)\,, & 0 < t \le a \\ X(t) = \eta(t)\,, & -1 \le t \le 0. \end{cases}$$

Use partitions $\{\pi_n(h)\}$ of $[-1, a]$ generated by a continuous (strictly positive) function $h : [0, a] \to (0, \infty)$. For each integer n, choose partition points $t_{k,n} \equiv t_k$ of $\pi_n(h)$ in $[0, a]$ such that

$$t_0 = 0, \quad \int_{t_k}^{t_{k+1}} h(s)\,ds = \frac{1}{n}, \quad k = 0, 1, \cdots, n-1.$$

i.e. subdivide interval in such a way that the areas under h over each subinterval are all equal to $1/n$. Then

$$\lim_{n\to\infty, t_k \to t} n(t_{k+1} - t_k) = 1/h(t).$$

e.g. $h(t) \equiv 1 \implies (t_{k+1} - t_k) = 1/n,\ k = 0, 1, \cdots, n-1.$

Euler scheme gives

$$X^{\pi_n}(t) = \begin{cases} X^{\pi_n}(t_k) + b(t_k) X^{\pi_n}(t_k - 1)(W(t) - W(t_k)), \\ \qquad\qquad\qquad\qquad t_k \leq t < t_{k+1}, \\ \eta(t), \quad t \in J := [-1, 0], \end{cases}$$

for $0 \leq k \leq n-1$. By Theorem 1, there is a positive constant C (independent of n) such that

$$nE|X(t) - X^{\pi_n}(t)|^2 \leq C,$$

for all $n \geq 1, t \in [0, a]$. Theorem 2 (below) shows that the left hand side of the above inequality has a limit (as $n \to \infty$) satisfying a *deterministic* DDE.

Theorem 2.

Suppose $\eta \in C^\gamma(J, \mathbf{R}^m), 1/2 < \gamma \leq 1$. *Let* $a \geq 1$. *Suppose* $b : [0, a] \to \mathbf{R}$ *satisfies*

$$|b(t) - b(s)| \leq K|t - s|^{(1/2)+\alpha}$$

for all $s, t \in [0, a]$ *and some* $K, \alpha > 0$. *Let* X *be the solution of the SDDE and* X^{π_n} *its Euler approximation. Then* $\mathcal{Z}(t) := \lim_{n\to\infty} n\ E|X(t) - X^{\pi_n}(t)|^2$ *exists for each* $t \in [-1, a]$. *Furthermore,* $\mathcal{Z}(t)$ *satisfies the following deterministic linear DDE*

$$\mathcal{Z}'(t) = b^2(t)\mathcal{Z}(t-1) + b^2(t)b^2(t-1)EX^2(t-2)/h(t),\ 1 < t < a,$$

$$\mathcal{Z}(t) = 0, \qquad -1 \leq t \leq 1,$$

where $EX^2(t)$ *is given by the integral equation*

$$EX^2(t) = \begin{cases} \eta(0)^2 + \int_0^t b^2(s)EX^2(s-1)\,ds, & t \in [0, a], \\ \eta(t)^2, & t \in [-1, 0). \end{cases}$$

Milstein Scheme

Strong second order scheme for SDDE:

$$X(t) = \begin{cases} \eta(0) + \int_0^t g(s, \Pi_1(X_s))\, dW(s) \\ \qquad\quad + \int_0^t h(s, \Pi_2(X_s))\, ds,\ t \in T := [0, a] \\ \eta(t), \qquad -r \leq t < 0. \end{cases}$$

$$g : T \times \mathbf{R}^{mk_1} \to L(\mathbf{R}^d, \mathbf{R}^m), \qquad h : T \times \mathbf{R}^{mk_2} \to \mathbf{R}^m.$$

Requires infinite-dimensional Itô formula for "tame" functions of segments of semimartingales or (anticipating) processes. Proof based on Nualart-Pardoux anticipating calculus techniques.

$l, n =$ positive integers, $T := [0, a]$, $a > 0$, $J := [-r, 0]$.

Partitions: $\pi_n := \{t_i : -l \leq i \leq n\}$ of $[-r, a]$, mesh $|\pi_n|$. $\quad X^n := X^{\pi_n}$; $\quad (x_{i_1 j_1}) \in \mathbf{R}^{mk_1}$.

Milstein approximations for SDDE:

$$X^{i,n}(t) = X^{i,n}(t_k) + h^i(t_k, \Pi_2(X^n_{t_k}))(t - t_k)$$
$$+ \sum_j g^{ij}(t_k, \Pi_1(X^n_{t_k}))(W^j(t) - W^j(t_k))$$
$$+ \sum_{i_1, j_1, j} \frac{\partial g^{ij}}{\partial x_{i_1 j_1}}(t_k, \Pi_1(X^n_{t_k}))g^{i_1 j_1}(t_k + s_{1,j_1}, \Pi_1(X^n_{t_k + s_{1,j_1}}))\times$$
$$\times 1_{[0,T]}(t_k + s_{1,j_1}) \times I_{j,j_1}(t_k + s_{1,j_1}, t + s_{1,j_1}; s_{1,j_1}),$$

for $t_k < t \leq t_{k+1}$, $i, i_1 = 1, 2, \cdots, m$, $1 \leq j \leq d$, $1 \leq j_1 \leq k_1$, where

$$I_{j,j_1}(t_k + s_{1,j_1}, t + s_{1,j_1}; s_{1,j_1}) := \int_{t_k}^{t} \int_{t_k + s_{1,j_1}}^{t_1 + s_{i,j_1}} \circ dW^j(t_2) \circ dW^{j_1}(t_1).$$

X^i, h^i, $g^{ij} =$ coordinates of X, h and g with respect to standard bases in Euclidean space.

Milstein scheme has strong order of convergence 1.

Theorem 3.

Consider the Milstein scheme for the SDDE. Let $0 < \gamma \leq 1$. *Suppose that* $\eta : [-r, 0] \to L^2(\Omega, \mathbf{R}^m)$ *is Hölder continuous with exponent* $\frac{\gamma}{2}$, *i.e. there is a positive constant* K *such that*

$$E|\eta(s) - \eta(t)|^2 \leq K|s - t|^\gamma$$

for all $s, t \in J$. *Suppose that* $g \in C^{1,2}(T \times \mathbf{R}^{mk_1}, L(\mathbf{R}^d, \mathbf{R}^m)$, $h \in C^{1,2}(T \times \mathbf{R}^{mk_2}, \mathbf{R}^m)$ *and have bounded first and second spatial derivatives. Let*

$$\begin{cases} Z^n(t) := X^n(t) - X(t), & 0 \leq t \leq a, \\ Z_0^n := \eta_0^n - \eta. \end{cases}$$

Assume that

$$\sup_{-r\le s\le 0} E(|Z^n(s)|^2) \le C'|\pi_n|^{2\gamma}$$

for some positive constant C'*. Then there exists a constant* $C > 0$ *(depending on* a *and independent of* π_n*) such that*

$$\sup_{-r\le t\le a} E|Z^n(t)|^2 \le C|\pi_n|^{2\gamma}$$

for any $n \ge 1$*.*

Surprise! Proof requires use of *anticipating* calculus techniques:

Example:

One-dimensional SDDE:

$$\begin{aligned} dX(t) &= g(X(t-1), X(t))\, dW(t), \quad t \geq 0 \\ X(t) &= W(t), \quad -1 \leq t < 0. \end{aligned}$$

$g : \mathbf{R}^2 \to \mathbf{R}$ smooth function. For second-order scheme, formally seek a stochastic differential of the coefficient $g(X(t-1), X(t))$ on RHS of SDDE.

For $t \in (0,1]$, formally expect something like:

$$
\begin{aligned}
&dg(W(t-1), W(t)) \\
&= \frac{\partial g}{\partial x_2}(W(t-1), W(t))\, dW(t) \\
&+ \frac{\partial g}{\partial x_1}(W(t-1), W(t))\, dW(t-1) \;\; (\textit{anticipating!}) \\
&+ \frac{1}{2}\left(\frac{\partial^2 g}{\partial x_1^2}(W(t-1), W(t))\, dt + \frac{\partial^2 g}{\partial x_2^2}(W(t-1), W(t))\, dt\right) \\
&+ \frac{1}{2}\frac{\partial^2 g}{\partial x_1 \partial x_2}(W(t-1), W(t))\, dW(t-1)\, dW(t) (= 0!)
\end{aligned}
$$

- *LHS is adapted but anticipating integral on RHS.*
- $(\mathcal{F}_t)_{0 \le t \le 1}$-adapted process

  $$[0,1] \ni t \to (X(t-1), X(t)) \in \mathbf{R}^2$$

  is not a semimartingale with respect to any natural filtration.

- The components $X(t-1)$ and $X(t)$ are not independent. Existing anticipating versions of Itô's formula do not apply (cf. [AN], [AP] and [NP]). Hence need new Itô formula for tame functions:

$$g(W(t-1), W(t)) = g(W_t(-1), W_t(0)).$$

- Last second-order Itô integral on RHS is zero:

Proof.

$(\Omega, \mathcal{F}, (\mathcal{F}_t), P) :=$ filtered probability space.

$\pi := \{t_i\}$ any partition of $[0, T]$, f any $(\mathcal{F}_t)$-adapted (a.s. bounded) process on $[0, T]$. Then

$$\int_0^T f(t)\, dW(t-1)\, dW(t) = \lim_{|\pi| \to 0} \sum_i f(t_i) \Delta_i W(\cdot - 1) \Delta_i W$$

where

$$\Delta_i W := W(t_{i+1}) - W(t_i),$$

$$\Delta_i W(\cdot - 1) := W(t_{i+1} - 1) - W(t_i - 1)$$

$$E|\sum_i f(t_i)\Delta_i W(\cdot - 1)\Delta_i W|^2 = \sum_{i,j} EX_{i,j}$$

$$X_{i,j} := f(t_i)f(t_j)\Delta_i W(\cdot - 1)\Delta_j W(\cdot - 1)\Delta_i W \Delta_j W$$

For $i < j$,

$$E(X_{i,j}) = E\{E(X_{i,j}|\mathcal{F}_{t_j})\}$$

and

$$E(X_{i,j}|\mathcal{F}_{t_j})$$

$$= f(t_i)f(t_j)\Delta_i W(\cdot - 1)\Delta_j W(\cdot - 1)\Delta_i W \cdot E(\Delta_j W|\mathcal{F}_{t_j})$$

$$= 0$$

By symmetry,

$$\begin{aligned}\sum_{i,j} EX_{i,j} &= \sum_i EX_{i,i}\\ &= \sum_i Ef(t_i)^2[\Delta_i W(\cdot - 1)]^2[\Delta_i W]^2\\ &\leq K\sum_i E[\Delta_i W(\cdot - 1)]^2 \cdot E[\Delta_i W]^2\\ &\leq KT|\pi|\end{aligned}$$

Hence

$$E\left|\int_0^T f(t)\, dW(t-1)\, dW(t)\right|^2 = \lim_{|\pi|\to 0}\sum_{i,j} EX_{i,j} = 0. \quad \square$$

Shorthand:

	$dW(t-1)$	$dW(t)$
$dW(t-1)$	dt	0
$dW(t)$	0	dt

Proof.

Exercise.

Projection $\Pi : C \to \mathbf{R}^{mk}$ associated with $s_1, \cdots, s_k \in [-r, 0]$:

$$\Pi(\eta) := (\eta(s_1), \cdots, \eta(s_k)) \in \mathbf{R}^{mk}, \quad \eta \in C$$

Definition.

$\Phi \in C(T \times C(J; \mathbf{R}^m); \mathbf{R})$ is *tame* if there exist $\phi \in C(T \times \mathbf{R}^{mk}, \mathbf{R})$ and a projection Π such that

$$\Phi(t, \eta) = \phi(t, \Pi(\eta)).$$

for all $t \in T$ and $\eta \in C$.

Proof (Milstein Scheme).

Itô's formula for "tame" functionals

$$T \times C(J, \mathbf{R}^m) \to \mathbf{R}$$

of the segment X_t. Use formula + moment estimates on weak derivatives of X to get global error estimate for the Milstein approximations. $\square$

$W(t) := (W^1(t), \cdots, W^d(t))$, $t \geq 0$:= d-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, P)$.

$D := (D_1, \cdots, D_d)$:= Malliavin differentiation operator associated with $\{W(t) : t \geq 0\}$.

Pathwise-continuous process:

$$X(t) := \begin{cases} \eta(0) + \int_0^t u(s)\, dW(s) + \int_0^t v(s)\, ds, & t > 0, \\ \eta(t), & -r \leq t \leq 0, \end{cases}$$

Skorohod integral. $\eta \in C$, BV.

$u = (u^1, \cdots, u^m)^T$, $u^i \in \mathbb{L}^{2,4}_{d,loc}$;

$v = (v^1, \cdots, v^m)^T$, $v^i \in \mathbb{L}^{1,4}_{loc}$ ([Nualart]).

u and v may not be adapted to the Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$. Set $u(t) := 0$ for $t < 0$ or $t > a$,

$$v(t) := \begin{cases} 0, & t > a \\ \eta'(t), & -r \leq t \leq 0. \end{cases}$$

$W(t) := 0$ if $t < 0$ or $t > a$.

$$U(t) := \int_0^t u(s)\, dW(s),\ V(t) := \begin{cases} \eta(0) + \int_0^t v(s)\, ds, & t > 0 \\ \eta(t), & -r \leq t \leq 0. \end{cases}$$

Then

$$D_s X(t) = u(s) 1_{[0,a]}(t-s) + D_s \eta(0) + \int_0^t D_s v(r')\, dr' + \int_0^t D_s u(r')\, dW(r'), \quad t > 0$$

$\Pi :=$ projection associated with $s_1, \cdots, s_k \in J$.

Cannot apply multi-dimensional Itô formula to $\phi(t, \Pi(X_t))$ because $\Pi(U_t)$ is of the form

$$\left(\int_0^t u(s+s_1)\, dW(s+s_1), \cdots, \int_0^t u(s+s_k)\, dW(s+s_k) \right),$$

and the components $(W(t+s_1), \cdots, W(t+s_k))$ are not independent. Use anticipating calculus (Nualart-Pardoux) to derive an Itô formula for $\phi(t, \Pi(X_t))$.

Assume $\phi \in C^{1,2}(T \times \mathbf{R}^{mk})$, $\vec{x} = (\vec{x}_1, \cdots, \vec{x}_m)$, $\vec{x}_i = (x_{i1}, \cdots, x_{ik}) \in \mathbf{R}^k$. Write

$$\phi(t, \vec{x}) = \phi(t, \vec{x}_1, \cdots, \vec{x}_m).$$

Theorem 4. *(Itô's formula).*

Suppose X *satisfies above conditions and let* $\phi \in C^{1,2}(T \times \mathbf{R}^{mk}, \mathbf{R})$. *Then*

$$\phi(t, \Pi(X_t)) - \phi(0, \Pi(X_0))$$
$$= \int_0^t \frac{\partial \phi}{\partial s}(s, \Pi(X_s))\, ds + \int_0^t \frac{\partial \phi}{\partial \vec{x}}(s, \Pi(X_s))\, d(\Pi(X_s)) +$$
$$\frac{1}{2} \sum_{i,j=1}^{k} \sum_{i_1, j_1=1}^{m} \int_0^t \frac{\partial^2 \phi}{\partial x_{i,i_1} \partial x_{j,j_1}}(s, \Pi(X_s)) u^{i_1}(s+s_i) D_{s+s_i} X^{j_1}(s+s_j)\, ds$$

a.s. for all $t \in T$.

Example (Revisited)

$$
\begin{aligned}
g(W(t-1), W(t)) - g(W(-1), W(0)) &= \int_0^t \frac{\partial g}{\partial x_1}(W(s-1), W(s))\, dW(s) \\
&+ \int_0^t \frac{\partial g}{\partial x_2}(W(s-1), W(s))\, dW(s-1) \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x_1^2}(W(s-1), W(s)) ds \\
&+ \frac{1}{2}\int_0^t \frac{\partial^2 g}{\partial x_2^2}(W(s-1), W(s)) 1_{(1,\infty)}(s)\, ds
\end{aligned}
$$

for $t > 0$.

Weak differentiability of solutions of SDDE's.

Cf. Bell and Mohammed, Nualart.

$\mathbb{D}_m^{k,\infty} := \cap_{p\geq 2}\mathbb{D}_m^{k,p}, \ k \in N.$

$D_u^l, 1 \leq l \leq d$, $0 \leq u \leq a$, weak differentiation with respect to l-th component of W.

Proposition.

In the Itô SDDE, assume that $g \in C_b^{0,1}(T\times\mathbf{R}^{k_1 m}; L(\mathbf{R}^d, \mathbf{R}^m)$ *and* $h \in C_b^{0,1}(T \times \mathbf{R}^{k_2 m}; \mathbf{R}^m)$. *Let* X *be the solution of the SDDE. Then* $X(t) \in \mathbb{D}_m^{1,\infty}$ *for all* $t \in T$, *and*

$$\sup_{0\leq u\leq a} E(\sup_{u\leq s\leq a} |D_u X(s)|^p) < \infty$$

for all $p \geq 2$. *Furthermore, the "partial" weak derivatives* $D_u^l X^j(t)$ *with respect to the* l*-th coordinate of* W *satisfy* *the following linear SDDE's a.s.:*

$$
\begin{aligned}
D_u^l X^j(t) = g^{jl}(u, \Pi_1(X_u^j)) + \\
&\int_u^t \sum_{i=1}^{k_1} \frac{\partial g^{jl}}{\partial \vec{x}_i}(s, \Pi_1(X_s)) D_u^l X^j(s + s_{1,i})\, dW^l(s) \\
&+ \int_0^t \sum_{i=1}^{k_2} \frac{\partial h^j}{\partial \vec{x}_i}(s, \Pi_2(X_s)) D_u^l X^j(s + s_{2,i})\, ds, \quad t \geq u, \\
&= 0, \qquad t < u,\ l = 1, \cdots, d,\ j = 1, \cdots, m
\end{aligned}
$$

$g^{jl} = (j, l)$ *entry of the* $m \times d$ *matrix* g,

$h^j = j$*-th coordinate of* h.

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