

I. EXISTENCE

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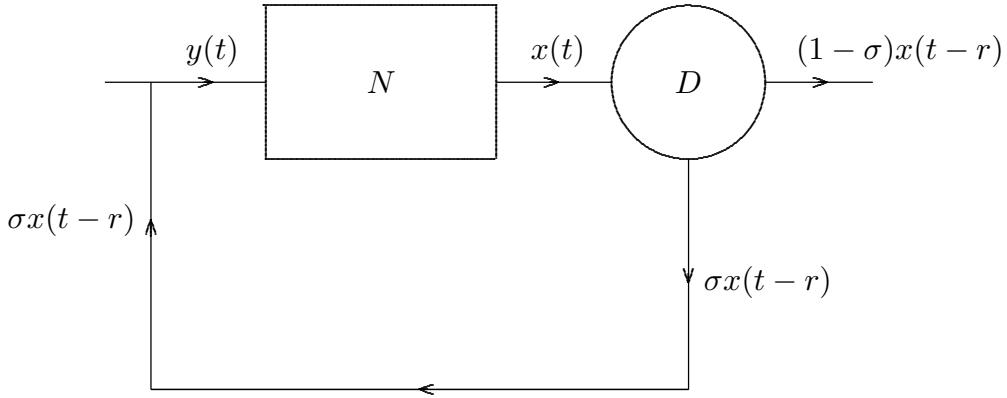
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I. EXISTENCE

1. Examples

Example 1. (*Noisy Feedbacks*)



Box N: Input = $y(t)$, output = $x(t)$ at time $t > 0$ related by

$$x(t) = x(0) + \int_0^t y(u) dZ(u) \quad (1)$$

where $Z(u)$ is a semimartingale noise.

Box D: Delays signal $x(t)$ by $r (> 0)$ units of time. A proportion σ ($0 \leq \sigma \leq 1$) is transmitted through D and the rest $(1 - \sigma)$ is used for other purposes.

Therefore

$$y(t) = \sigma x(t - r)$$

Take $\dot{Z}(u) :=$ white noise = $\dot{W}(u)$

Then substituting in (1) gives the Itô integral equation

$$x(t) = x(0) + \sigma \int_0^t x(u - r) dW(u)$$

or the stochastic differential delay equation (sdde):

$$dx(t) = \sigma x(t-r)dW(t), \quad t > 0 \quad (I)$$

To solve (I), need an *initial process* $\theta(t)$, $-r \leq t \leq 0$:

$$x(t) = \theta(t) \quad \text{a.s.,} \quad -r \leq t \leq 0$$

r = 0: (I) becomes a linear stochastic ode and has closed form solution

$$x(t) = x(0)e^{\sigma W(t) - \frac{\sigma^2 t}{2}}, \quad t \geq 0.$$

r>0: Solve (I) by successive Itô integrations over steps of length r :

$$\begin{aligned} x(t) &= \theta(0) + \sigma \int_0^t \theta(u-r) dW(u), \quad 0 \leq t \leq r \\ x(t) &= x(r) + \sigma \int_r^t [\theta(0) + \sigma \int_0^{(v-r)} \theta(u-r) dW(u)] dW(v), \quad r < t \leq 2r, \\ \dots &= \dots \quad 2r < t \leq 3r, \end{aligned}$$

No closed form solution is known (even in deterministic case).

Curious Fact!

In the sdde (I) the Itô differential dW may be replaced by the Stratonovich differential $\circ dW$ *without changing the solution x*. Let x be the solution of (I) under an Itô differential dW . Then using finite partitions $\{u_k\}$ of the interval $[0, t]$:

$$\int_0^t x(u-r) \circ dW(t) = \lim \sum_k \frac{1}{2} [x(u_k - r) + x(u_{k+1} - r)][W(u_{k+1}) - W(u_k)]$$

where the limit in probability is taken as the mesh of the partition $\{u_k\}$ goes to zero. Compare the Stratonovich and Itô integrals using the corresponding partial sums:

$$\begin{aligned}
& \lim E \left(\sum_k \frac{1}{2} [x(u_k - r) + x(u_{k+1} - r)][W(u_{k+1}) - W(u_k)] \right. \\
& \quad \left. - \sum_k [x(u_k - r)][W(u_{k+1}) - W(u_k)] \right)^2 \\
& = \lim E \left(\sum_k \frac{1}{2} [x(u_{k+1} - r) - x(u_k - r)][W(u_{k+1}) - W(u_k)] \right)^2 \\
& = \lim \sum_k \frac{1}{4} E[x(u_{k+1} - r) - x(u_k - r)]^2 E[W(u_{k+1}) - W(u_k)]^2 \\
& = \lim \sum_k \frac{1}{4} E[x(u_{k+1} - r) - x(u_k - r)]^2 (u_{k+1} - u_k) \\
& = 0
\end{aligned}$$

because W has independent increments, x is adapted to the Brownian filtration, $u \mapsto x(u) \in L^2(\Omega, \mathbf{R})$ is continuous, and the delay r is positive. Alternatively

$$\int_0^t x(u - r) \circ dW(u) = \int_0^t x(u - r) dW(u) + \frac{1}{2} \langle x(\cdot - r, W) \rangle(t)$$

and $\langle x(\cdot - r, W) \rangle(t) = 0$ for all $t > 0$.

Remark.

When $r > 0$, the solution process $\{x(t) : t \geq -r\}$ of (I) is a martingale but is *non-Markov*.

Example 2. (Simple Population Growth)

Consider a large population $x(t)$ at time t evolving with a constant birth rate $\beta > 0$ and a constant death rate α per capita. Assume immediate removal of the dead from the population. Let $r > 0$ (fixed,

non-random= 9, e.g.) be the development period of each individual and assume there is migration whose overall rate is distributed like white noise $\sigma \dot{W}$ (mean zero and variance $\sigma > 0$), where W is one-dimensional standard Brownian motion. The change in population $\Delta x(t)$ over a small time interval $(t, t + \Delta t)$ is

$$\Delta x(t) = -\alpha x(t)\Delta t + \beta x(t-r)\Delta t + \sigma \dot{W}\Delta t$$

Letting $\Delta t \rightarrow 0$ and using Itô stochastic differentials,

$$dx(t) = \{-\alpha x(t) + \beta x(t-r)\} dt + \sigma dW(t), \quad t > 0. \quad (II)$$

Associate with the above affine sdde the initial condition $(v, \eta) \in \mathbf{R} \times L^2([-r, 0], \mathbf{R})$

$$x(0) = v, \quad x(s) = \eta(s), \quad -r \leq s < 0.$$

Denote by $M_2 = \mathbf{R} \times L^2([-r, 0], \mathbf{R})$ the Delfour-Mitter Hilbert space of all pairs (v, η) , $v \in \mathbf{R}$, $\eta \in L^2([-r, 0], \mathbf{R})$ with norm

$$\|(v, \eta)\|_{M_2} = \left(|v|^2 + \int_{-r}^0 |\eta(s)|^2 ds \right)^{1/2}.$$

Let $W : \mathbf{R}^+ \times \Omega \rightarrow \mathbf{R}$ be defined on the canonical filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbf{R}^+}, P)$ where

$$\Omega = C(\mathbf{R}^+, \mathbf{R}), \quad \mathcal{F} = \text{Borel } \Omega, \quad \mathcal{F}_t = \sigma\{\rho_u : u \leq t\}$$

$\rho_u : \Omega \rightarrow \mathbf{R}$, $u \in \mathbf{R}^+$, are evaluation maps $\omega \mapsto \omega(u)$, and P = Wiener measure on Ω .

Example 3. (*Logistic Population Growth*)

A single population $x(t)$ at time t evolving logistically with *development (incubation) period* $r > 0$ under Gaussian type noise (e.g. migration on a molecular level):

$$\dot{x}(t) = [\alpha - \beta x(t-r)] x(t) + \gamma x(t) \dot{W}(t), \quad t > 0$$

i.e.

$$dx(t) = [\alpha - \beta x(t-r)] x(t) dt + \gamma x(t) dW(t) \quad t > 0. \quad (III)$$

with *initial condition*

$$x(t) = \theta(t) \quad -r \leq t \leq 0.$$

For positive delay r the above sdde can be solved *implicitly* using forward steps of length r , i.e. for $0 \leq t \leq r$, $x(t)$ satisfies the *linear* sode (without delay)

$$dx(t) = [\alpha - \beta \theta(t-r)] x(t) dt + \gamma x(t) dW(t) \quad 0 < t \leq r. \quad (III')$$

$x(t)$ is a semimartingale and is non-Markov (Scheutzow [S], 1984).

Example 4. (Heat bath)

Model proposed by R. Kubo (1966) for physical Brownian motion. A molecule of mass m moving under random gas forces with position $\xi(t)$ and velocity $v(t)$ at time t ; cf classical work by Einstein and Ornstein and Uhlenbeck. Kubo proposed the following modification of the Ornstein-Uhenbeck process

$$\left. \begin{aligned} d\xi(t) &= v(t) dt \\ mdv(t) &= -m \left[\int_{t_0}^t \beta(t-t') v(t') dt' \right] dt + \gamma(\xi(t), v(t)) dW(t), \end{aligned} \right\} \quad t > t_0. \quad (IV)$$

m = mass of molecule. No external forces.

β = viscosity coefficient function with compact support.

γ a function $\mathbf{R}^3 \times \mathbf{R}^3 \rightarrow \mathbf{R}$ representing the random gas forces on the molecule.

$\xi(t)$ = position of molecule $\in \mathbf{R}^3$.

$v(t)$ = velocity of molecule $\in \mathbf{R}^3$.

W = 3- dimensional Brownian motion.

([Mo], Pitman Books, RN # 99, 1984, pp. 223-226).

Further Examples

Delay equation with Poisson noise:

$$\left. \begin{aligned} dx(t) &= x((t-r)-) dN(t) & t > 0 \\ x_0 &= \eta \in D([-r, 0], \mathbf{R}) \end{aligned} \right\} \quad (V)$$

$N :=$ Poisson process with iid interarrival times ([S], Hab. 1988).
 $D([-r, 0], \mathbf{R}) =$ space of all cadlag paths $[-r, 0] \rightarrow \mathbf{R}$, with sup norm.

Simple model of dye circulation in the blood (or pollution) (cf. Bailey and Williams [B-W], JMAA, 1966, Lenhart and Travis ([L-T], PAMS, 1986)).

$$\left. \begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r)\} dt + \sigma x(t) dW(t) & t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R} \times L^2([-r, 0], \mathbf{R}), \end{aligned} \right\} \quad (VI)$$

([Mo], Survey, 1992; [M-S], II, 1995.)

In above model:

$x(t) :=$ dye concentration (gm/cc)

$r =$ time taken by blood to traverse side tube (vessel)

Flow rate (cc/sec) is Gaussian with variance σ .

A fixed proportion of blood in main vessel is pumped into side vessel(s). Model will be analysed in Lecture V (Theorem V.5).

$$\left. \begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r)\} dt + \left\{ \int_{-r}^0 x(t+s) \sigma(s) ds \right\} dW(t), \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R} \times L^2([-r, 0], \mathbf{R}), t > 0. \end{aligned} \right\} \quad (VII)$$

([Mo], Survey, 1992; [M-S], II, 1995.)

Linear d -dimensional systems driven by m -dimensional Brownian motion $W := (W_1, \dots, W_m)$ with constant coefficients.

$$\left. \begin{aligned} dx(t) &= H(x(t-d_1), \dots, x(t-d_N), x(t), x_t) dt \\ &\quad + \sum_{i=1}^m g_i x(t) dW_i(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (VIII)$$

$H := (\mathbf{R}^d)^N \times M_2 \rightarrow \mathbf{R}^d$ linear functional on $(\mathbf{R}^d)^N \times M_2$; g_i $d \times d$ -matrices ([Mo], Stochastics, 1990).

Linear systems driven by (helix) semimartingale noise (N, L) , and memory driven by a (stationary) measure-valued process ν and a (stationary) process K ([M-S], I, AIHP, 1996):

$$\left. \begin{aligned} dx(t) &= \left\{ \int_{[-r, 0]} \nu(t)(ds) x(t+s) \right\} dt \\ &\quad + dN(t) \int_{-r}^0 K(t)(s) x(t+s) ds + dL(t) x(t-), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 = \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (IX)$$

Multidimensional affine systems driven by (helix) noise Q ([M-S], Stochastics, 1990):

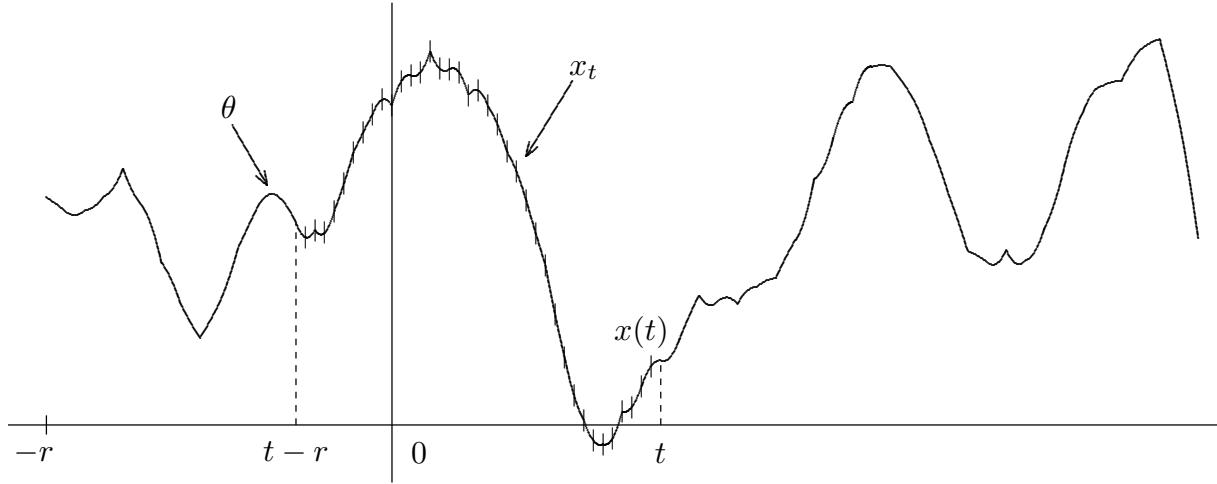
$$\left. \begin{aligned} dx(t) &= \left\{ \int_{[-r, 0]} \nu(t)(ds) x(t+s) \right\} dt + dQ(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d) \end{aligned} \right\} \quad (X)$$

Memory driven by white noise:

$$\left. \begin{aligned} dx(t) &= \left\{ \int_{[-r,0]} x(t+s) dW(s) \right\} dW(t) & t > 0 \\ x(0) &= v \in \mathbf{R}, \quad x(s) = \eta(s), & -r < s < 0, \quad r \geq 0 \end{aligned} \right\} \quad (XI)$$

([Mo], Survey, 1992).

Formulation



Slice each solution path x over the interval $[t - r, t]$ to get *segment* x_t as a process on $[-r, 0]$:

$$x_t(s) := x(t + s) \quad \text{a.s., } t \geq 0, s \in J := [-r, 0].$$

Therefore sdde's (I), (II), (III) and (XI) become

$$\left. \begin{aligned} dx(t) &= \sigma x_t(-r) dW(t), \quad t > 0 \\ x_0 &= \theta \in C([-r, 0], \mathbf{R}) \end{aligned} \right\} \quad (I)$$

$$\left. \begin{aligned} dx(t) &= \{-\alpha x(t) + \beta x_t(-r)\} dt + \sigma dW(t), \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in \mathbf{R} \times L^2([-r, 0], \mathbf{R}) \end{aligned} \right\} \quad (II)$$

$$\left. \begin{aligned} dx(t) &= [\alpha - \beta x_t(-r)]x_t(0) dt + \gamma x_t(0) dW(t) \\ x_0 &= \theta \in C([-r, 0], \mathbf{R}) \end{aligned} \right\} \quad (III)$$

$$\left. \begin{aligned} dx(t) &= \left\{ \int_{[-r, 0]} x_t(s) dW(s) \right\} dW(t) \quad t > 0 \\ (x(0), x_0) &= (v, \eta) \in \mathbf{R} \times L^2([-r, 0], \mathbf{R}), \quad r \geq 0 \end{aligned} \right\} \quad (XI)$$

Think of R.H.S.'s of the above equations as functionals of x_t (and $x(t)$) and generalize to *stochastic functional differential equation* (sfde)

$$\left. \begin{aligned} dx(t) &= h(t, x_t)dt + g(t, x_t)dW(t) \quad t > 0 \\ x_0 &= \theta \end{aligned} \right\} \quad (XII)$$

on filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual conditions:

$(\mathcal{F}_t)_{t \geq 0}$ right-continuous and each \mathcal{F}_t contains all P -null sets in \mathcal{F} .

$C := C([-r, 0], \mathbf{R}^d)$ Banach space, sup norm.

$W(t) = m$ -dimensional Brownian motion.

$L^2(\Omega, C) :=$ Banach space of all $(\mathcal{F}, \text{Borel } C)$ -measurable L^2 (Bochner sense) maps $\Omega \rightarrow C$ with the L^2 -norm

$$\|\theta\|_{L^2(\Omega, C)} := \left[\int_{\Omega} \|\theta(\omega)\|_C^2 dP(\omega) \right]^{1/2}$$

Coefficients:

$$h : [0, T] \times L^2(\Omega, C) \rightarrow L^2(\Omega, \mathbf{R}^d) \quad (\text{Drift})$$

$$g : [0, T] \times L^2(\Omega, C) \rightarrow L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d)) \quad (\text{Diffusion}).$$

Initial data:

$$\theta \in L^2(\Omega, C, \mathcal{F}_0).$$

Solution:

$x : [-r, T] \times \Omega \rightarrow \mathbf{R}^d$ measurable and sample-continuous, $x|[0, T]$ $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted and $x(s)$ is \mathcal{F}_0 -measurable for all $s \in [-r, 0]$.

Exercise: $[0, T] \ni t \mapsto x_t \in C([-r, 0], \mathbf{R}^d)$ is $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted.

(Hint: Borel C is generated by all evaluations.)

Hypotheses (E_1).

- (i) h, g are jointly continuous and uniformly Lipschitz in the second variable with respect to the first:

$$\|h(t, \psi_1) - h(t, \psi_2)\|_{L^2(\Omega, \mathbf{R}^d)} \leq L\|\psi_1 - \psi_2\|_{L^2(\Omega, C)}$$

for all $t \in [0, T]$ and $\psi_1, \psi_2 \in L^2(\Omega, C)$. Similarly for the diffusion coefficient g .

- (ii) For each $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted process $y : [0, T] \rightarrow L^2(\Omega, C)$,

the processes $h(\cdot, y(\cdot)), g(\cdot, y(\cdot))$ are also $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted.

Theorem I.1. ([Mo], 1984) (Existence and Uniqueness).

Suppose h and g satisfy Hypotheses (E_1). Let $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

Then the sfde (XII) has a unique solution ${}^\theta x : [-r, \infty) \times \Omega \rightarrow \mathbf{R}^d$ starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ with $t \mapsto {}^\theta x_t$ continuous and ${}^\theta x \in L^2(\Omega, C([-r, T]; \mathbf{R}^d))$ for all $T > 0$. For a given θ , uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted processes in $L^2(\Omega, C([-r, T], \mathbf{R}^d))$.

Proof.

[Mo], Pitman Books, 1984, Theorem 2.1, pp. 36-39. \square

Theorem I.1 covers equations (I), (II), (IV), (VI), (VII), (VIII), (XI) and a large class of sfde's driven by white noise. Note that (XI) *does not satisfy the hypotheses underlying the classical results* of Doleans-Dade [Dol], 1976, Metivier and Pellaumail [Met-P], 1980, Protter, Ann. Prob. 1987, Lipster and Shirayev [Lip-Sh], [Met], 1982. This is because the coefficient

$$\eta \rightarrow \int_{-r}^0 \eta(s) dW(s)$$

on the RHS of (XI) *does not admit almost surely Lipschitz (or even linear) versions $C \rightarrow \mathbf{R}$* . This will be shown later.

When the coefficients h, g factor through functionals

$$H : [0, T] \times C \rightarrow \mathbf{R}^d, \quad G : [0, T] \times C \rightarrow \mathbf{R}^{d \times m}$$

we can impose the following local Lipschitz and global linear growth conditions on the sfde

$$\left. \begin{aligned} dx(t) &= H(t, x_t) dt + G(t, x_t) dW(t) & t > 0 \\ x_0 &= \theta \end{aligned} \right\} \quad (XIII)$$

with W m -dimensional Brownian motion:

Hypotheses (E_2)

- (i) H, G are Lipschitz on bounded sets in C : For each integer $n \geq 1$ there exists $L_n > 0$ such that

$$|H(t, \eta_1) - H(t, \eta_2)| \leq L_n \|\eta_1 - \eta_2\|_C$$

for all $t \in [0, T]$ and $\eta_1, \eta_2 \in C$ with $\|\eta_1\|_C \leq n$, $\|\eta_2\|_C \leq n$. Similarly for the diffusion coefficient G .

- (ii) There is a constant $K > 0$ such that

$$|H(t, \eta)| + \|G(t, \eta)\| \leq K(1 + \|\eta\|_C)$$

for all $t \in [0, T]$ and $\eta \in C$.

Note that the adaptability condition is not needed (explicitly) because H, G are deterministic and because the sample-continuity and adaptability of x imply that the segment $[0, T] \ni t \mapsto x_t \in C$ is also adapted.

Exercise: Formulate the heat-bath model (IV) as a sfde of the form (XIII). (β has compact support in \mathbf{R}^+ .)

Theorem I.2. ([Mo], 1984) (Existence and Uniqueness).

Suppose H and G satisfy Hypotheses (E_2) and let $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

Then the sfde (XIII) has a unique $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted solution ${}^\theta x : [-r, T] \times \Omega \rightarrow \mathbf{R}^d$ starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ with $t \mapsto {}^\theta x_t$ continuous and ${}^\theta x \in L^2(\Omega, C([-r, T], \mathbf{R}^d))$ for all $T > 0$. For a given θ , uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0 \leq t \leq T}$ -adapted processes in $L^2(\Omega, C([-r, T], \mathbf{R}^d))$.

Furthermore if $\theta \in L^{2k}(\Omega, C; \mathcal{F}_0)$, then ${}^\theta x_t \in L^{2k}(\Omega, C; \mathcal{F}_t)$ and

$$E \|{}^\theta x_t\|_C^{2k} \leq C_k [1 + \|\theta\|_{L^{2k}(\Omega, C)}^{2k}]$$

for all $t \in [0, T]$ and some positive constants C_k .

Proofs of Theorems I.1, I.2.(Outline)

[Mo], pp. 150-152. Generalize some proofs in Gihman and Skorohod ([G-S], 1973) or Friedman ([Fr], 1975):

- (1) Truncate coefficients outside bounded sets in C . Reduce to globally Lipschitz case.
- (2) Successive approx. in globally Lipschitz situation.
- (3) Use local uniqueness ([Mo], Theorem 4.2, p. 151) to “patch up” solutions of the truncated sfde’s.

For (2) consider globally Lipschitz case and $h \equiv 0$.

We look for solutions of (XII) by successive approximation in $L^2(\Omega, C([-r, a], \mathbf{R}^d))$. Let $J := [-r, 0]$.

Suppose $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$ is \mathcal{F}_0 -measurable. Note that this is equivalent to saying that $\theta(\cdot)(s)$ is \mathcal{F}_0 -measurable for all $s \in J$, because θ has a.a. sample paths continuous.

We prove by induction that there is a sequence of processes ${}^k x : [-r, a] \times \Omega \rightarrow \mathbf{R}^d$, $k = 1, 2, \dots$ having the

Properties P(k):

(i) ${}^k x \in L^2(\Omega, C([-r, a], \mathbf{R}^d))$ and is adapted to $(\mathcal{F}_t)_{t \in [0, a]}$.

(ii) For each $t \in [0, a]$, ${}^k x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is \mathcal{F}_t -measurable.

(iii)

$$\left. \begin{aligned} \|{}^{k+1}x - {}^k x\|_{L^2(\Omega, C)} &\leq (ML^2)^{k-1} \frac{a^{k-1}}{(k-1)!} \|{}^2 x - {}^1 x\|_{L^2(\Omega, C)} \\ \|{}^{k+1}x_t - {}^k x_t\|_{L^2(\Omega, C)} &\leq (ML^2)^{k-1} \frac{t^{k-1}}{(k-1)!} \|{}^2 x - {}^1 x\|_{L^2(\Omega, C)} \end{aligned} \right\} \quad (1)$$

where M is a “martingale” constant and L is the Lipschitz constant of g .

Take ${}^1 x : [-r, a] \times \Omega \rightarrow \mathbf{R}^d$ to be

$${}^1 x(t, \omega) = \begin{cases} \theta(\omega)(0) & t \in [0, a] \\ \theta(\omega)(t) & t \in J \end{cases}$$

a.s., and

$${}^{k+1} x(t, \omega) = \begin{cases} \theta(\omega)(0) + (\omega) \int_0^t g(u, {}^k x_u) dW(\cdot)(u) & t \in [0, a] \\ \theta(\omega)(t) & t \in J \end{cases} \quad (2)$$

a.s.

Since $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is \mathcal{F}_0 -measurable, then ${}^1 x \in L^2(\Omega, C([-r, a], \mathbf{R}^d))$ and is trivially adapted to $(\mathcal{F}_t)_{t \in [0, a]}$. Hence ${}^1 x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is \mathcal{F}_t -measurable for all $t \in [0, a]$. P(1) (iii) holds trivially.

Now suppose $P(k)$ is satisfied for some $k > 1$. Then by Hypothesis $(E_1)(i), (ii)$ and the continuity of the slicing map (*stochastic memory*), it follows from $P(k)(ii)$ that the process

$$[0, a] \ni u \longmapsto g(u, {}^k x_u) \in L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d))$$

is continuous and adapted to $(\mathcal{F}_t)_{t \in [0, a]}$. $P(k+1)(i)$ and $P(k+1)(ii)$ follow from the continuity and adaptability of the stochastic integral. Check $P(k+1)(iii)$, by using Doob's inequality.

For each $k > 1$, write

$${}^k x = {}^1 x + \sum_{i=1}^{k-1} ({}^{i+1} x - {}^i x).$$

Now $L_A^2(\Omega, C([-r, a], \mathbf{R}^d))$ is closed in $L^2(\Omega, C([-r, a], \mathbf{R}^d))$; so the series

$$\sum_{i=1}^{\infty} ({}^{i+1} x - {}^i x)$$

converges in $L_A^2(\Omega, C([-r, a], \mathbf{R}^d))$ because of (1) and the convergence of

$$\sum_{i=1}^{\infty} \left[(ML^2)^{i-1} \frac{a^{i-1}}{(i-1)!} \right]^{1/2}.$$

Hence $\{{}^k x\}_{k=1}^{\infty}$ converges to some $x \in L_A^2(\Omega, C([-r, a], \mathbf{R}^d))$.

Clearly $x|J = \theta$ and is \mathcal{F}_0 -measurable, so applying Doob's inequality to the Itô integral of the difference

$$u \longmapsto g(u, {}^k x_u) - g(u, x_u)$$

gives

$$\begin{aligned} E\left(\sup_{t \in [0, a]} \left| \int_0^t g(u, {}^k x_u) dW(\cdot)(u) - \int_0^t g(u, x_u) dW(\cdot)(u) \right|^2\right) \\ < ML^2 a \| {}^k x - x \|_{L^2(\Omega, C)}^2 \\ \longrightarrow 0 \text{ as } k \rightarrow \infty. \end{aligned}$$

Thus viewing the right-hand side of (2) as a process in $L^2(\Omega, C([-r, a], \mathbf{R}^d))$ and letting $k \rightarrow \infty$, it follows from the above that x must satisfy the sfde (XII) a.s. for all $t \in [-r, a]$.

For uniqueness, let $\tilde{x} \in L_A^2(\Omega, ([-r, a], \mathbf{R}^d))$ be also a solution of (XII) with initial process θ . Then by the Lipschitz condition:

$$\|x_t - \tilde{x}_t\|_{L^2(\Omega, C)}^2 < ML^2 \int_0^t \|x_u - \tilde{x}_u\|_{L^2(\Omega, C)}^2 du$$

for all $t \in [0, a]$. Therefore we must have $x_t - \tilde{x}_t = 0$ for all $t \in [0, a]$; so $x = \tilde{x}$ in $L^2(\Omega, C([-r, a], \mathbf{R}^d))$ a.s. \square

Remarks and Generalizations.

- (i) In Theorem I.2 replace the process $(t, W(t))$ by a (square integrable) semimartingale $Z(t)$ satisfying appropriate conditions. ([Mo], 1984, Chapter II).
- (ii) Results on existence of solutions of sfde's driven by white noise were first obtained by Itô and Nisio ([I-N], J. Math. Kyoto University, 1968) and then Kushner (JDE, 197).
- (iii) Extensions to sfde's with *infinite* memory. Fading memory case: work by Mizel and Trützer [M-T], JIE, 1984, Marcus and Mizel [M-M], Stochastics, 1988; general infinite memory: Itô and Nisio [I-N], J. Math. Kyoto University, 1968.
- (iii) Pathwise local uniqueness holds for sfde's of type (XIII) under a global Lipschitz condition: If coefficients of two sfde's agree on an open set in C , then the corresponding trajectories leave the open set at the same time and agree almost surely up to the time they leave the open set ([Mo], Pitman Books, 1984, Theorem 4.2, pp. 150-151.)

(iv) Replace the state space C by the Delfour-Mitter Hilbert space

$M_2 := \mathbf{R}^d \times L^2([-r, 0], \mathbf{R}^d)$ with the Hilbert norm

$$\|(v, \eta)\|_{M_2} = \left(|v|^2 + \int_{-r}^0 |\eta(s)|^2 ds \right)^{1/2}$$

for $(v, \eta) \in M_2$ (T. Ahmed, S. Elsanousi and S. Mohammed, 1983).

(v) Have Lipschitz and smooth dependence of ${}^\theta x_t$ on the initial process $\theta \in L^2(\Omega, C)$ ([Mo], 1984, Theorems 3.1, 3.2, pp. 41-45).