I. EXISTENCE

Campinas, Brazil

June 26, 2002

Salah-Eldin A. Mohammed

Southern Illinois University

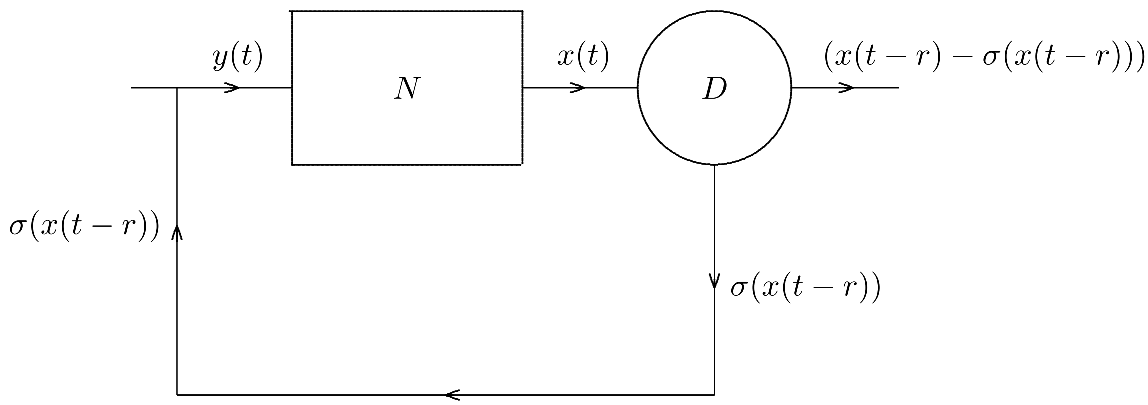
Carbondale, IL 62901–4408 USA

Web page: **http://sfde.math.siu.edu**

I. EXISTENCE

1. Examples

Example 1. (*Noisy Feedbacks*)



Box N: Input $= y(t)$,   output $= x(t)$ at time $t > 0$ related by

$$x(t) = x(0) + \int_0^t y(u)\, dZ(u) \tag{1}$$

where $Z(u)$ is a semimartingale noise.

Box D: Delays signal $x(t)$ by $r\,(> 0)$ units of time. A proportion σ $(0 \le \sigma \le 1)$ is transmitted through D and the rest $(1-\sigma)$ is used for other purposes.

Therefore

$$y(t) = \sigma x(t-r)$$

Take $\dot{Z}(u) :=$ white noise $= \dot{W}(u)$

Then substituting in (1) gives the Itô integral equation

$$x(t) = x(0) + \sigma \int_0^t x(u-r) dW(u)$$

or the stochastic differenial delay equation (sdde):

$$dx(t) = \sigma x(t-r)dW(t), \qquad t > 0 \tag{I}$$

To solve (I), need an *initial process* $\theta(t)$, $-r \leq t \leq 0$:

$$x(t) = \theta(t) \quad \text{a.s.}, \qquad -r \leq t \leq 0$$

$\mathbf{r = 0}$: (I) becomes a linear stochastic ode and has closed form solution

$$x(t) = x(0)e^{\sigma W(t) - \frac{\sigma^2 t}{2}}, \qquad t \geq 0.$$

$\mathbf{r>0}$: Solve (I) by successive Itô integrations over steps of length r:

$$x(t) = \theta(0) + \sigma \int_0^t \theta(u-r)\, dW(u), \quad 0 \leq t \leq r$$
$$x(t) = x(r) + \sigma \int_r^t [\theta(0) + \sigma \int_0^{(v-r)} \theta(u-r)\, dW(u)]\, dW(v),\ r < t \leq 2r,$$
$$\cdots = \cdots \qquad 2r < t \leq 3r,$$

No closed form solution is known (even in deterministic case).

Curious Fact!

In the sdde (I) the Itô differential dW may be replaced by the Stratonovich differential $\circ dW$ *without changing the solution* x. Let x be the solution of (I) under an Itô differential dW. Then using finite partitions $\{u_k\}$ of the interval $[0,t]$:

$$\int_0^t x(u-r) \circ dW(t) = \lim \sum_k \frac{1}{2}[x(u_k - r) + x(u_{k+1} - r)][W(u_{k+1}) - W(u_k)]$$

where the limit in probability is taken as the mesh of the partition $\{u_k\}$ goes to zero. Compare the Stratonovich and Itô integrals using the corresponding partial sums:

$$
\begin{aligned}
\lim E\bigg(\sum_k \frac{1}{2}[x(u_k - r) + x(u_{k+1} - r)][W(u_{k+1}) - W(u_k)] \\
- \sum_k [x(u_k - r)][W(u_{k+1}) - W(u_k)]\bigg)^2 \\
= \lim E\bigg(\sum_k \frac{1}{2}[x(u_{k+1} - r) - x(u_k - r)][W(u_{k+1}) - W(u_k)]\bigg)^2 \\
= \lim \sum_k \frac{1}{4} E[x(u_{k+1} - r) - x(u_k - r)]^2 \, E[W(u_{k+1}) - W(u_k)]^2 \\
= \lim \sum_k \frac{1}{4} E[x(u_{k+1} - r) - x(u_k - r)]^2 \, (u_{k+1} - u_k) \\
= 0
\end{aligned}
$$

because W has independent increments, x is adapted to the Brownian filtration, $u \mapsto x(u) \in L^2(\Omega, \mathbf{R})$ is continuous, and the delay r is positive. Alternatively

$$
\int_0^t x(u-r) \circ dW(u) = \int_0^t x(u-r)\, dW(u) + \frac{1}{2} < x(\cdot - r, W > (t)
$$

and $< x(\cdot - r, W > (t) = 0$ for all $t > 0$.

Remark.

When $r > 0$, the solution process $\{x(t) : t \geq -r\}$ of (I) is a martingale but is *non-Markov*.

Example 2. (*Simple Population Growth*)

Consider a large population $x(t)$ at time t evolving with a constant birth rate $\beta > 0$ and a constant death rate α per capita. Assume immediate removal of the dead from the population. Let $r > 0$ (fixed, non-random$=9$, e.g.) be the development period of each individual and assume there is migration whose overall rate is distributed like white noise $\sigma\dot{W}$ (mean zero and variance $\sigma > 0$), where W is one-dimensional standard Brownian motion. The change in population $\Delta x(t)$ over a small time interval $(t, t+\Delta t)$ is

$$\Delta x(t) = -\alpha x(t)\Delta t + \beta x(t-r)\Delta t + \sigma\dot{W}\Delta t$$

Letting $\Delta t \to 0$ and using Itô stochastic differentials,

$$dx(t) = \{-\alpha x(t) + \beta x(t-r)\}\, dt + \sigma dW(t), \quad t > 0. \tag{II}$$

Associate with the above affine sdde the initial condition $(v, \eta) \in \mathbf{R} \times L^2([-r, 0], \mathbf{R})$

$$x(0) = v, \quad x(s) = \eta(s), \quad -r \leq s < 0.$$

Denote by $M_2 = \mathbf{R} \times L^2([-r, 0], \mathbf{R})$ the Delfour-Mitter Hilbert space of all pairs (v, η), $v \in \mathbf{R}$, $\eta \in L^2([-r, 0], \mathbf{R})$ with norm

$$\|(v, \eta)\|_{M_2} = \left(|v|^2 + \int_{-r}^{0} |\eta(s)|^2\, ds \right)^{1/2}.$$

Let $W : \mathbf{R}^+ \times \Omega \to \mathbf{R}$ be defined on the canonical filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbf{R}^+}, P)$ where

$$\Omega = C(\mathbf{R}^+, \mathbf{R}),\ \mathcal{F} = \text{ Borel } \Omega,\ \mathcal{F}_t = \sigma\{\rho_u : u \leq t\}$$

$\rho_u : \Omega \to \mathbf{R}, u \in \mathbf{R}^+$, are evaluation maps $\omega \mapsto \omega(u)$, and $P =$ Wiener measure on Ω.

Example 3. (*Logistic Population Growth*)

A single population $x(t)$ at time t evolving logistically with *development (incubation) period* $r > 0$ under Gaussian type noise (e.g. migration on a molecular level):

$$\dot{x}(t) = [\alpha - \beta x(t-r)]\, x(t) + \gamma x(t)\dot{W}(t), \quad t > 0$$

i.e.

$$dx(t) = [\alpha - \beta x(t-r)]\, x(t)\, dt + \gamma x(t) dW(t) \quad t > 0. \tag{III}$$

with *initial condition*

$$x(t) = \theta(t) \qquad -r \leq t \leq 0.$$

For positive delay r the above sdde can be solved *implicitly* using forward steps of length r, i.e. for $0 \leq t \leq r$, $x(t)$ satisfies the *linear* sode (without delay)

$$dx(t) = [\alpha - \beta\theta(t-r)]\, x(t)\, dt + \gamma x(t) dW(t) \qquad 0 < t \leq r. \tag{III'}$$

$x(t)$ is a semimartingale and is non-Markov (Scheutzow [S], 1984).

Example 4. (*Heat bath*)

Model proposed by R. Kubo (1966) for physical Brownian motion. A molecule of mass m moving under random gas forces with position $\xi(t)$ and velocity $v(t)$ at time t; cf classical work by Einstein and Ornestein and Uhlenbeck. Kubo proposed the following modification of the Ornstein-Uhenbeck process

$$\left.\begin{aligned} d\xi(t) &= v(t)\, dt \\ m dv(t) &= -m[\int_{t_0}^{t} \beta(t-t')v(t')\, dt']\, dt + \gamma(\xi(t), v(t))\, dW(t),\ t > t_0. \end{aligned}\right\} \tag{IV}$$

m =mass of molecule. No external forces.

$\beta =$ viscosity coefficient function with compact support.

γ a function $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ representing the random gas forces on the molecule.

$\xi(t) =$ position of molecule $\in \mathbf{R}^3$.

$v(t) =$ velocity of molecule $\in \mathbf{R}^3$.

$W = 3-$ dimensional Brownian motion.

([Mo], Pitman Books, RN # 99, 1984, pp. 223-226).

Further Examples

Delay equation with Poisson noise:

$$\left.\begin{aligned} dx(t) &= x((t-r)-)\,dN(t) \quad t>0 \\ x_0 &= \eta \in D([-r,0],\mathbf{R}) \end{aligned}\right\} \qquad (V)$$

N := Poisson process with iid interarrival times ([S], Hab. 1988). $D([-r,0],\mathbf{R})$ = space of all cadlag paths $[-r,0] \to \mathbf{R}$, with sup norm.

Simple model of dye circulation in the blood (or pollution) (cf. Bailey and Williams [B-W], JMAA, 1966, Lenhart and Travis ([L-T], PAMS, 1986).

$$\left.\begin{aligned} dx(t) &= \{\nu x(t) + \mu x(t-r))\}\,dt + \sigma x(t)\,dW(t) \quad t>0 \\ (x(0), x_0) &= (v,\eta) \in M_2 = \mathbf{R} \times L^2([-r,0],\mathbf{R}), \end{aligned}\right\} \qquad (VI)$$

([Mo], Survey, 1992; [M-S], II, 1995.)

In above model:

$x(t)$:= dye concentration (gm/cc)

r = time taken by blood to traverse side tube (vessel)

Flow rate (cc/sec) is Gaussian with variance σ.

A fixed proportion of blood in main vessel is pumped into side vessel(s). Model will be analysed in Lecture V (Theorem V.5).

$$
\left.\begin{aligned}
dx(t) = \{\nu x(t) + \mu x(t-r))\}\, dt + \{\int_{-r}^{0} x(t+s)\sigma(s)\, ds\}\, dW(t), \\
(x(0), x_0) = (v, \eta) \in M_2 = \mathbf{R} \times L^2([-r,0], \mathbf{R}),\ t > 0.
\end{aligned}\right\} \qquad (VII)
$$

([Mo], Survey, 1992; [M-S], II, 1995.)

Linear d-dimensional systems driven by m-dimensional Brownian motion $W := (W_1, \cdots, W_m)$ with constant coefficients.

$$
\left.\begin{aligned}
dx(t) = H(x(t-d_1), \cdots, x(t-d_N), x(t), x_t)dt \\
+ \sum_{i=1}^{m} g_i x(t)\, dW_i(t), \quad t > 0 \\
(x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d)
\end{aligned}\right\} \qquad (VIII)
$$

$H := (\mathbf{R}^d)^N \times M_2 \to \mathbf{R}^d$ linear functional on $(\mathbf{R}^d)^N \times M_2$; g_i $d \times d$-matrices ([Mo], Stochastics, 1990).

Linear systems driven by (helix) semimartingale noise (N, L), and memory driven by a (stationary) measure-valued process ν and a (stationary) process K ([M-S], I, AIHP, 1996):

$$
\left.\begin{aligned}
dx(t) = \Big\{\int_{[-r,0]} \nu(t)(ds)\, x(t+s)\Big\}\, dt \\
+ dN(t) \int_{-r}^{0} K(t)(s)\, x(t+s)\, ds + dL(t)\, x(t-), \quad t > 0 \\
(x(0), x_0) = (v, \eta) \in M_2 = \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d)
\end{aligned}\right\} \qquad (IX)
$$

Multidimensional affine systems driven by (helix) noise Q ([M-S], Stochastics, 1990):

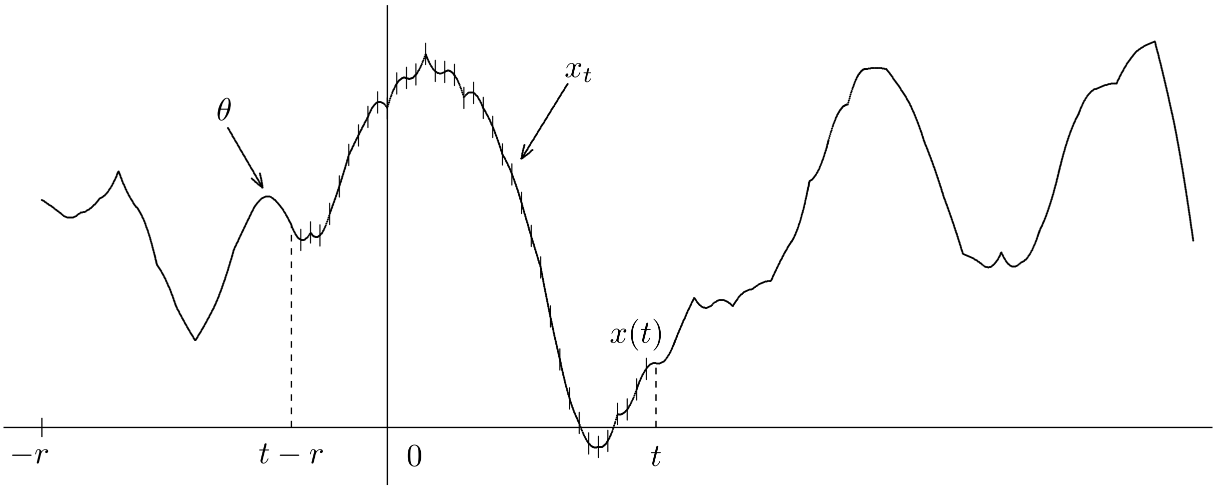
$$
\left.\begin{aligned}
dx(t) = \Big\{\int_{[-r,0]} \nu(t)(ds)\, x(t+s)\Big\}\, dt + dQ(t), \quad t > 0 \\
(x(0), x_0) = (v, \eta) \in M_2 := \mathbf{R}^d \times L^2([-r,0], \mathbf{R}^d)
\end{aligned}\right\} \qquad (X)
$$

Memory driven by white noise:

$$\left.\begin{aligned} dx(t) &= \left\{\int_{[-r,0]} x(t+s)\,dW(s)\right\} dW(t) \quad t > 0 \\ x(0) &= v \in \mathbf{R}, \quad x(s) = \eta(s), \quad -r < s < 0, \quad r \geq 0 \end{aligned}\right\} \qquad (XI)$$

([Mo], Survey, 1992).

Formulation



Slice each solution path x over the interval $[t-r,t]$ to get *segment* x_t as a process on $[-r,0]$:

$$x_t(s) := x(t+s) \qquad \text{a.s.},\ t \geq 0, s \in J := [-r,0].$$

Therefore sdde's (I), (II), (III) and (XI) become

$$\left.\begin{aligned} dx(t) = \sigma x_t(-r)dW(t), \quad t>0 \\ x_0 = \theta \in C([-r,0],\mathbf{R}) \end{aligned}\right\} \qquad (I)$$

$$\left.\begin{aligned} dx(t) = \{-\alpha x(t) + \beta x_t(-r)\}\, dt + \sigma dW(t), \quad t>0 \\ (x(0),x_0) = (v,\eta) \in \mathbf{R}\times L^2([-r,0],\mathbf{R}) \end{aligned}\right\} \qquad (II)$$

$$\left.\begin{array}{l} dx(t) = [\alpha - \beta x_t(-r)]x_t(0)\,dt + \gamma x_t(0)\,dW(t) \\ \\ x_0 = \theta \in C([-r,0],\mathbf{R}) \end{array}\right\} \qquad (III)$$

$$\left.\begin{array}{r} dx(t) = \left\{\int_{[-r,0]} x_t(s)\,dW(s)\right\} dW(t) \quad t > 0 \\ (x(0), x_0) = (v,\eta) \in \mathbf{R} \times L^2([-r,0],\mathbf{R}), \quad r \geq 0 \end{array}\right\} \qquad (XI)$$

Think of R.H.S.'s of the above equations as functionals of x_t (and $x(t)$) and generalize to *stochastic functional differential equation* (sfde)

$$\left.\begin{array}{l} dx(t) = h(t,x_t)dt + g(t,x_t)dW(t) \quad t > 0 \\ \\ x_0 = \theta \end{array}\right\} \qquad (XII)$$

on filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions:

$(\mathcal{F}_t)_{t\geq 0}$ right-continuous and each $\mathcal{F}_t$ contains all P-null sets in $\mathcal{F}$.

$C := C([-r,0],\mathbf{R}^d)$ Banach space, sup norm.

$W(t) = m$–dimensional Brownian motion.

$L^2(\Omega, C) :=$ Banach space of all $(\mathcal{F}, Borel C)$-measurable L^2 (Bochner sense) maps $\Omega \to C$ with the L^2-norm

$$\|\theta\|_{L^2(\Omega,C)} := \left[\int_\Omega \|\theta(\omega)\|_C^2 \, dP(\omega)\right]^{1/2}$$

Coefficients:

$$h : [0,T] \times L^2(\Omega, C) \to L^2(\Omega, \mathbf{R}^d) \qquad \text{(Drift)}$$

$$g : [0,T] \times L^2(\Omega, C) \to L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d) \qquad \text{(Diffusion)}.$$

Initial data:

$$\theta \in L^2(\Omega, C, \mathcal{F}_0).$$

Solution:

$x : [-r, T] \times \Omega \to \mathbf{R}^d$ measurable and sample-continuous, $x|[0,T]$ $(\mathcal{F}_t)_{0 \le t \le T}$-adapted and $x(s)$ is $\mathcal{F}_0$-measurable for all $s \in [-r, 0]$.

Exercise: $[0,T] \ni t \mapsto x_t \in C([-r,0], \mathbf{R}^d)$ is $(\mathcal{F}_t)_{0 \le t \le T}$-adapted.

(*Hint:* $Borel\, C$ is generated by all evaluations.)

Hypotheses (E_1)**.**

(i) h, g are jointly continuous and uniformly Lipschitz in the second variable with respect to the first:

$$\|h(t,\psi_1) - h(t,\psi_2)\|_{L^2(\Omega,\mathbf{R}^d)} \leq L\|\psi_1 - \psi_2\|_{L^2(\Omega,C)}$$

for all $t \in [0,T]$ and $\psi_1, \psi_2 \in L^2(\Omega, C)$. Similarly for the diffusion coefficent g.

(ii) For each $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted process $y : [0,T] \to L^2(\Omega, C)$, the processes $h(\cdot, y(\cdot)), g(\cdot, y(\cdot))$ are also $(\mathcal{F}_t)_{0\leq t\leq T}$- adapted.

Theorem I.1. ([Mo], 1984) (Existence and Uniqueness).

Suppose h and g satisfy Hypotheses (E_1). Let $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

Then the sfde (XII) has a unique solution ${}^\theta x : [-r,\infty) \times \Omega \to \mathbf{R}^d$ starting off at $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ with $t \longmapsto {}^\theta x_t$ continuous and ${}^\theta x \in L^2(\Omega, C([-r,T]\mathbf{R}^d))$ for all $T > 0$. For a given θ, uniqueness holds up to equivalence among all $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted processes in $L^2(\Omega, C([-r,T], \mathbf{R}^d))$.

Proof.

[Mo], Pitman Books, 1984, Theorem 2.1, pp. 36-39. □

Theorem I.1 covers equations (I), (II), (IV), (VI), (VII), (VIII), (XI) and a large class of sfde's driven by white noise. Note that (XI) *does not satisfy the hypotheses underlying the classical results* of Doleans-Dade [Dol], 1976, Metivier and Pellaumail [Met-P], 1980, Protter, Ann. Prob. 1987, Lipster and Shiryayev [Lip-Sh], [Met], 1982. This is because the coefficient

$$\eta \to \int_{-r}^{0} \eta(s)\, dW(s)$$

on the RHS of (XI) *does not admit almost surely Lipschitz (or even linear) versions* $C \to \mathbf{R}$! This will be shown later.

When the coeffcients h, g factor through functionals

$$H : [0,T] \times C \to \mathbf{R}^d, \qquad G : [0,T] \times C \to \mathbf{R}^{d \times m}$$

we can impose the following local Lipschitz and global linear growth conditions on the sfde

$$\left.\begin{array}{ll} dx(t) = H(t, x_t)\, dt + G(t, x_t)\, dW(t) & t > 0 \\ x_0 = \theta & \end{array}\right\} \qquad (XIII)$$

with W m-dimensional Brownian motion:

Hypotheses (E_2)

(i) H, G are Lipschitz on bounded sets in C: For each integer $n \geq 1$ there exists $L_n > 0$ such that

$$|H(t, \eta_1) - H(t, \eta_2)| \leq L_n \|\eta_1 - \eta_2\|_C$$

for all $t \in [0, T]$ and $\eta_1, \eta_2 \in C$ with $\|\eta_1\|_C \leq n$, $\|\eta_2\|_C \leq n$. Similarly for the diffusion coefficent G.

(ii) There is a constant $K > 0$ such that

$$|H(t, \eta)| + \|G(t, \eta)\| \leq K(1 + \|\eta\|_C)$$

for all $t \in [0, T]$ and $\eta \in C$.

Note that the adaptability condition is not needed (explicitly) because H, G are deterministic and because the sample-continuity and adaptability of x imply that the segment $[0, T] \ni t \mapsto x_t \in C$ is also adapted.

Exercise: Formulate the heat-bath model (IV) as a sfde of the form (XIII).(β has compact support in $\mathbf{R}^+$.)

Theorem I.2. ([Mo], 1984) (Existence and Uniqueness).

Suppose H *and* G *satisfy Hypotheses* (E_2) *and let* $\theta \in L^2(\Omega, C; \mathcal{F}_0)$.

Then the sfde (XIII) has a unique $(\mathcal{F}_t)_{0\le t\le T}$-adapted *solution* ${}^\theta x : [-r,T] \times \Omega \to \mathbf{R}^d$ *starting off at* $\theta \in L^2(\Omega, C; \mathcal{F}_0)$ *with* $t \longmapsto {}^\theta x_t$ *continuous and* ${}^\theta x \in L^2(\Omega, C([-r,T], \mathbf{R}^d))$ *for all* $T > 0$. *For a given* θ, *uniqueness holds up to equivalence among all* $(\mathcal{F}_t)_{0\le t\le T}$-adapted *processes in* $L^2(\Omega, C([-r,T], \mathbf{R}^d))$.

Furthermore if $\theta \in L^{2k}(\Omega, C; \mathcal{F}_0)$, *then* ${}^\theta x_t \in L^{2k}(\Omega, C; \mathcal{F}_t)$ *and*

$$E\|{}^\theta x_t\|_C^{2k} \le C_k[1 + \|\theta\|_{L^{2k}(\Omega,C)}^{2k}]$$

for all $t \in [0,T]$ *and some positive constants* C_k.

Proofs of Theorems I.1, I.2.(Outline)

[Mo], pp. 150-152. Generalize sode proofs in Gihman and Skorohod ([G-S], 1973) or Friedman ([Fr], 1975):

(1) Truncate coefficients outside bounded sets in C. Reduce to globally Lipschitz case.

(2) Successive approx. in globally Lipschitz situation.

(3) Use local uniqueness ([Mo], Theorem 4.2, p. 151) to "patch up" solutions of the truncated sfde's.

For (2) consider globally Lipschitz case and $h \equiv 0$.

We look for solutions of (XII) by successive approximation in $L^2(\Omega, C([-r,a], \mathbf{R}^d))$. Let $J := [-r, 0]$.

Suppose $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$ is $\mathcal{F}_0$-measurable. Note that this is equivalent to saying that $\theta(\cdot)(s)$ is $\mathcal{F}_0$-measurable for all $s \in J$, because θ has a.a. sample paths continuous.

We prove by induction that there is a sequence of processes ${}^k x : [-r, a] \times \Omega \to \mathbf{R}^d$, $k = 1, 2, \cdots$ having the

Properties $P(k)$*:*

(i) ${}^k x \in L^2(\Omega, C([-r,a], \mathbf{R}^d))$ and is adapted to $(\mathcal{F}_t)_{t\in[0,a]}$.

(ii) For each $t \in [0,a]$, ${}^k x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_t$-measur-able.

(iii)

$$\left.\begin{aligned} \|{}^{k+1}x - {}^k x\|_{L^2(\Omega,C)} &\leq (ML^2)^{k-1}\frac{a^{k-1}}{(k-1)!}\|{}^2 x - {}^1 x\|_{L^2(\Omega,C)} \\ \|{}^{k+1}x_t - {}^k x_t\|_{L^2(\Omega,C)} &\leq (ML^2)^{k-1}\frac{t^{k-1}}{(k-1)!}\|{}^2 x - {}^1 x\|_{L^2(\Omega,C)} \end{aligned}\right\} \quad (1)$$

where M is a "martingale" constant and L is the Lipschitz constant of g.

Take ${}^1 x : [-r,a] \times \Omega \to \mathbf{R}^d$ to be

$${}^1 x(t,\omega) = \begin{cases} \theta(\omega)(0) & t \in [0,a] \\ \theta(\omega)(t) & t \in J \end{cases}$$

a.s., and

$${}^{k+1}x(t,\omega) = \begin{cases} \theta(\omega)(0) + (\omega)\displaystyle\int_0^t g(u, {}^k x_u)dW(\cdot)(u) & t \in [0,a] \\ \theta(\omega)(t) \qquad t \in J \end{cases} \quad (2)$$

a.s.

Since $\theta \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_0$-measurable, then ${}^1 x \in L^2(\Omega, C([-r,a], \mathbf{R}^d))$ and is trivially adapted to $(\mathcal{F}_t)_{t\in[0,a]}$. Hence ${}^1 x_t \in L^2(\Omega, C(J, \mathbf{R}^d))$ and is $\mathcal{F}_t$-measurable for all $t \in [0,a]$. $P(1)$ (iii) holds trivially.

Now suppose $P(k)$ is satisfied for some $k > 1$. Then by Hypothesis $(E_1)(i), (ii)$ and the continuity of the slicing map (*stochastic memory*), it follows from $P(k)(ii)$ that the process

$$[0,a] \ni u \longmapsto g(u, {}^k x_u) \in L^2(\Omega, L(\mathbf{R}^m, \mathbf{R}^d))$$

is continuous and adapted to $(\mathcal{F}_t)_{t\in[0,a]}$. $P(k+1)(i)$ and $P(k+1)(ii)$ follow from the continuity and adaptability of the stochastic integral. Check $P(k+1)(iii)$, by using Doob's inequality.

For each $k > 1$, write

$${}^k x = {}^1 x + \sum_{i=1}^{k-1} ({}^{i+1}x - {}^i x).$$

Now $L^2_A(\Omega, C([-r,a], \mathbf{R}^d))$ is closed in $L^2(\Omega, C([-r,a], \mathbf{R}^d))$; so the series

$$\sum_{i=1}^{\infty} ({}^{i+1}x - {}^i x)$$

converges in $L^2_A(\Omega, C([-r,a], \mathbf{R}^d))$ because of (1) and the convergence of

$$\sum_{i=1}^{\infty} \left[(ML^2)^{i-1} \frac{a^{i-1}}{(i-1)!} \right]^{1/2}.$$

Hence $\{{}^k x\}_{k=1}^{\infty}$ converges to some $x \in L^2_A(\Omega, C([-r,a], \mathbf{R}^d))$.

Clearly $x|J = \theta$ and is $\mathcal{F}_0$-measurable, so applying Doob's inequality to the Itô integral of the difference

$$u \longmapsto g(u, {}^k x_u) - g(u, x_u)$$

gives

$$\begin{aligned} E\Big(\sup_{t\in[0,a]}\Big|\int_0^t g(u, {}^k x_u)\, dW(\cdot)(u) - \int_0^t g(u, x_u)\, dW(\cdot)(u)\Big|^2\Big) \\ < ML^2 a \|{}^k x - x\|^2_{L^2(\Omega,C)} \\ \longrightarrow 0 \text{ as } k \to \infty. \end{aligned}$$

Thus viewing the right-hand side of (2) as a process in $L^2(\Omega, C([-r,a], \mathbf{R}^d))$ and letting $k \to \infty$, it follows from the above that x must satisfy the sfde (XII) a.s. for all $t \in [-r, a]$.

For uniqueness, let $\tilde{x} \in L^2_A(\Omega, ([-r,a], \mathbf{R}^d))$ be also a solution of (XII) with initial process θ. Then by the Lipschitz condition:

$$\|x_t - \tilde{x}_t\|^2_{L^2(\Omega,C)} < ML^2 \int_0^t \|x_u - \tilde{x}_u\|^2_{L^2(\Omega,C)}\, du$$

for all $t \in [0,a]$. Therefore we must have $x_t - \tilde{x}_t = 0$ for all $t \in [0,a]$; so $x = \tilde{x}$ in $L^2(\Omega, C([-r,a], \mathbf{R}^d))$ a.s. □

Remarks and Generalizations.

(i) In Theorem I.2 replace the process $(t, W(t))$ by a (square integrable) semimartingale $Z(t)$ satisfying appropriate conditions.([Mo], 1984, Chapter II).

(ii) Results on existence of solutions of sfde's driven by white noise were first obtained by Itô and Nisio ([I-N], J. Math. Kyoto University, 1968) and then Kushner (JDE, 197).

(iii) Extensions to sfde's with *infinite* memory. Fading memory case: work by Mizel and Trützer [M-T],JIE, 1984, Marcus and Mizel [M-M], Stochastics, 1988; general infinite memory: Itô and Nisio [I-N], J. Math. Kyoto University, 1968.

(iii) Pathwise local uniqueness holds for sfde's of type (XIII) under a global Lipschitz condition: If coeffcients of two sfde's agree on an open set in C, then the corresponding trajectories leave the open set at the same time and agree almost surely up to the time they leave the open set ([Mo], Pitman Books, 1984, Theorem 4.2, pp. 150-151.)

(iv) Replace the state space C by the Delfour-Mitter Hilbert space $M_2 := \mathbf{R}^d \times L^2([-r,0],\mathbf{R}^d)$ with the Hilbert norm

$$\|(v,\eta)\|_{M_2} = \left(|v|^2 + \int_{-r}^{0} |\eta(s)|^2 \, ds\right)^{1/2}$$

for $(v,\eta) \in M_2$ (T. Ahmed, S. Elsanousi and S. Mohammed, 1983).

(v) Have Lipschitz and smooth dependence of ${}^{\theta}x_t$ on the initial process $\theta \in L^2(\Omega, C)$ ([Mo], 1984, Theorems 3.1, 3.2, pp. 41-45).